Technological Change, Population Dynamics, and Natural Resource Depletion

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Abstract
How does technological change affect the depletion of non-renewable natural resources? We argue that to answer this question it is necessary to account for the interaction between skill-biased technological change, population dynamics and their effects on natural resource use. Skill-biased technological change leads in a non-malthusian world to a decline in the long-run growth rate of the population. Furthermore, skill-biased technological change is associated with a lower long-run depletion rate of natural resources, whereas the transition to a long-run equilibrium is characterized by an inverse relationship between the growth rate of the population and the depletion rate of natural resources. In the context of a shrinking or stationary population, the emergence of sustained economic growth depends on the existence of positive intertemporal knowledge spillovers with respect to research and development.

Keywords: OLG-Model, Endogenous Fertility, Directed Technological Change, Non-renewable Natural Resources

JEL: J13, O40, O41, Q32
1 Introduction

How does technological change affect the depletion of non-renewable natural resources? We argue that to answer this question it is necessary to account for the interaction between skill-biased technological change, population dynamics and their effects on natural resource use. Recent history in the developed world has been characterized by an increase in the supply of skilled labor relative to unskilled labor inducing both skill-biased technological change and - among other factors - a decline in population growth rates. Additionally this process coincides with an increasing demand for natural resources. In recent years several economists clarified the economic link between population dynamics, education and economic growth, where a formal analysis of the non-malthusian interaction between population, technology and resource depletion within the context of dynamic general equilibrium models is still missing in the literature. Our research suggests that the non-malthusian interaction between population and technology in terms of skill biased technological change induces a decline in population growth and a decline in the depletion of natural resources in the long-run. During the transition to a long-run equilibrium, population growth and natural resource depletion are inversely related.

One of the major stylized facts that characterized the development process of industrialized countries is a decline in fertility rates. In developed economies, the transition from rapid population growth to low net fertility rates began at the start of the second phase of industrialization in the nineteenth century. Birth rates declined faster than mortality rates, yielding a substantial reduction in net population growth, inducing the so-called demographic transition (Galor, 2005). During the last couple of decades, net fertility rates have reached exceptionally low levels, and have fallen short of the replacement threshold even in countries that have traditionally exhibited quite high fertility rates - e.g. Spain and Italy. In less developed countries, the fertility transition started in the mid-1960s, and it was particularly rapid in East Asia.\(^1\) Since we do not observe a demographic transition without economic development or vice versa, it seems that demographic transition can be considered to be an inherent factor of economic development. Several economic channels that are responsible for the observed fertility decline have been isolated. The most important ones with respect to the interaction between technology and population are proposed by Galor and Weil (2000,1996) stating that: (a) declining fertility rates may be due to technological progress that, via its impact on the demand for human capital, reverses the relationship between income and population growth with respect to the regime of Malthusian stagnation and (b) increasing real wages raise the opportunity cost of having children, where lower fertility generates positive feedback effects on economic growth by means of capital accumulation.\(^2\)

In the past sixty years, the relative supply of skilled labor has increased sharply in the U.S. as well

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\(^1\)For a comprehensive overview over aspects of the demographic transition, see Lee (2003). For the emergence of lowest-low fertility rates in Europe, see Kohler et al. (2002).

\(^2\)For a more detailed discussion of other channels see for example Schäfer and Valente (2010).
as in other industrialized countries. Moreover, and contrary to the predictions of a neoclassical framework with concave production technologies, there has been a sharp increase in the skilled wage premium since the 1970s. The standard explanation for this pattern is an acceleration in the skill bias of capital-embodied technological change (Autor et al., 1998 and Hornstein et al., 2005). At the same time, increasing demand for human capital and increasing wages can be seen as being responsible for the decline in average fertility rates, since better educated families tend to have fewer children and provide more education per child, due to a trade off between the number of children parents wish to raise and the amount of resources they spend on education per child (see for example de la Croix and Doepke 2003). Given a high intergenerational persistence, fertility decisions and investments in education per child are transferred from one generation to another and interact with macroeconomic aggregates.\(^3\)

