1 Set Up

In a standard New Keynesian business cycle framework prices are sticky and only adjust slowly to exogenous shocks. Every period there is a constant share of randomly selected firms which can adjust their prices. All the other firms simply keep their prices from last period. This follows the framework first introduced by Calvo (1983). The probability whether a single firm can adjust its prices in a given period is determined by a Poisson distribution, hence it is independent of the history of the firm. The aggregate price level can be formulated by

\[ P_t = \left[ \int_{S(t)} P_{t-1}(i)^{1-\Theta} di + (1 - \omega)(p^*_t)^{1-\Theta} \right]^{\frac{1}{1-\Theta}}, \]

(1)

where \( P_t \) is the average price level today, \( P_{t-1}(i) \) is the price level of a single firm \( i \) from last period, \( p^*_t \) is the price level chosen by price adjusting firms, \( \Theta \) is the elasticity of demand between differentiated goods, and \( \omega \) is the constant share of firms that cannot adjust their prices today.

The price level is therefore determined by two elements. First, the distribution of prices of all firms that are not allowed to adjust their prices today. Second, the optimal price chosen by all firms that can adjust their prices today. Since the production function is symmetric across firms, all price adjusting firms choose the same price.

We can rewrite (1) by

\[ P_t = \left[ \omega P_{t-1}^{1-\Theta} + (1 - \omega)(p^*_t)^{1-\Theta} \right]^{\frac{1}{1-\Theta}}, \]

(2)
where the aggregation of the firms keeping the price level from last period follows from two assumptions. First, there is mass one of identical firms. This is, there are infinitely many firms on the unit interval. Second, whether firms can adjust their prices or not in a given period is determined by a Poisson distribution. Therefore, by the law of large numbers, it follows that the price distribution is equal across both groups of firms (before the adjusting firms choose a new price) and hence equal to the average price level last period.

One can rewrite (2) by dividing through $P_t$

$$1 = (1 - \omega) \left( \frac{p_t}{P_t} \right)^{1-\Theta} + \omega \left( \frac{P_{t-1}}{P_t} \right)^{1-\Theta}$$

$$\Leftrightarrow \frac{p_t}{P_t} = \left[ \frac{1 - \omega \pi_t^{\Theta-1}}{1 - \omega} \right]^{\frac{1}{1-\Theta}},$$

where $\pi_t$ is the inflation rate today. This reformulation will become useful later on.

Each firm that can adjust its price this period optimizes it by maximizing the present-discounted value of all future profits for the expected time this price is active, hence as long as the firm cannot re-adjust its prices in average. Thus, each firm maximizes

$$\max_{p_{j,t}} E_t \sum_{i=0}^{\infty} \omega^i \beta_{t,t+i} \Pi_{j,t+i} = \max_{p_{j,t}} E_t \sum_{i=0}^{\infty} \omega^i \beta_{t,t+i} \left[ \left( \frac{p_{j,t}}{P_{t+i}} \right)^{1-\Theta} - \varphi_{t+i} \right] C_{j,t+i},$$

where $\beta_{t,t+i} = \beta^i \left( \frac{\lambda_{t+i}}{\lambda_t} \right)$ is the stochastic discount factor, $\Pi_{j,t+i}$ is the profit function of firm $j$ in period $t+i$, $C_{j,t+i}$ is the demand function for the differentiated good of firm $j$, $p_{j,t}$ is the price set by firm $j$ in period $t$, and $\varphi_{t+i}$ is the real marginal cost of production in period $t+i$. Substituting for $c_{j,t+i}$ with the demand function one gets

$$\max_{p_{j,t}} E_t \sum_{i=0}^{\infty} \omega^i \beta_{t,t+i} \left[ \left( \frac{p_{j,t}}{P_{t+i}} \right)^{1-\Theta} - \varphi_{t+i} \left( \frac{p_{j,t}}{P_{t+i}} \right)^{-\Theta} \right] C_{t+i},$$

which is a profit function depending on each firm’s price and marginal cost only, while the aggregate price level and consumption are given.

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1Since we assume that firms are owned by households, the firms discount the future using the household’s discount function.

