Solow Model (Feb 21, 2019)

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Introduction: Why thinking about “economic growth”? (1)

- Why are there **poor** and **rich countries**?
  - The richest countries are about 30 times richer than the poorest countries (in 1988). (Parente and Prescott, 2000)

- Why do some economies experience **rapid growth**, while some other **stagnate** at the same time?
  - Niger (NER) versus South Korea (KOR); see PWT!

- Manipulating an economy’s (long run) growth rate seems to have **massive welfare consequences**.
  - If per capita income grows at a growth rate of 1 percent it takes about 70 years until per capita income is doubled.
  - If per capita income grows at a growth rate of 2 percent it takes about 35 years until per capita income is doubled.

- Lucas (1987, 2003) has argued that the **complete removal of consumption volatility** (→ business cycle phenomenon) would imply a **welfare gain** which is equivalent to a **permanent increase in consumption of about 0.1 percent to 1 percent**.
  - Taking heterogeneity into account should further increase the welfare gain.
  - Follow up studies found substantially larger welfare gains of about 10 percent to 30 percent (Lucas, 2003, p. 7).

- Potential welfare gain resulting from the implementation of **optimal growth policies** appears to be substantially higher compared to results mentioned above.
  - An appropriate policy reform could achieve a welfare gain which is equivalent to a permanent doubling of per capita consumption (Grossmann, Steger, Trimborn, 2010).
The consequences for human welfare involved in questions like these are simply staggering: Once one starts thinking about them, it’s hard to think about anything else. Robert Lucas (1988)

How long does it take until \( \chi(t) \) is twice its initial value?

Solve \( x_0 e^{gt} = 2x_0 \) for \( t \) to get \( t = \ln(2)/g \).
Reminder (1): The Kaldor Facts

- In 1961 **Nicolas Kaldor** listed **6 stylized facts** that describe economic growth in advanced economies:

  1. $Y/L$ (output per worker) exhibits continual growth.
  2. $K/L$ (capital per worker) exhibits continual growth.
  3. $r$ (real interest rate) is roughly constant.
  4. $K/Y$ (capital-output ratio) is roughly constant.
  5. $rK/Y, wL/Y$ (factor shares) are roughly constant.
  6. There are wide differences in the rate of growth of productivity across countries.

- Today the Kaldor facts are considered to be outdated for at least two reasons:


There are \( L \) households, every HH is endowed with one unit of time which is supplied inelastically to the labor market.

The numeraire good is \( Y \). Notice that wealth \((a)\) and capital \((K)\) are both measured in units of \( Y \).

\[ s : \text{exogenous saving rate} \]
The stock of physical capital per capita seems to matter for the standard of living!

This is, in a nutshell, the story of the neoclassical model.

The Solow model has some more insights:
- role of technological progress
- transitional dynamics
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Model setup (3): two versions

- **Real economy (Robinson Crusoe)**
  - A consumer-producer household with technology
    \[ Y = K^\alpha L^{1-\alpha} \]
  - The resource constraint reads
    \[ Y = \dot{K} + \delta K + C \]
  - Define the (gross) saving rate
    \[ s = \frac{Y - C}{Y} \]
  - The \( K \)-equation is then given by
    \[ \dot{K} = sY - \delta K \]

- **Terminology**
  - \( Y \): gross output (= gross factor income)
  - \( Y - \delta K \): net output (= net factor income)
  - \( \dot{K} + \delta K \): gross investment
  - \( \dot{K} \): net investment

