1 The Ramsey Model

Consider the household’s problem from a standard Ramsey growth model

\[
\max_{\{c\}} \int_0^{\infty} \frac{c^{1-\sigma} - 1}{1 - \sigma} e^{-\rho t} dt
\]

s.t. \( \dot{a} = ra + w - c, \quad a(0) = a_0 \) \hspace{1cm} (1) \hspace{1cm} (2)

where \( c \) denotes consumption, \( a \) financial wealth, \( w \) the wage rate, \( r \) the interest rate, \( \rho \) the time preference rate and \( t \in \mathbb{R} \) the time index, respectively.

a) Determine the Keynes-Ramsey rule of optimal consumption and provide a sound economic interpretation.

b) *Mass one of firms produce under perfect competition one homogeneous output good with the production function

\[
Y = AK^\alpha L^{1-\alpha}. \hspace{1cm} (3)
\]

The population size is normalized to one and each household supplies one unit of labor inelastically. Derive the reduced system of equations.\(^1\)

c) *Determine the steady state(s).

d) Illustrate the dynamics of the model with a phase diagram.

\(^1\)Hints: The reduced system of equations are two differential equations in \( K \) and \( C \). In equilibrium it holds that \( a = K \).
2 Status Consumption

There is mass one of identical households. The utility function of the typical household is given by

$$u(c(i)) = \left( \frac{c(i)^\beta \left( \frac{c(i)}{\bar{c}} \right)^{1-\beta}}{1-\sigma} \right)^{1-\sigma} - 1$$

with $c(i), \sigma > 0, 0 < \beta < 1$. Also $\bar{c}$ denotes the average level of consumption, taken as exogenous from individual perspective. Notice that this setup implies that aggregate consumption $C = \int_0^1 c(i)di$ coincides with individual consumption $c(i)$. In addition, due to symmetry average consumption $\bar{c}$ must coincide with individual consumption $c(i)$. The household’s intertemporal budget constraint reads

$$\dot{a}(i) = ra(i) - c(i) \quad \text{with } a(0) = a_0$$

where $a(i)$ denotes financial wealth and $r$ the constant rate of return per period of time.

a) Determine the Keynes-Ramsey rule and provide an economic interpretation.

3 The OLG Model

Consider a dynamic model of a perfectly competitive, closed economy. On the household side there are overlapping generations. Time is discrete. Each individual lives for two periods ("youth" and "retirement") only. Life-time utility of an individual born at $t \in \mathbb{N}$ is given by

$$U_t = \ln(c_{1,t}) + \beta \ln(c_{2,t+1})$$

where $c_{1,t}$ denotes the amount of goods consumed in period $t$ by a young individual born at the beginning of period $t$ and, similarly, $c_{2,t+1}$ is the amount of goods consumed in period $t+1$ by an old individual born at $t$, and $0 < \beta < 1$. Each individual supplies one unit of labor when young and divides the resulting labor income between current consumption ($c_{1,t}$) and savings ($s_t$). In the second period, the individual consumes the
savings and the interest payments. The representative HH solves the following problem

\[
\max_{c_{1,t},c_{2,t+1}} U_t \quad \text{(2)}
\]
\[
\text{s.t. } s_t + c_{1,t} \leq w_t \quad \text{(3)}
\]
\[
c_{2,t+1} \leq (1 + r_{t+1})s_t. \quad \text{(4)}
\]

where \(w_t\) denotes the wage rate and \(r_t\) the interest rate.

a) Determine the amount of savings \(s_t\), the amount of first-period consumption \(c_{1,t}\) of young individuals born at \(t\), and write down the Euler equation.

b) How does the savings rate depend on the interest rate \(r\)? Provide a concise economic interpretation of the result.

c) The perfectly competitive firm sector is described by the production function \(y = Ak^\alpha\), where \(k\) is capital in per-worker units and \(A\) is a constant TFP parameter. In equilibrium it must hold that \(s_t = k_{t+1}\). Given this information solve the model further such that it reduces to one difference equation in only one variable, \(k_t\).

d) Determine the steady state(s).

e) *Assume the economy is in its steady state. Now total factor productivity \(A\) increases permanently to a higher level \(A' > A\). Describe the transition with the help of a graphical illustration.

