1 Setup - Lagrange Function

\[ L = \max_{c_0, c_1, c_2, s_0, s_1, \lambda_1, \lambda_2, \lambda_3} \left[ (u(c_0) + \beta u(c_1) + \beta^2 u(c_2)) \right. \]
\[ \left. - \lambda_0 (w_0 - s_0 - c_0) \right. \]
\[ \left. - \lambda_1 (w_1 + (1 + r)s_0 - s_1 - c_1) \right. \]
\[ \left. - \lambda_2 ((1 + r)s_1 - c_2) \right] \]

2 First-order conditions

\[ L_{c_0} = u'(c_0) + \lambda_0 = 0 \] (1)
\[ L_{c_1} = \beta u'(c_1) + \lambda_1 = 0 \] (2)
\[ L_{c_2} = \beta^2 u'(c_2) + \lambda_2 = 0 \] (3)
\[ L_{s_0} = \lambda_0 - \lambda_1 (1 + r) = 0 \] (4)
\[ L_{s_1} = \lambda_1 - \lambda_2 (1 + r) = 0 \] (5)
\[ L_{\lambda_0} = w_0 - s_0 - c_0 = 0 \] (6)
\[ L_{\lambda_1} = w_1 + (1 + r)s_0 - s_1 - c_1 = 0 \] (7)
\[ L_{\lambda_2} = (1 + r)s_1 - c_2 = 0 \] (8)

Plug (1) and (2) into (4):
\[ u'(c_0) = (1 + r)\beta u'(c_1) \] (9)

Plug (2) and (3) into (5):
\[ \beta u'(c_1) = (1 + r)\beta^2 u'(c_2) \]
\[ u'(c_1) = (1 + r)\beta u'(c_2) \] (10)

Solve (6) for \( s_0 \) and plug into (7):
\[ s_1 = w_1 - c_1 + (1 + r)(w_0 - c_0) \] (11)
Solve (11) for $s_1$ and plug into (8):

$$c_2 = (1 + r)(w_1 - c_1) + (1 + r)^2(w_0 - c_0)$$

(12)

3 Summary and interpretation

A full set of FOC’s contains three equations (9),(10),(12) with three unknowns $c_0, c_1, c_2$:

$$u'(c_0) = (1 + r)\beta u'(c_1)$$
$$u'(c_1) = (1 + r)\beta u'(c_2)$$
$$c_2 = (1 + r)(w_1 - c_1) + (1 + r)^2(w_0 - c_0)$$

The Lagrange function has up to three Lagrange variables ($\lambda_0, \lambda_1, \lambda_2$). The exact number depends on whether you summarize the constraints before you set up the Lagrange function. Contrary to the example in the slides, the Lagrange variables have a negative interpretation here, as it is set up differently:

- $\lambda_0$: If the first constraint is marginally more slack, utility increases by $(-1)\lambda_0$. E.g. if wealth in period 0 $w_0$ increases by one marginal unit, utility increases by $(-1)\lambda_0$ units.

Similar arguments can be made for $\lambda_1$ and $\lambda_2$. 