1 Logistic difference equation

a) The steady states are given by

\[ x = \lambda x(1 - x). \]

One solution is \( x = 0 \). If \( x \neq 0 \) we can divide both sides by \( x \) and we get \( 1 = \lambda(1 - x) \).

This gives a second steady state of \( x = \frac{\lambda - 1}{\lambda} \).

b) No.

c) i) The steady state \( x = \frac{\lambda - 1}{\lambda} \) is locally stable iff \( \left. \frac{dx_{t+1}}{dx_t} \right|_{x_t=x} < 1 \). The derivative at the steady state is given by

\[
\frac{dx_{t+1}}{dx_t} = \lambda(1 - 2x_t) \\
= \lambda \left( 1 - 2\frac{\lambda - 1}{\lambda} \right) \\
= \lambda - 2\lambda + 2 \\
= 2 - \lambda.
\]

Hence, for \( 1 < \lambda < 3 \) the steady state is locally stable.

ii) Oscillations occur iff \( \left. \frac{dx_{t+1}}{dx_t} \right|_{x_t=x} < 0 \). That is, \( 2 - \lambda \) has to be negative, or, \( \lambda > 2 \).

Figure 1 depicts one graph for \( 1 < \lambda < 2 \), and one for \( 2 < \lambda < 3 \).

d) See accompanying spreadsheet.
2 Linear, 2-dimensional, 1st order differential equation

a) The steady states are \( x = y = 1 \).

b) The \( \dot{x} = 0 \)-locus is given by

\[
y = \frac{1}{2} + \frac{1}{2}x
\]

and the \( \dot{y} = 0 \)-locus by

\[
y = 2 - x.
\]

The phase diagram is shown in Figure 2.

c) It is convenient to rewrite the differential equations system in matrix notation as

\[
\dot{u} = Au + b,
\]

where \( A := \begin{pmatrix} -2 & 4 \\ 1 & 1 \end{pmatrix} \), \( u := \begin{pmatrix} x \\ y \end{pmatrix} \), and \( b := \begin{pmatrix} 1 \\ 1 \end{pmatrix} \). This system of linear, first-order differential equations is non-homogeneous (i.e. \( b \neq 0 \)). It is easier to solve homogeneous differential equations. We therefore transform the system by defining the deviation from the steady state \( u^* = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) as \( z := u - u^* \). The differential equations system eq. (2.1)
then becomes

\[ \dot{z} = Az. \] (2.2)

For the solution (as well for the stability analysis) we need to know the eigenvalues of \( A \). The eigenvalues of a square matrix \( A \) are defined as follows: the so-called eigenvectors \( v \) solve

\[ Av = \lambda v, \]

where \( A \) is a square matrix with \( n \) rows and columns, \( v \) is a row-vector with dimension \( n \) and \( \lambda \) is a scalar. This equation can be rearranged to

\[ (A - \lambda I)v = 0, \] (2.3)

In order for a non-trivial (i.e. \( v \neq 0 \)) solution to exist, the matrix in brackets has to be singular, i.e. its determinant has to be zero. In our example \( n = 2 \) and the determinant is given by

\[ \det (A - \lambda I) = \det \begin{pmatrix} -2 - \lambda & 4 \\ 1 & 1 - \lambda \end{pmatrix} = \lambda^2 + \lambda - 6 = 0 \]

The last equation is called characteristic equation. It has two solutions which are the “roots of the characteristic equation” or, equivalently, the “eigenvalues of \( A \)”, \( \lambda = r = 2 \)
and $\lambda = s = -3$. Note, the eigenvalues are real (they could be complex), distinct (they could be the same), and of opposite sign. We will come back to the eigenvalues later.

Now we need to obtain the solution to eq. (2.3), i.e. the eigenvectors. Since we have two eigenvalues this yields two eigenvectors, denoted by $v^r$ and $v^s$.

First, for $\lambda = r = 2$ eq. (2.3) reads

$$(A - 2I) v^r = \begin{pmatrix} -4 & 4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v^r_1 \\ v^r_2 \end{pmatrix} = 0$$

We normalize (any other constant is possible) the eigenvector by setting $v^r_1 = 1$. The solution to the previous equation is then given by $v^r_2 = 1$. Hence, the first eigenvector is given by $v^r = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Second, for $\lambda = s = -3$ eq. (2.3) reads

$$(A + 3I) v^s = \begin{pmatrix} 1 & 4 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} v^s_1 \\ v^s_2 \end{pmatrix} = 0$$

We normalize the eigenvector by setting $v^s_1 = 1$. The solution to the previous equation is then given by $v^s_2 = -0.25$. Hence, the first eigenvector is given by $v^s = \begin{pmatrix} 1 \\ -0.25 \end{pmatrix}$.