4 Optimal Capital Subsidization

Consider a perfectly competitive economy in the form of a deterministic, time-continuous Ramsey model. There is a continuum of length one of identical, infinitely lived households. Each HH is endowed with one unit of time per period, which is supplied inelastically to the labor market. The representative household is assumed to solve the following problem

\[
\max_{\{C(t)\}} \int_0^\infty \frac{C(t)^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} dt \quad \text{(1)}
\]
\[
\text{s.t. } \dot{a} = r(t)a(t) + w(t)L - C(t) - T(t), \quad a(0) = a_0 \quad \text{(2)}
\]
where \( a(t) \) denotes financial wealth, \( r(t) \) the competitive interest rate, \( w(t) \) the competitive wage rate, \( C(t) \) denotes consumption, and \( T(t) \) lump sum taxes.

There is a continuum of length one of identical final output firms. Each firm has access to the following technology

\[
Y(i, t) = AK(i, t)^\alpha \bar{K}(t)^\beta L(i, t)^{1-\alpha}
\]  

(3)

where \( Y(i, t) \) denotes final output of firm \( i \) at time \( t \), \( A > 0 \) a constant TFP parameter, \( K(i, t) \) capital of firm \( i \) at time \( t \), \( L(i, t) \) labor input of firm \( i \) at time \( t \), \( \bar{K} = \int_{i=0}^{1} K(i, t)di \) the average level (across firms) of capital, and \( 0 < \alpha < 1 \), \( \beta \geq 0 \) constant technology parameters. Profit of the individual firm \( \pi(i, t) \) may be expressed as

\[
\pi(i, t) = pY(i, t) - (1 - s_K)(r(t) + \delta)K(i, t) - w(t)L(i, t)
\]  

(4)

where \( 0 < s_K < 1 \) denotes a capital subsidization rate. Finally, we assume that the government runs a balanced budget such that

\[
T(t) = s_K(r(t) + \delta)K(t) \quad \forall t
\]  

(5)

where \( K(t) \) is the aggregate stock of capital given by \( K(t) = \int_{i=0}^{1} K(i, t)di \).

a) Write down the reduced system of equations for the market economy.

b) Write down the reduced system of equations for the centralized economy (Social Planner’s solution).

c) *Determine the first-best subsidization rate \( s_K \), i.e. the subsidization rate which maximizes intertemporal welfare.

5 Dixit-Stiglitz (1977) Model

Consider a static economy with two sectors. Production technologies read

Final Output: \[ X = H^{1-\alpha} \sum_{i=1}^{N} x_i^\alpha \quad 0 < \alpha < 1 \]  

(1)
Intermediate Goods: \[ x_i = L_i \quad \forall i \in \{1, \ldots, N\} \] (2)

where \( H \) is simply a "fixed factor". Production takes place according to the following sequence: i) \( x \)-production; ii) \( X \)-Production.

There is a continuum of mass one of identical households. Each household is endowed with \( L \) units of time which are supplied inelastically to the labor market. Total labor supply hence is \( L \). Total labor demand is \( \sum_{i=1}^{N} L_i \) (in symmetric equilibrium \( NL_i \)). The price of intermediate goods is \( p_i \) and the price of final output \( X \) is \( P \).

**Further Assumptions:**

1. There is an infinite number of possible varieties \( x_i \)
2. Setting up a new firm is associated with fixed costs \( F > 0 \)
3. \( N \) (which is endogenous) is sufficiently large such that the \( x \)-market can be described as monopolistically competitive implying that the individual \( x_i \)-producer takes the price index \( P \) as given

**Observation:** The reduced form technology (under symmetry, noting that \( Nx = L \) implies \( x = \frac{L}{N} \)) reads

\[ X = H^{1-\alpha} \sum_{i=1}^{N} x_i^\alpha = H^{1-\alpha} Nx^\alpha = H^{1-\alpha} N \left( \frac{L}{N} \right)^\alpha = H^{1-\alpha} N^{1-\alpha} L^\alpha \] (3)

Labor productivity \( \frac{X}{L} \) is positively associated with the number of varieties \( N \).