**Figure 1 about here**

Since the world-wide oil crisis of the mid 1970s at the very latest, western societies have been increasingly concerned about the sustainable development of their economies at large, and we observe a strong positive relationship between income per capita and demand for natural resources. In 2007, OECD petroleum consumption amounted to 57% of the world petroleum consumption (U.S. Energy Information Administration, 2008) and per capita energy use differ between the richest and the poorest group of countries by a factor ten (Weil, 2005). In addition the world depletion rate of crude oil is increasing, see Figure 1, where OECD petroleum consumption increased by a factor greater than two between 1960 and 2005 (U.S. Energy Information Administration, 2006). From the outset of economic theory, scarcity of natural resources and population dynamics have been at its core. Whereas Malthus and Ricardo primarily had land in mind as causing diminishing returns to other (possibly reproducible) inputs, the concerns of the Club of Rome and the report "Limits to Growth" by Meadows dealt with non-renewable resources as essential inputs for production, while neglecting any interaction between scarcity of natural resources, relative prices and technology. As regards the long-run development of the world, their predictions have proven worse than Malthusian stagnation, since they arrived at the conclusion that achieving even a stage of at least constant per capita incomes would not be possible. In this context, Solow (1974), Stiglitz (1974), and Dasgupta and Heal (1974) provided the foundations for the formal analyses of non-renewable resources within the context of models of endogenous growth that emerged in the last two decades.\(^4\)

This paper integrates the features of skill-biased technological change, fertility decline and natural resource use into a comprehensive framework. More in detail, we consider an overlapping

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\(^3\)For empirical and theoretical evidence, see Kremer and Chen (2000); Rosenzweig and Wolpin (1980); de la Croix and Doepke (2003,2004).

\(^4\)For a comprehensive overview on the theory of endogenous economic growth and the role of non-renewable natural resources see Groth (2007) and Pittel (2002).
generations (OLG) economy populated by skilled and unskilled households. The educational choices of parents for their offspring generate differential fertility between skilled and unskilled households, in the spirit of Galor and Mountford (2006), and de la Croix and Doepke (2003). The production side of the model is characterized by a scale invariant two-sector growth model with multiple production stages. Moreover, the direction of technological change is determined by the relative supply of skilled and unskilled labor (Acemoglu 1998, 2002) resulting from the educational choices of parents for their offspring. In addition, production is subject to a non-renewable natural resource as an essential input.\footnote{Models of directed technical change have been applied in environmental economics by Andre and Smulders (2005), Di Maria and Smulders (2004), Di Maria and van der Werf (2005), and Di Maria and Valente (2006). Di Maria and Smulders (2004) analyze the effects of pollution and Andre and Smulders (2005) study a labor-resource economy. Di Maria and Valente (2006) are probably the first who provide a micro-foundation of purely resource-augmenting technical progress.} The focus of our analysis will be the interaction between skill-biased technological change, population dynamics and resource depletion in the long-run and during the transition to a long-run equilibrium. In order to address the transitional dynamics, we perform three numerical experiments under realistic parameters, dealing with temporary and/or long-run increases in the skilled-unskilled population ratio.

The remainder of the paper is organized as follows: Section 2 introduces the optimization problem of households. Section 3 describes the production side of the model. Section 4 presents the equilibrium structure, the dynamic system and the properties of the balanced growth path (long-run equilibrium). In Section 5, we calibrate the model and explore the dynamic behavior of the model. Finally, Section 6 provides a summary and a conclusion.

## 2 Households

### 2.1 Preliminary Remarks

We consider two groups of households, skilled and unskilled. In accordance with empirical observations reporting a high intergenerational persistence in education, we assume that the group of skilled households raises only skilled offspring. The group of unskilled households, in turn, raises both types of children, but on the household level a single household raises either skilled or unskilled offspring. The fraction of unskilled households raising skilled offspring will be determined endogenously in equilibrium. Hence, there exist three types of agents, skilled households raising skilled offspring, unskilled households raising unskilled offspring and unskilled households raising skilled offspring. The setting we present here generates upward mobility or no mobility at all. In other words, downward mobility and discrimination among offspring with respect to educational choices are absent as we focus on developed economies with low fertility rates, which makes the emergence of the former and latter unlikely.
2.2 Preferences and Budget Constraints