- **Decentralized market economy**
  - Firms have access to the following technology
    \[ Y = K^\alpha L^{1-\alpha} \]
  - Competitive factor prices read
    \[ r = \alpha \frac{Y}{K} - \delta, \quad w = (1 - \alpha) \frac{Y}{L} \]
  - There are \( L \) identical HH. Every HH is endowed with one unit of time per period, which is supplied inelastically to the labor market. HH have also access to a perfect capital market.
  - Individual (gross) household income \( X \) is given by
    \[ X = (r + \delta) a + w \]
  - Capital market equilibrium requires
    \[ \frac{K}{K_{\text{supply}}} = \frac{aL}{K_{\text{demand}}} \]
  - Total household income (assuming \( K = aL \)) reads
    \[ XL = (r + \delta) aL + wL = \alpha \frac{Y}{K} K + (1 - \alpha) \frac{Y}{L} L = Y \]
  - Individual stock of assets evolves according to
    \[ \dot{a} = \frac{X - \delta a - C}{\text{net factor income}} \]
  - Defining the (gross) saving rate \( s := (X - C)/X \), we have
    \[ \dot{a} = sX - \delta a \]
  - From \( K = aL \) and \( LX = Y \) we get
    \[ \dot{K} = \dot{a}L = sXL - \delta aL = sY - \delta K \]
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Model setup (4)

- The typical **final output firm** has access to the following **Cobb-Douglas technology**

\[ Y(t) = F[K(t), L(t)] = K(t)^\alpha L(t)^{1-\alpha} \]

- This technology exhibits constant returns to scale (CRS), i.e. \( F(\lambda K, \lambda L) = \lambda Y \) for all \( \lambda \in \mathbb{R} \).

- The **problem of the typical firm** reads (time index \( t \) is omitted in what follows)

\[
\max_{K,L} \left\{ p_f F(K, L) - \underbrace{(r + \delta)}_{\text{user cost of capital}} K - wL \right\}
\]

- **Implied inverse demand functions**

\[
w = \frac{\partial F(K, L)}{\partial L} = (1-\alpha)K^\alpha L^{-\alpha} = (1-\alpha)\frac{Y}{L}
\]

\[
r = \frac{\partial F(K, L)}{\partial K} - \delta = \alpha K^{\alpha-1} L^{-\alpha} - \delta = \alpha \frac{Y}{K} - \delta
\]

- Notice that we have set \( p_y = 1 \).

- Labor income share: \( (wL)/Y = 1 - \alpha \); capital income share: \( (r + \delta)K/Y = \alpha \).

- Output is exactly exhausted: \( Y = \tau K + wL \) (⇒ Euler's Theorem).
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Intensive production function

- **Output per worker** (=labor productivity)

\[ \frac{Y}{L} = K^\alpha L^{-\alpha} = \left( \frac{K}{L} \right)^\alpha \]

\[ y = k^\alpha \quad \text{with } k := \frac{K}{L} \text{ and } y := \frac{Y}{L} \]

- 1\(^{st}\) derivative is positive \((\partial y / \partial k = \alpha k^{\alpha-1} > 0)\) and 2\(^{nd}\) derivative is negative \((\partial^2 y / \partial k^2 = \alpha (\alpha - 1) k^{\alpha-2} < 0)\), i.e. there are diminishing returns to capital per worker.

- Notice

\[ \frac{\partial Y}{\partial K} = \alpha K^{\alpha-1} L^{-\alpha} = \alpha \left( \frac{K}{L} \right)^{\alpha-1} = \alpha k^{\alpha-1} = \frac{\partial y}{\partial k} \]

- In words: The marginal product of capital equals the marginal product of capital per worker, evaluated at corresponding levels of capital and capital per worker.
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Capital accumulation

- **Capital accumulation equation** reads

\[ \dot{K} = sY - \delta K \]

- Equation of motion for \( k := K/L \) is obtained as follows

\[ \frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L} \quad \Rightarrow \quad \frac{\dot{k}}{k} = s \frac{Y}{K} - \delta - \frac{\dot{L}}{L} \]

\[ \dot{k} = s \frac{Y}{K} \frac{K}{L} - \left( \delta + \frac{\dot{L}}{L} \right) k \quad \Rightarrow \quad \dot{k} = sy - (\delta + n)k \]