**Definition:** A (general) equilibrium in this economy consists of quantities \( \{L_i, x_i\} \) and prices \( \{P, p_i, w\} \) such that

1. \( X \)-producers maximize profits
2. \( x \)-producers maximize profits
3. The labor market clears, i.e. \( L^D = L^S \) (where \( L^D = \sum_{i=1}^{N} L_i \) and \( L^S = L \))
4. The intermediate goods market clears, i.e. \( \sum_{i=1}^{N} x_i^S = \sum_{i=1}^{N} x_i^D \)
5. The market for the final output good clears, i.e. \( X^S = X^D \) (not discussed)
Exercises:

a) Determine the demand schedule for \( x_i \) and compare this demand schedule to the demand schedule that would result in the "original setup" where the \( X \)-technology is characterized by complementarity among the intermediate goods, i.e.
\[
X = \left( \sum_{i=1}^{N} x_i^\alpha \right)^{\frac{1}{\alpha}}.
\]

b) *Determine the number of intermediate goods varieties/firms \( N \) in equilibrium. Discuss how \( N \) depends on \( F \). Provide an economic reasoning.

6 IRS and Competitive Equilibrium

6.1 Basic Setup

**Firms:** Consider a static one sector economy. There is continuum of length one of identical firms. Each firm has access to the following technology
\[
Y(i) = AK(i)^{\alpha}L(i)^{\beta} \quad A > 0; \quad \alpha + \beta > 1; \quad i \in [0, \ldots, 1] \quad (1)
\]
where \( Y(i) \) denotes output of firm \( i \), \( K(i) \) physical capital employed by firm \( i \), and \( L(i) \) is labor employed by firm \( i \). Notice that there are increasing returns to scale (IRS) in \( Y \)-production at the level of the individual firm.

**Households:** On the household side there is mass one of identical households who own the capital stock \( K \) and are endowed with \( L \) units of labor, which are supplied inelastically to the labor market.

a) *Does total household income equal total output? Stated differently, is the assumption of IRS compatible with a competitive equilibrium? Provide a concise economic reasoning.

6.2 Marshallian Externalities

**Firms:** Consider a perfectly competitive, one sector economy. There is a continuum of length one of identical firms. The output technology of the individual firm reads as follows
\[
Y(i) = K(i)^{\alpha}L(i)^{\beta}K^{\alpha-1}L^{\beta} \quad \alpha + \beta = 1; \quad a, b > 0; \quad i \in [0, \ldots, 1] \quad (2)
\]
where $\overline{K} = \int_0^1 K(i)di$ and $\overline{L} = \int_0^1 L(i)di$ denote the average (across firms) levels of capital and labor, respectively.\textsuperscript{2} The implicit assumption is that there are positive spillover effects in the production sphere. Total factor productivity of the individual firm is higher the higher is the overall input of capital and labor. Notice that there are constant returns to scale (CRS) at the level of the individual firm but IRS at the aggregate level.

**Households:** On the household side there is mass one of identical households who own the capital stock $K$ and are endowed with $L$ units of labor, which are supplied inelastically to the labor market.

\[ \text{a) } *\text{Does total household income equal total output? Stated differently, is the assumption of IRS compatible with a competitive equilibrium? Provide a concise economic reasoning.}\]

6.3 Monopolistic Competition

**Firms:** The economy comprises two sectors. In the final output sector (CRS, perfectly competitive) there is mass one of identical firms. The output technology reads

\[ Y(j) = \left( \int_0^1 x(i)^\lambda di \right)^{\frac{1}{\lambda}} \quad 0 < \lambda < 1 ; j \in [0, \ldots, 1] \quad (3) \]

In the intermediate goods sector (IRS, monopolistic competition) there is mass one of identical firms. Each firm has access to the following technology

\[ x(i) = K(i)^\alpha L(i)^\beta \quad \alpha, \beta > 0; \ \alpha + \beta > 1 \quad (4) \]

**Households:** On the household side there is mass one of identical households who own the capital stock $K$ and are endowed with $L$ units of labor, which are supplied inelastically to the labor market. Households are the owners of the firms. Hence, total earnings of the

\textsuperscript{2}Notice that $\overline{K}$ doesn’t change if $K(i)$ changes by one unit since the individual is of mass zero. The discrete analogue of this statement is as follows. Imagine an economy which comprises, say, 10,000 firms. If one firm would increase its $K(i)$ by one unit, then the average stock of capital, $\frac{1}{10000} \sum_{i=1}^{10000} K(i) = \frac{1}{10000} K(1) + \cdots + \frac{1}{10000} K(10000)$, remains approximately constant because the weight of any individual firm, $\frac{1}{10000}$, is small. More specifically, if an individual firm increases its capital stock by one unit, then the average capital stock increases by $\frac{1}{10000}$ units. The same reasoning applies to $\overline{L}$ and $L(i)$. 
representative household is given by

\[
\text{Earnings} = \pi_Y + \pi(i) + rK + wL \tag{5}
\]

**Market structure**: Factor markets are perfectly competitive. The final output sector is perfectly competitive. The intermediate goods sector is monopolistically competitive.

a) Does total household income equal total output? Stated differently, is the assumption of IRS compatible with a competitive equilibrium? Provide a concise economic reasoning.