The economy under consideration is populated by a continuum of overlapping generations. In this setting, time is discrete, indexed by $t$ and ranges from 0 to $\infty$. Households live for three periods: childhood, adulthood, and old age. All economically relevant decisions are made in the second period of life, adulthood. Adult agents supply one unit of work time each to firms, raise children and save, whereas old agents consume their savings. The economy is populated by two groups of households, skilled and unskilled, denoted by $L^i_t$ ($i = u, s$), respectively, earning a wage income $w^i_t$ in their second period of life. Moreover, there are two types of children, $n^{i,j}_t$, either trained to be skilled workers or unskilled workers in $t + 1$, denoted by $j = u, s$. Preferences of a member $i = u, s$ of generation $t$ that is born in $t - 1$ are defined over consumption in $t$ and $t + 1$ as well as the potential aggregate income of her children. They are specified as

$$u^i_t = \ln c^i_t + \nu \ln (w^j_{t+1} n^{i,j}_t) + \rho \ln c^i_{t+1},$$

where $c_t$ and $c_{t+1}$ are present and future consumption. $w^j_{t+1} n^{i,j}_t$ reflects total potential income of an individual’s offspring weighted by the altruism factor $\nu$. Moreover, $\rho$ represents as usual the individual discount factor of future consumption.

We denote the fraction of unskilled households investing in education for their children by $\theta_t \in [0,1]$. Hence, the two population groups evolve according to

$$L^{s}_{t+1} = \theta_t n^{u,s}_t L^u_t + n^{s,s}_t L^s_t$$

and

$$L^{u}_{t+1} = (1 - \theta_t) n^{u,u}_t L^u_t.$$

The education system is privately funded. As teaching requires skilled labor, $L^E_t$, schooling fees depend on the wage rate for skilled labor, $w^s_t$, and the exogenously fixed teacher-student ratio, $\phi$.

$$w^s_t \phi (n^{s,s}_t L^s_t + n^{u,s}_t \theta_t L^u_t) = w^s_t L^E_t.$$

Consequently the education cost per child amounts to $w^s_t \phi$. Regardless of the type of children parents wish to raise, fertility is subject to forgone wage earnings, in terms of opportunity costs and consumption needs of children. To this end, parents have to relinquish the fraction $z$ of their wage income per child. Therefore, child-rearing costs for unskilled children amount to $zw^i_t n^{i,u}_t$ with $i = u$ because only unskilled households raise unskilled offspring. Total child-rearing costs

\[\text{5} \text{For similar assumptions regarding the schooling sector see Eicher (1996,1999), and Bhagwati and Srinivasan (1977).}\]
for skilled offspring on the other hand amount to \((zw^i_t + w^i_t \phi)n^{i,s}_t\), with \(i = u, s\).

In order to cover old age consumption, members of generation \(t\) can buy property rights to natural resources (natural capital) and invest in the capital market (man-made capital). We denote the stock of the exhaustible natural resource in period \(t\) by \(M_t^7\), and its extraction allocated to production by \(R_t\). The economy is initially endowed with a resource stock \(M_0 > 0\). At the beginning of the current period, \(t\), the stock of exhaustible natural resources is determined by the past resource stock minus extraction in the current period, hence, \(M_t = M_{t-1} - R_t\). Each member of generation \(t\) buys \(m^i_t\) units of natural resources from the old age generation at the competitive price \(p^R_t\). When she retires in \(t + 1\), she sells a part of her natural resources to firms for use in production and the property rights of the remaining part to the adult cohort born in period \(t\) at price \(p^R_{t+1}\) per unit of natural resources. The level of future consumption equals revenues from investments in man-made capital on the capital market \(((1 + r_{t+1})s^i_t)\), plus earnings from selling natural resources to firms (extraction) and the selling of the property rights to natural resources to the adult cohort born in \(t\) \((p^R_{t+1}m^i_t)\), hence

\[
c^i_{t+1} = (1 + r_{t+1})s^i_t + p^R_{t+1}m^i_t,
\]

with \(i = u, s\).