- **Two basic questions**
  - How does long-run output per worker compare in two economies that have different investment rates and population growth rates?
  - How does output per worker evolve over time?
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Capital accumulation (a)

- **Possibility 1**
  
  - Differentiating $k(t) = \frac{K(t)}{L(t)}$ w.r.t. time by applying the quotient rule, noting $\frac{dK(t)}{dt} = \dot{K}$, gives

  $$\frac{d}{dt} \left( \frac{K(t)}{L(t)} \right) = \frac{\dot{K}(t) L(t) - K(t) \dot{L}(t)}{L(t)^2}$$

  Divide both sides of this equation by $k(t) = K(t)/L(t)$ to get

  $$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} \frac{1}{L} - \frac{\dot{L}}{L}$$

- **Possibility 2**
  
  - Take the natural logarithm on both sides of $k(t) = K(t)/L(t)$

    $$\ln[k(t)] = \ln[K(t)] - \ln[L(t)]$$

    Next differentiate w.r.t. time

    $$\frac{d}{dt} \ln[k(t)] = \frac{d}{dt} \ln[K(t)] - \frac{d}{dt} \ln[L(t)]$$

    Recall

    $$\frac{d}{dt} \ln[X(t)] = \frac{d}{dt} \frac{X(t)}{X(t)} = 1 X(t) \frac{X(t)}{X(t)}$$

    $$\frac{d}{dt} \ln[X(t)] = \frac{d}{dt} \frac{X(t)}{X(t)} = 1 X(t) \frac{X(t)}{X(t)}$$

    $$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} \frac{1}{L} - \frac{\dot{L}}{L}$$
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The steady state

- We need a **dynamic equilibrium concept** that describes the long-run state of a dynamic economy.
  - Usual equilibrium concept is that of **steady state equilibrium**.
  - Steady state: state of the economy such that the growth rates of the endogenous variables are constant.

- From the $\dot{k}$-equation the **growth rate of capital per worker** is

$$\dot{k} = sk^{\alpha-1} - (\delta + n)$$

- Since $0 < \alpha < 1$, $sk^{\alpha-1}$ declines as $k$ increases, $\lim_{k\to0} sk^{\alpha-1} = \infty$ and $\lim_{k\to\infty} sk^{\alpha-1} = 0$ (see next slide).
- This indicates that there is a level of $k$ such that $k = 0$ such that $k = \text{const}$.

- The **steady state level of $k$** is as follows (by setting $\dot{k} = 0$ we ignore the trivial steady state $k^* = 0$)

$$\dot{k} = sk^{\alpha-1} - (\delta + n) = 0$$

$$k^* = \left(\frac{s}{n + \delta}\right)^{\frac{1}{1-\alpha}} \quad \Rightarrow \quad y^* = \left(\frac{s}{n + \delta}\right)^{\frac{\alpha}{1-\alpha}} \quad c^* = (1-s)\left(\frac{s}{n + \delta}\right)^{\frac{\alpha}{1-\alpha}}$$
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Graphical representations of the steady state

\[ \begin{align*}
  k^* &= \frac{y}{(n+\delta)k} \\
  y &= k^\alpha \\
  (n+\delta)k &\quad \text{per capita consumption} \\
  sY &
\end{align*} \]
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Technological progress (1)

- The **production function** now reads

\[ Y = F(K, AL) = K^\alpha (AL)^{1-\alpha} \]

- Technological progress \((\dot{A} > 0)\) is **labor-augmenting** and exogenous

\[
\frac{\dot{A}}{A} = g \quad \text{has} \quad \text{solution} \quad A = A_0 e^{gt}
\]

- The production function in **intensive form** is then given by

\[ y = k^\alpha A^{1-\alpha} \]

- Expressed in **growth rates** the preceding equation reads as follows

\[
\frac{\dot{y}}{y} = \alpha \frac{\dot{k}}{k} + (1-\alpha) \frac{\dot{A}}{A} \quad (*)
\]
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Technological progress (2)