2The demand function for a differentiated good follows from the first stage optimization of the households when firms are monopolistic competitors.
2 First-order Conditions

By maximizing (6) one receives the following first-order-condition

$$\frac{\partial \Pi_{j,t+i}}{\partial p_{j,t}} = E_t \sum_{i=0}^{\infty} \omega^i \beta_{t,t+i} \left[ (1 - \Theta) + \Theta \varphi_{t+i} \left( \frac{p_{j,t}}{P_{t+i}} \right)^{-1} \right] \left( \frac{p_{j,t}}{P_{t+i}} \right)^{-\Theta} \frac{C_{t+i}}{P_{t+i}} = 0$$ (7)

$$= E_t \sum_{i=0}^{\infty} \omega^i \beta_{t,t+i} \left[ (1 - \Theta) + \Theta \varphi_{t+i} \left( \frac{P_{t+i}}{p_{j,t} P_{t+i}} \right) \left( \frac{P_{t+i}}{P_{t}} \right)^{-\Theta} \frac{C_{t+i}}{P_{t+i}} \right] = 0$$ (8)

$$= \left( \frac{P_{t+i}}{P_{j,t}} \right)^{\Theta} E_t \sum_{i=0}^{\infty} \omega^i \beta_{t,t+i} \left[ (1 - \Theta) + \Theta \varphi_{t+i} \pi_{t,t+i} \left( \frac{P_{t+i}}{p_{j,t} P_{t+i}} \right) \pi_{t,t+i} \left( \frac{P_{t+i}}{P_{t}} \right)^{-\Theta} \frac{C_{t+i}}{P_{t+i}} \right] = 0$$ (9)

$$= -(\Theta - 1) E_t \sum_{i=0}^{\infty} \omega^i \beta_{t,t+i} \pi_{t,t+i}^{\Theta-1} C_{t+i} + \Theta E_t \left[ \sum_{i=0}^{\infty} \omega^i \beta_{t,t+i} \pi_{t,t+i}^{\Theta} C_{t+i} \varphi_{t+i} \right] = 0$$ (10)

where $\pi_{t,t+i} = \left( \frac{P_{t+i}}{P_{j,t}} \right)$. The first line directly follows from taking the first derivative, the second line rewrites the aggregate prices as inflation rates, the third line takes all terms that are common to both sums and do not depend on the sum operator i out of the sum, the last line splits the two sums into two separate sum operators.

Since the problem is symmetric for all firms that can re-adjust their prices, we drop the index j, this is $p_{j,t} = p^*_t$. Solving for $p^*_t / P_t$ we receive

$$\frac{p^*_t}{P_t} = \frac{\Theta}{\Theta - 1} \frac{E_t \sum_{i=0}^{\infty} \omega^i \beta_{t,t+i} \pi_{t,t+i}^{\Theta} C_{t+i} \varphi_{t+i}}{E_t \sum_{i=0}^{\infty} \omega^i \beta_{t,t+i} \pi_{t,t+i}^{\Theta-1} C_{t+i}}.$$ (11)

The optimal price is given by (11). It is determined by the expected present discounted value of all future costs divided by the expected present discounted value of all future returns. Time discounting is determined by the time preferences $\beta$ and the average probability of the price still being active in a certain period $\omega^i$. Due to monopolistic competition, the firm tries to keep an average mark-up $\Theta$ on its price.

3 Recursive Framework

So far, the first-order condition contains two infinite sums. To solve the model both analytically and numerically, we need a more tractable version of this equation. By rewriting (11) recursively, we get a first-order difference equation which we can use in a canonical representation of the system.

First, we split the problem into two problems by introducing two auxiliary variables, so
each infinite sum is treated separately. Hence, (11) can be rewritten as

\[ \frac{p_t^*}{P_t} = \frac{\Theta}{\Theta - 1} Ax_t, \]  

where \( Ax_t = E_t \sum_{i=0}^{\infty} \omega^i \beta_{t,t+i} \pi_{t,t+i}^\theta C_{t+i} \varphi_{t+i} \) and \( Ay_t = E_t \sum_{i=0}^{\infty} \omega^i \beta_{t,t+i} \pi_{t,t+i}^{\theta-1} C_{t+i} \).