Finally, let’s solve the system of differential equations. A solution for eq. (2.2) consists of two parts:

$$z = c_1 w^1 + c_2 w^2, \quad (2.4)$$

where $c_1$ and $c_2$ are constants that we will determine below. Now that we know that both eigenvalues are real and distinct we apply the following guess for the solution (look that up in a relevant textbook, e.g. Shone, 2002):

$$w^i = e^{\lambda t} v^i \quad i \in \{1, 2\}.$$ 

Taking the derivative wrt to time and inserting the result in eq. (2.2) yields (I omit the
\[ \lambda e^{\lambda t}v = A e^{\lambda t}v \]
\[ \Leftrightarrow \lambda v = Av \]
\[ \Leftrightarrow (A - \lambda I)v = 0 \]

The last equation is identical to eq. (2.3) which has the two eigenvalues \( r = 2 \) and \( s = -3 \) with corresponding eigenvectors \( v^r \) and \( v^s \). The two parts of the solution, \( w^1 \) and \( w^1 \) are then given by

\[ w^1 = e^{rt}v^r = e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

\[ w^2 = e^{st}v^s = e^{-3t} \begin{pmatrix} 1 \\ -0.25 \end{pmatrix} \]

Inserting that into eq. (2.4) yields the solution

\[ z = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ -0.25 \end{pmatrix} \]

Finally, we need to transform this solution back to the non-homogeneous system. Note, \( u = z + u^* \), such that we get

\[ u = \begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ -0.25 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (2.5) \]

Lastly, we can determine the constants \( c_1 \) and \( c_2 \) by making use of the initial conditions for \( x \) and \( y \). Set \( t = 0 \) in eq. (2.5) and replace \((x(t = 0), y(t = 0))\) with \((x^0, y^0)\) and solve for \( c_1 \) and \( c_2 \) to get

\[ x(t) = 0.8(x^0 - y^0)e^{-3t} + (0.2x^0 + 0.8y^0 - 1)e^{2t} + 1 \]

\[ y(t) = -0.2(x^0 - y^0)e^{-3t} + (0.2x^0 + 0.8y^0 - 1)e^{2t} + 1 \]
Note, the solution contains a stable and an unstable part. The stable part is the term with $e^{-3t}$ because for $t \to \infty$ this expression converges to a constant (to zero). The unstable part is the term with $e^{2t}$ because this part can approach infinity. Only if the expression $(0.2x^0 + 0.8y^0 - 1)$ is equal to zero does the system converge to a stable point, a steady state. This trajectory is the saddle-path, i.e. there is only one combination of $x$ and $y$ such that the system converges to the steady state. Otherwise, the system does not converge.
Alternative way of obtaining a solution:

We need to transform the system into a second order differential equation. Take the derivative of the first differential equation wrt time:

\[ \ddot{x} = -2\dot{x} + 4\dot{y}. \]

We need to get rid of \( y \). Therefore, rearrange the first differential equation wrt \( y \), yielding

\[ y = \frac{1}{4}(\dot{x} + 2 + 2x). \] (2.6)

Inserting the latter into the second differential equation gives

\[ \dot{y} = x + \frac{1}{4}(\dot{x} + 2 + 2x) - 2. \]

Insert this expression into the \( \ddot{x} \) equation:

\[ \ddot{x} = -\dot{x} + 6x - 6. \] (2.7)

Now, we need to solve this 2\(^{nd}\) order differential equation. The solution will consist of a complementary and a partial solution, i.e. \( x = x_c + x_p \).

Let’s guess the partial solution as \( x_p = k \), a constant. Then, \( \dot{x}_p = \ddot{x}_p = 0 \), and \( x_p = k = 1 \).

The complementary solution is defined as the solution to the homogeneous part of the differential equations, i.e. of \( \ddot{x}_c = -\dot{x}_c + 6x_c \). Let’s guess the complementary solution as \( x_c = Ae^{rt} \). It follows that \( \dot{x}_c = rAe^{rt} \) and \( \ddot{x}_c = r^2Ae^{rt} \). Inserting that into the (homogeneous) \( \ddot{x}_c \) equation and rearranging gives the characteristic equation:

\[ r^2 + r - 6 = 0. \]

The solution consists of two distinct roots, \( r_1 = -3 \) and \( r_2 = 2 \).