- To determine the **steady state growth rate of** $y$, we first consider the growth rate of $k$

  $$\hat{k} = s\frac{y}{k} - (\delta + n)$$

- Hence, $\hat{k} = \text{const.}$ requires $\hat{y} = \hat{k}$. Using (*) we find

  $$\hat{y} = \alpha \hat{y} + (1 - \alpha) \frac{\dot{A}}{A}$$

  $$\hat{y} = \frac{\dot{A}}{A} = g$$

- The growth rate of output per worker equals the rate of technical progress.

- Moreover, since $\hat{y} = \hat{k}$ the following relation holds

  $$\hat{y} = \hat{k} = g$$

- Growth path along which all variables grow at the same constant rate is called a **balanced growth path** (BGP).
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Technological progress (3)

- We now solve the model for the steady state. Since there is permanent growth in per capita variables \((y, k)\), we define new variables

\[
\tilde{y} := \frac{Y}{AL} \quad \text{(output per efficiency units of labor)}
\]

\[
\tilde{k} := \frac{K}{AL} \quad \text{(capital per efficiency units of labor)}
\]

- To derive the equation of motion of \(\tilde{k}\), we differentiate \(\tilde{k} := \frac{K}{AL}\) w.r.t. time

\[
\hat{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L} - \frac{\dot{A}}{A} \quad \Rightarrow \quad \hat{k} = \frac{sY - \delta K}{K} - n - g
\]

\[
\hat{k} = \frac{sY}{K} - \delta - n - g \quad \Rightarrow \quad \hat{k} = \frac{sY / (AL)}{K / (AL)} - \delta - n - g
\]

\[
\hat{k} = s\tilde{y} - \delta - n - g \quad \Rightarrow \quad \hat{k} = s\tilde{y} - (\delta + n + g)\tilde{k}
\]

Logic behind detrending

\[
\lim_{t \to \infty} \tilde{x}(t) = g
\]

\[
x(t) := \frac{X(t)}{e^{gt}}
\]

\[
\lim_{t \to \infty} x(t) = \text{const.}
\]
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Technological progress (4)

- Noting $\tilde{y} = \tilde{k}^\alpha$, the **steady state** in terms of $\tilde{k}$ and $\tilde{y}$ reads as

$$\tilde{k}^* = \left( \frac{s}{n + \delta + g} \right)^{\frac{1}{1-\alpha}} \Rightarrow \tilde{y}^* = \left( \frac{s}{n + \delta + g} \right)^{\frac{\alpha}{1-\alpha}} (***)$$

- To give a clear **economic interpretation**, rewrite (***) as

$$y^*(t) = A(t) \left( \frac{s}{n + \delta + g} \right)^{\frac{\alpha}{1-\alpha}}$$

Notice: $y = A\tilde{y}$ (from $y = \frac{\gamma}{L}$ and $\tilde{y} = \frac{\gamma}{\tilde{L}}$)

→ Along the BGP output per worker $y$ grows at the constant rate $g$

→ The level of the BGP is determined by economic fundamentals ($s$, $n$ etc.)

→ An increase in $s$ does not affect the long-run growth rate. It does, however, affect the level of the BGP and hence the growth rate along the transition to the (new) BGP.
We now turn to the speed at which the economy converges to its steady state. This is measured by the rate of convergence (ROC). The ROC of any variable \( x(t) \) is defined by

\[
\psi_x(t) := -\frac{\dot{x}(t)}{x(t) - x^*}
\]

\( \psi_x > 0 \): convergence
\( \psi_x < 0 \): divergence

### Important tool: linearization

Consider the following general, possibly non-linear, differential equation (DE)

\[
\dot{x}(t) = F[x(t)]
\]

This DE is assumed to possess a stationary equilibrium defined by \( F(x^*) = 0 \).