7 Romer (1990) Model: Social Planner’s Problem

The social planner’s problem reads as follows

\[
\max_{\{C,L,A\}} \int_0^\infty \frac{C^{1-\sigma} - 1}{1 - \sigma} e^{-\rho t} dt \tag{1}
\]

s.t. \( \dot{K} = K^\alpha (AL)^{1-\alpha} - C \)  
\[ L_y + L_a = L \tag{2} \]
\[ \dot{A} = \eta AL \]  
\[ K(0) = K_0; \ A(0) = A_0 \]  
\[ L_y + L_a = L \tag{2} \]
\[ \dot{A} = \eta AL \]  
\[ K(0) = K_0; \ A(0) = A_0 \]  
\[ \dot{A} = \eta AL \]  
\[ K(0) = K_0; \ A(0) = A_0 \]  
\[ \dot{A} = \eta AL \]  
\[ K(0) = K_0; \ A(0) = A_0 \]  
\[ \dot{A} = \eta AL \]  
\[ K(0) = K_0; \ A(0) = A_0 \]  
\[ \dot{A} = \eta AL \]  
\[ K(0) = K_0; \ A(0) = A_0 \]

The notation is the same as in the lecture notes on the Romer (1990) model.\(^3\)

a) Explain why this dynamic optimization problem is indeed the social planner’s problem which corresponds to the market economy in the Romer (1990) model.

b) In which sense is this model (Romer (1990) - social planner’s problem) simpler compared to the Romer (1990) model of a decentralized economy?

c) Determine the steady state growth rate resulting from the social planner’s solution. Provide a concise economic interpretation. Compare this steady state growth rate to the decentralized steady state growth rate.

\(^3\)On terminology: ”decentralized economy” and ”market economy” are used synonymous.
8 Investment under Capital Adjustment Costs

Consider a firm that produces a homogenous final output good \( Y_t \) under perfect competition. The output technology reads

\[
Y_t = A (K_t)^{\alpha} (L_t)^{1-\alpha}
\]

where \( K_t \) is the amount of capital at time \( t \in \{0, 1, \ldots\} \), \( L_t \) the amount of labor, and \( A > 0, 0 < \alpha < 1 \). The firm’s planning horizon is infinity. There are investment costs (capital adjustment costs). Installing the amount of \( I_t \) additional capital goods (gross investment) requires

\[
IC_t = I_t \left[ 1 + \theta \left( \frac{I_t}{K_t} \right) ^{\eta} \right]
\]

with \( \theta \geq 0 \) and \( \eta \geq 1 \) units of final output. Notice that the usual assumption is \( \theta = 0 \) (standard Ramsey model). Let

\[
CF_t = Y_t - w_t L_t - IC_t
\]

denote the firm’s cash flow (or the capitalist’s / entrepreneur’s residual income). We assume that the firm maximizes the PDV of cash flow, i.e.

\[
\max_{\{I_t, L_t\}} \sum_{t=0}^{\infty} \frac{1}{1 + r}^t CF_t
\]

\[
\text{s.t. } K_{t+1} = I_t + (1 - \delta) K_t, \ (1), \ (2), \ K_0 = \text{given}
\]

where \( w_t \) denotes the wage rate, \( r \) the fixed interest rate, and \( \delta \geq 0 \) the capital depreciation rate.

a) Verify that (2) and (3) are indeed equivalent formulations for the firm’s net cash flow.

b) Determine the firm’s investment demand (i.e. the demand for final output devoted to capital investment).

c) *Determine the shadow price of capital in the steady state.
Remark: The investment demand results from the solution of the above stated dynamic optimization problem and will be a function $I_t = I_t(q_t)$, where $q_t$ denotes the shadow price of installed capital goods.