Second, we have to get rid of the infinite sums. Here, I will show how to rewrite infinite sums into a recursive framework for the first auxiliary variable \( Ax_t \). The procedure for \( Ay_t \) works exactly the same way and is left as an exercise. We start by writing out the infinite sum for some periods

\[ Ax_t = C_t \varphi_t + \omega E_t \beta_{t,t+1} \pi_{t,t+1}^\theta C_{t+1} \varphi_{t+1} + \omega^2 E_t \beta_{t,t+2} \pi_{t,t+2}^\theta C_{t+2} \varphi_{t+2} + \ldots \]  

(13)

Now, shift \( Ax_t \) one period into the future, hence

\[ Ax_{t+1} = C_{t+1} \varphi_{t+1} + \omega E_{t+1} \beta_{t+1,t} \pi_{t+1,t}^\theta C_{t+1} \varphi_{t+1} + \omega^2 E_{t+1} \beta_{t+1,t+2} \pi_{t+1,t+2}^\theta C_{t+2} \varphi_{t+2} + \ldots \]  

(14)

Then, we manipulate (14) such that the terms for each time period \((t+1, t+2, t+3, \ldots)\) on the right-hand side are the same in (13) and (14). In the later equation we do not have period \( t \), but we neglect that for the moment. This results in

\[ \omega \beta_{t,t+1} \pi_{t,t+1}^\theta Ax_{t+1} = \omega \beta_{t,t+1} \pi_{t,t+1}^\theta C_{t+1} \varphi_{t+1} + \omega^2 E_{t+1} \beta_{t+1,t+2} \pi_{t+1,t+2}^\theta C_{t+2} \varphi_{t+2} + \ldots \]  

(15)

where \( \pi_{t,t+2} = \pi_{t+1,t+2} \cdot \pi_{t+1,t+1} = \frac{P_{t+2}}{P_{t+1}} \cdot \frac{P_{t+1}}{P_t} \) and \( \beta_{t,t+2} = \beta_{t,t+1} \cdot \beta_{t+1,t+2} = \beta \left[ \frac{\lambda_{t+1}}{\lambda_t} \right] \left[ \frac{\lambda_{t+2}}{\lambda_{t+1}} \right] \).

As a last step we have to get the expectations operator right. This is, we multiply both sides of (15) by \( E_t \). By the law of iterated expectations \( E_t \cdot E_{t+1} = E_t \) we receive

\[ \omega E_t \beta_{t,t+1} \pi_{t,t+1}^\theta Ax_{t+1} = \omega E_t \beta_{t,t+1} \pi_{t,t+1}^\theta C_{t+1} \varphi_{t+1} + \omega^2 E_t \beta_{t,t+2} \pi_{t,t+2}^\theta C_{t+2} \varphi_{t+2} + \ldots \]  

(16)

As the sum operator runs to infinity, the right hand side of (16) and the right-hand side of (13) minus \( C_t \varphi_t \) are the same. Thus we can plug in and get

\[ Ax_t = C_t \varphi_t + \omega E_t \beta_{t,t+1} \pi_{t,t+1}^\theta Ax_{t+1} \]  

(17)

The same procedure works for \( Ay_t \), which results in

\[ Ay_t = C_t + \omega E_t \beta_{t,t+1} \pi_{t,t+1}^{\theta-1} Ay_{t+1} \]  

(18)
A framework where some variable in period \( t \) depends on itself in period \( t+1 \) is called a recursive framework. This is used quite extensively in business cycle theory to get a more tractable framework and be able to calculate a solution for the system.

### 4 Linearization

For computational convenience we linearize our non-linear system around the deterministic steady state. Therefore, we use a first-order Taylor approximation given by

\[
h(x, y) \approx h(\bar{x}, \bar{y}) + \frac{\partial h}{\partial x}(\bar{x}, \bar{y})(x - \bar{x}) + \frac{\partial h}{\partial y}(\bar{x}, \bar{y})(y - \bar{y}) = h(\bar{x}, \bar{y}) + h_x \bar{x} \hat{x} + h_y \bar{y} \hat{y}, \quad (19)
\]

where \( \bar{x} \) and \( \bar{y} \) are the deterministic steady states and \( \hat{x} = \frac{x - \bar{x}}{\bar{x}} \) and \( \hat{y} = \frac{y - \bar{y}}{\bar{y}} \) are percentage deviations thereof. To calculate the deterministic steady state in a time-discrete system without deterministic growth, simply drop all time indexes and simplify. For (12), (17), and (18) this results in

\[
Ay = \frac{\Theta}{\Theta - 1} Ax, \quad (20)
\]

\[
Ax = \frac{C \varphi}{1 - \omega \beta}, \quad (21)
\]

\[
Ay = \frac{C}{1 - \omega \beta}, \quad (22)
\]

where we can solve for the steady state real marginal costs \( \varphi = \frac{\Theta - 1}{\Theta} \), which is the inverse of the price mark-up. This is a common result of monopolistic competition.