Linearization of \( F(x) \) around \( x^* \) by means of a first-order Taylor approximation gives

\[
\dot{x}(t) \approx F(x^*) + F'(x^*)[x(t) - x^*]
\]

Hence, noting that \( F(x^*) = 0 \), \( x(t) \) converges at the following rate against \( x^* \)

\[
\psi_x(t) := -\frac{\dot{x}}{x - x^*} = -F''(x^*)
\]
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Speed of convergence (2)

- To determine the **ROC for the Solow model** recall that

\[
\dot{k} = s \tilde{k}^\alpha - (\delta + n + g)\tilde{k} \quad \text{and} \quad \tilde{k}^* = \left( \frac{s}{n + \delta + g} \right)^{\frac{1}{1-\alpha}}
\]

- **Linearizing the above DE around** \( \tilde{k}^* \) gives

\[
\dot{k} = \left[ \alpha s \tilde{k}^{\alpha-1} - (\delta + n + g) \right] (\tilde{k} - \tilde{k}^*)
\]

\[
\dot{k} = \left[ \alpha s \left( \frac{s}{n + \delta + g} \right)^{\frac{\alpha-1}{1-\alpha}} - (\delta + n + g) \right] (\tilde{k} - \tilde{k}^*)
\]

\[
\dot{k} = (\alpha - 1)(\delta + n + g) \left( \frac{(\tilde{k} - \tilde{k}^*)}{-\psi_k} \right) \quad (*)
\]

- **Recall**

\[\dot{x}(t) \equiv \mathcal{F}'(x^*) [x(t) - x^*]\]

- **Recall**

\[\psi_k := -\frac{\dot{k}}{\tilde{k} - \tilde{k}^*}\]

- **Hence, the (local) rate of convergence reads** \( \psi_k = (1 - \alpha)(\delta + n + g) > 0 \).
To see the **quantitative implication**, we assign empirically plausible parameter values: $\alpha = 0.35; \delta = 0.01; n = 0.015; g = 0.02$.

$\psi_k = (1 - \alpha)(\delta + n + g) = 0.0293$

More intuitive is the implied **half life** $(t_{0.5})$, which results from: $[k^* - k(0)]\exp(-0.0293 \cdot t_{0.5}) = 0.5[k^* - k(0)]$

$t_{0.5} = -\ln(0.5)/0.0293 = 23.70$
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Summary

- Solow model is compatible with the Kaldor facts
  - output per worker and capital per worker grow over time
  - the ratio of capital over output is largely constant
  - the real wage is growing over time
  - the rate of interest is largely constant
  - the labor and capital income shares are largely constant

- The long run level of GDP per worker is determined by economic fundamentals, i.e. the saving rate and the population growth rate.

- Long run growth in GDP per worker is primarily driven by technological progress accompanied by capital accumulation.

- Growth rate of GDP per worker may temporarily exceeds its long run level. This is due to transitional dynamics in response to, say,
  - a destruction in the stock of capital or
  - an increase in the in the saving rate.
Basic Concepts

Basic concepts and tools: Differential equations (0)

- **Two points on differential equations**

1. Differential equations (DE) are functional equations. That is, the solution of a DE $\dot{y}(t) = f[y(t)]$ with $y(0) = y_0$ is itself a function of time, i.e. $y = g(t, y_0)$, that satisfies the law of motion $\dot{y}(t) = f[y(t)]$ at any point in time.

2. If the DE under study satisfies the Lipschitz condition, then there exists a unique solution. This implies that if a solution can be found, we know that this solution is the unique solution.