Hence,

\[ x_c = A_1e^{-3t} + A_2e^{2t}. \]
The general solution, \( x = x_c + x_p \), reads then

\[
x = 1 + A_1 e^{-3t} + A_2 e^{2t}
\]  
(2.8)

You can verify that the solution is indeed correct by inserting eq. (2.8) and the derivative of eq. (2.8) wrt time into the right hand side of eq. (2.7). The resulting expression should be equal to the second derivative of eq. (2.8) wrt time.

The solution for \( y \) is obtained by taking the derivative wrt time of eq. (2.8), which is given by \( \dot{x} = -3A_1 e^{-3t} + 2A_2 e^{2t} \), and inserting this expression and eq. (2.8) in (2.6):

\[
y = 1 - \frac{1}{4} A_1 e^{-3t} + A_2 e^{2t}.
\]  
(2.9)

We obtain \( A_1 \) and \( A_2 \) (constants) by making use of the start values of \( y \) and \( x \). Set \( t = 0 \) in eq. (2.8) and (2.9) to get

\[
x^0 = 1 + A_1 + A_2
\]
\[
y^0 = 1 - \frac{1}{4} A_1 + A_2.
\]

This are two equations in two unknowns. The solutions are \( A_1 = \frac{4}{5} (x^0 - y^0) \) and \( A_2 = \frac{1}{5} (4y^0 + x^0 - 5) \).

The particular solution now reads:

\[
x(t) = 1 + \frac{4}{5} (x^0 - y^0) e^{-3t} + \frac{1}{5} (4y^0 + x^0 - 5) e^{2t}
\]
\[
y(t) = 1 - \frac{1}{5} (x^0 - y^0) e^{-3t} + \frac{1}{5} (4y^0 + x^0 - 5) e^{2t}.
\]

**Summary**

1. Transform the 2-dimensional system into a second order differential equation \( \ddot{x}(t) \).

2. Find the complementary solution \( x_c \) for the homogeneous part of the equation (hint: try \( x = Ae^{rt} \), solve the characteristic equation and choose the appropriate solution formula for \( x_c \)).
3. Find the particular solution \( x_p \) (hint: try \( x = k \)).

4. Set \( x(t) = x_c + x_p \).

5. Determine \( y(t) \).

6. Use \( x(0) = x^0 \) and \( y(0) = y^0 \) to get rid of undetermined coefficients.

3 Non-linear, 2-dimensional, 1st order difference equations

a) The steady states are \( x \) and \( y \) that solve

\[
\begin{align*}
    x &= \sqrt{x} - y \\
    y &= \frac{y}{2\sqrt{x}}.
\end{align*}
\]

First, \( y = 0 \) is one solution. Then, \( x = 0 \) or \( x = 1 \).

Second, if \( y \neq 0 \) we get \( x = y = \frac{1}{4} \). To sum up, the system possesses three steady states:
\( \{0, 0\}, \{1, 0\}, \{\frac{1}{4}, \frac{1}{4}\} \).

b) The \( \Delta x_{t+1} = 0 \) locus follows from

\[
\begin{align*}
    x_{t+1} - x_t &= \sqrt{x_t} - y_t - x_t = 0 \\
    \iff y &= \sqrt{x} - x.
\end{align*}
\]

The \( \Delta y_{t+1} = 0 \) locus follows from

\[
\begin{align*}
    y_{t+1} - y_t &= \frac{y_t}{2\sqrt{x_t} - y_t} - y_t = 0 \\
    \iff y &= -\frac{1}{4} + \sqrt{x},
\end{align*}
\]

and

\( y = 0 \).

Note, the \( \Delta y_{t+1} = 0 \) locus consists of two parts. The phase diagram is shown in fig. 3.
c) Approximating the difference equations $x_{t+1}$ by means of a first-order Taylor Series expansion around the steady state gives

$$x_{t+1} \approx x_t + \frac{\partial x_{t+1}}{\partial x_t} \bigg|_{x_t = \bar{x}} (x_t - \bar{x}) + \frac{\partial x_{t+1}}{\partial y_t} \bigg|_{y_t = \bar{y}} (y_t - \bar{y})$$

$$= \bar{x} + \frac{1}{2\sqrt{\bar{x}}} (x_t - \bar{x}) - (y_t - \bar{y})$$

$$= \bar{x} + (x_t - \bar{x}) - (y_t - \bar{y}).$$

d) See accompanying spreadsheet.