Now, we linearize around the deterministic steady state by applying the first-order Taylor approximation to (12), (17), and (18), which results in

\[
\hat{p}_t^* - \hat{P}_t = \hat{A}x_t - \hat{A}y_t, \quad (23)
\]

\[
\hat{A}x_t = (1 - \omega \beta)(\hat{C}_t + \hat{\varphi}_t) + \omega \beta(\hat{\beta}_{t,t+1} + \Theta \hat{\pi}_{t+1} + \hat{A}x_{t+1}) \quad (24)
\]

\[
\hat{A}y_t = (1 - \omega \beta)\hat{C}_t + \omega \beta(\hat{\beta}_{t,t+1} + (\Theta - 1) \hat{\pi}_{t+1} + \hat{A}y_{t+1}) \quad (25)
\]
where we used the steady states (21) and (22) to simplify. Additionally, we linearize the aggregate price level equation (2) around its deterministic steady state for later use. It is given by

\[ \hat{p}_t^* - \hat{P}_t = \frac{\omega}{1 - \omega} \hat{\pi}_t. \]  

(26)

5 Aggregate Supply: The NKPC

Now, we put everything back together. First, plug (24) and (25) into (23):

\[ \hat{p}_t^* - \hat{P}_t = (1 - \omega \beta) (\hat{C}_t + \hat{\phi}_t) + \omega \beta (\hat{\beta}_{t,t+1} + \Theta \hat{\pi}_{t+1} + \hat{A}x_{t+1}) \]

(27)

\[- (1 - \omega \beta) \hat{C}_t - \omega \beta (\hat{\beta}_{t,t+1} + (\Theta - 1) \hat{\pi}_{t+1} + \hat{A}y_{t+1}) \]

(28)

\[\begin{align*}
&= (1 - \omega \beta) \hat{\phi}_t + \omega \beta (\hat{\pi}_{t+1} + \hat{A}x_{t+1} - \hat{A}y_{t+1}) \\
&= (1 - \omega \beta) \hat{\phi}_t + \omega \beta (\hat{\pi}_{t+1} + \hat{A}x_{t+1} - \hat{A}y_{t+1}) \end{align*} \]

(29)

Now replace \((\hat{A}x_{t+1} - \hat{A}y_{t+1})\) by \((\hat{p}_{t+1}^* - \hat{P}_{t+1})\) (using (23) forwarded one period)

\[ \hat{p}_t^* - \hat{P}_t = (1 - \omega \beta) \hat{\phi}_t + \omega \beta (\hat{\pi}_{t+1} + \hat{p}_{t+1}^* - \hat{P}_{t+1}). \]

(30)

And finally use (26) to solve for the inflation rate

\[ \frac{\omega}{1 - \omega} \hat{\pi}_t = (1 - \omega \beta) \hat{\phi}_t + \omega \beta (\hat{\pi}_{t+1} + \frac{\omega}{1 - \omega} \hat{\pi}_{t+1}) \]

(31)

\[ \hat{\pi}_t = \kappa \hat{\phi}_t + \beta E_t \hat{\pi}_{t+1}, \]

(32)

where \(\kappa = \frac{(1 - \omega)(1 - \omega \beta)}{\omega}\). Thus, in a New Keynesian framework with Calvo pricing, inflation today depends on the change in real marginal costs today and expected inflation tomorrow. How important either effect is, depends on the price stickiness \(\omega\) and the household discount rate \(\beta\).

As a last exercise one can iterate forward and substitute for \(\hat{\pi}_{t+1}, \hat{\pi}_{t+2}, \ldots\), which results in

\[ \hat{\pi}_t = E_t \sum_{i=0}^{\infty} \beta^i \kappa \hat{\phi}_{t+i}. \]

(33)

From this equation one can see that today’s prices depend solely on all discounted expected real marginal costs times \(\kappa\), the factor determined by price-stickiness and the discount factor.
References