![Phase diagrams](image)

*Figure 6: Phase diagrams. Left panel: stable focus; right panel: saddle point.*
Basic Concepts

Basic concepts and tools: Differential equations (1)

- **First-order, linear, differential equations**

  \[ \dot{y}(t) + u(t) \cdot y(t) = w(t) \]

  - \( t \in \mathbb{R} \) denotes the independent variable (often the time index) and \( \dot{y} := \frac{dy(t)}{dt} \).
  - first-order DE: only the first derivative of \( y(t) \) w.r.t. time occurs
  - linear DE: both \( \dot{y}(t) \) and \( y(t) \) appear only in first degree and there is no such term \( \dot{y}(t)y(t) \)

- **Homogenous case:** \( u(t) = a \) (constant coefficient) and \( w(t) = 0 \ \forall \ t \) (homogenous DE)

  \[ \dot{y}(t) + a \cdot y(t) = 0 \]

- **Solution is easily found** (\( t \) suppressed)

  \[
  \frac{1}{y} \frac{dy}{dt} = -a \quad \Rightarrow \quad \int \frac{1}{y} \frac{dy}{dt} dt = \int -adt \tag{LHS}
  \]

  \[
  \text{LHS: } \int \frac{1}{y} \frac{dy}{dt} = \int \frac{d\ln y}{dt} dt = \int \frac{dy}{y} = \ln y + c_1 \tag{RHS}
  \]

  \[
  \text{RHS: } \int -adt = -at + c_2
  \]

  \( \Rightarrow \ln y = -at + c \quad \text{with} \quad c := c_2 - c_1 \)

  \( \Rightarrow y(t) = Ae^{-at} \) (general solution)

  \( \Rightarrow y(t) = y(0)e^{-at} \) (definite solution)
Basic Concepts

Basic concepts and tools: Differential equations (2)

- **Non-homogenous case:** \( u(t) = a \) (constant coefficient) and \( w(t) = b \ \forall \ t \)

\[
\dot{y}(t) + a \cdot y(t) = b
\]

⇒ compare to Solow model

- Solution comprises two terms: **complementary function** \( y_c \) and **particular integral** \( y_p \)
- **Complementary function** \( y_c \): general solution of the homogenous DE (i.e. \( b = 0! \))
- **Particular integral** \( y_p \): any particular solution of the non-homogenous equation

- **Complementary function** \( y_c \) (→ discussion of homogenous DE)

\[
y_c = Ae^{-at}
\]

- **Particular integral** \( y_p \): try simplest possible type of solution: \( y = \text{const.} \)

\[
y_p = b / a
\]

- \( y = \text{const.} \) implies \( \dot{y} = 0 \) and hence \( \dot{y} + ay = b \) yields \( y_p = b/a \) (assuming \( a \neq 0 \))

- **General solution of non-homogenous DE**

\[
y = y_c + y_p = \frac{b}{a} + Ae^{-at} \quad \text{(general solution)}
\]

\[
y = \left[ y(0) - \frac{b}{a} \right] e^{-at} + \frac{b}{a} \quad \text{(definite solution; from } t = 0 \text{)}
\]
The growth rate of a time-dependent variable $x(t)$ with $t \in \mathbb{R}$ is defined as

$$\dot{x}(t) := \frac{\dot{x}(t)}{x(t)} = \frac{dx(t)}{dt}$$

Consider the following dependent variable (time index $t$ is suppressed)

$$y := x_1^\alpha \cdot x_2^\beta$$

**Question**: What is the growth rate of $y$ in terms of $x_1$ and $x_2$?

**Answer**: The growth rate reads as follows (reasoning: next slide)

$$\dot{y} = \alpha \dot{x}_1 + \beta \dot{x}_2$$
Reminder (4): Basic concepts and tools - continuous-time growth rates (a)

- **Reasoning (1):** Differentiate both side of \( y = x_1^\alpha x_2^\beta \) w.r.t. time

\[
\frac{dy}{dt} = \frac{dx_1^\alpha}{dt} x_2^\beta + x_1^\alpha \frac{dx_2^\beta}{dt} \quad \text{(product rule)}
\]

\[
\dot{y} = \alpha x_1^{\alpha-1} x_2^\beta \dot{x}_1 + \beta x_1^\alpha x_2^{\beta-1} x_2 \quad \text{(chain rule)}
\]

- Subsequently, divide both sides by \( y = x_1^\alpha x_2^\beta \) to get

\[
\frac{\dot{y}}{y} = \frac{\alpha x_1^{\alpha-1} x_2^\beta \dot{x}_1 + \beta x_1^\alpha x_2^{\beta-1} x_2}{x_1^\alpha x_2^\beta} = \alpha \frac{\dot{x}_1}{x_1} + \beta \frac{\dot{x}_2}{x_2} \quad \Rightarrow \quad \dot{y} = \alpha \dot{x}_1 + \beta \dot{x}_2
\]

- **Reasoning (2):** Form the natural logarithm of \( y = x_1^\alpha x_2^\beta \) to get

\[
\ln(y) = \alpha \ln(x_1) + \beta \ln(x_2)
\]

- Subsequently, take the derivative w.r.t. time on both sides

\[
\frac{d}{dt} \ln(y) = \alpha \frac{d}{dt} \ln(x_1) + \beta \frac{d}{dt} \ln(x_2)
\]

- Noting

\[
\frac{d}{dt} \ln(x_i) = \frac{1}{x(t)} \frac{d}{dt} x(t) = \frac{\dot{x}}{x} = \dot{x} \quad \text{(chain rule)}
\]

- Gives

\[
\dot{y} = \alpha \dot{x}_1 + \beta \dot{x}_2
\]
Growth accounting is due to Solow (1957). This procedure decomposes the growth rate of GDP into several supply side components. As a by-product, growth accounting allows an assessment of the technological change component.

Point of departure: a **Cobb-Douglas output technology**

\[ Y(t) = A(t)K(t)^{\alpha} L(t)^{1-\alpha} \]

Expressed in terms of growth rates

\[ \dot{Y} = \dot{A} + \alpha \dot{K} + (1 - \alpha) \dot{L} \]

- \( Y, K, L \) and \( \alpha \) are observable or can be determined empirically.

- Growth rate of GDP \( Y \) is decomposed into the contribution of capital accumulation \( \alpha K \), the contribution of a change in labor input \( (1 - \alpha)L \) and the contribution of technical change \( \dot{A} \).
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Notation and Abbreviations

Notation

- **a**: wealth
- **0 < α < 1**: technology parameter
- **A > 0**: technology parameter (either TFP or labor efficiency)
- **C**: consumption
- **g**: growth rate of technology
- **K**: physical capital
- **k**: steady state value of \( k \)
- **\( k^* \)**: steady state value of \( \bar{k} \)
- **\( K = \frac{dK}{dt} \)**: derivative of \( K \) w.r.t. time
- **\( k = \frac{K}{AL} \)**: capital per efficiency units of labor
- **\( K = \frac{K}{K} \)**: growth rate of capital
- **\( k = \frac{K}{L} \)**: capital per worker
- **L**: labor input
- **n**: growth rate of population
- **\( p_Y \)**: price of \( Y \) (we use \( p_Y = 1 \))
- **r**: interest rate
- **0 < s < 1**: saving rate (investment rate)
- **t ∈ \( \mathbb{R} \)**: time index
- **w**: wage rate
- **Y**: final output
- **\( \bar{y} = \frac{Y}{AL} \)**: output per efficiency units of labor
- **\( y = \frac{Y}{L} \)**: output per worker
- **δ > 0**: capital depreciation rate
- **\( \psi_x \)**: rate of convergence of variable \( x \)

Abbreviations

- **BGP**: balanced growth path
- **DE**: differential equation
- **CRS**: constant returns to scale
- **GDP**: gross domestic product
- **OLS**: ordinary least squares
- **PWT**: Penn World Tables
- **ROC**: rate of convergence
- **TFP**: total factor productivity
- **i.i.d.**: independent identically distributed
- **w.r.t.**: with respect to