The budget constraint of a skilled household raising skilled offspring \((i, j = s)\) thus reads as

\[
w^s_t \geq zw^s_t n^{s,s}_t + w^s_t \phi n^{s,s}_t + c^s_t + p^R_t m^s_t + s^s_t,
\]

for unskilled households raising unskilled offspring, \((i, j = u)\)

\[
w^u_t \geq zw^u_t n^{u,u}_t + c^u_t + p^R_t m^u_t + s^u_t,
\]

and finally for unskilled households raising skilled offspring, \((i = u\) and \(j = s)\)

\[
w^u_t \geq (zw^u_t + \phi w^u_t)n^{u,s}_t + c^u_t + p^R_t m^u_t + s^u_t.
\]

### 2.3 Optimization

A member \(i = u, s\) of generation \(t\) chooses \(\{c^i_t, n^{i,j}_t, s^i_t, m^i_t\}\) so as to maximize the utility function given by Eq.(1), with \(j = s\) if \(i = s\) and \(j = u, s\), if \(i = u\). Regardless of the type of household and which type of children a household wishes to raise, the maximization over the two

\(^7\)We denote aggregate levels in capital letters and per capita levels in lower case letters.
assets, natural resources \((m_t^i)\), and investment in the capital market \((s_t^i)\) imply a non-arbitrage condition between the two assets that is known as Hotelling’s rule, such that marginal returns on both assets are equalized

\[
1 + r_{t+1} = \frac{p_{t+1}R}{p_t^R}.
\]  

(9)

Hence, the marginal return of investment on the exhaustible resource stock, \(\frac{p_{t+1}R}{p_t^R}\), must equal the marginal return of investment on the capital market used to finance research and development (R&D).

2.3.1 Skilled Households

Skilled households raise skilled offspring and maximize (1) subject to (6), implying that

\[
c_t^s = \frac{1}{1 + \nu + \rho w_t^s},
\]

(10)

\[
n_{t,s,s} = \frac{\nu}{(1 + \nu + \rho)(z + \phi)},
\]

(11)

\[
s_t^s = \frac{\rho}{1 + \nu + \rho} w_t^s - p_t^R m_t^s.
\]

(12)

Since skilled households finance education with their labor income, \(w_t^s\), heterogeneity does not influence them. However, it is worth noting that their fertility decisions also depend negatively on the fraction of wage income that is transferred to their offspring, \(z\), and the student-teacher ratio, \(\phi\).

2.3.2 Unskilled Households Raising Unskilled Offspring

Unskilled households raising unskilled offspring, maximize (1) subject to (7), implying that

\[
c_t^u = \frac{1}{1 + \nu + \rho w_t^u},
\]

(13)

\[
n_{t,u,u} = \frac{\nu}{(1 + \nu + \rho)z},
\]

(14)

\[
s_t^u = \frac{\rho}{1 + \nu + \rho} w_t^u - p_t^R m_t^u.
\]

(15)

Since this type of household does not invest in education, the number of unskilled children raised in unskilled households is not influenced by the parameter \(\phi\). Comparing (11) and (14) shows that skilled parents raise fewer children than unskilled parents, due to the existence of education costs. Moreover, when \(w_t^u < w_t^s\), unskilled households have less resources available for present and future consumption.
2.3.3 Unskilled Households Raising Skilled Offspring

Maximizing lifetime utility (1) subject to (8) implies

\[ c_t^u = \frac{1}{1 + \nu + \rho} w_t^u, \]  
(16)

\[ n_t^{u,s} = \frac{\nu}{(1 + \nu + \rho)} \left( w_t^u z + w_t^u \phi \right) = \frac{\nu}{(1 + \nu + \rho)} \left( z + \omega_t \phi \right), \]  
(17)

\[ s_t^u = \frac{\rho}{1 + \nu + \rho} w_t^u - p_t^R m_t^u, \]  
(18)

with \( \omega_t = \frac{w_t^s}{w_t^u} \).

Apparently, the wage differential between skilled and unskilled labor \( \omega_t \) plays a crucial role in the decision of unskilled households that wish to educate their offspring to skilled workers. The higher the skilled-wage premium, the higher the educational cost per child compared to the wage income of an unskilled household, such that the number of skilled children born in unskilled households is inversely related to \( \omega_t \).

2.3.4 The Share of Unskilled Households Raising Skilled Offspring

Unskilled households raise either unskilled or possibly skilled children. Given that discrimination among offspring is absent, the fraction of unskilled households raising skilled offspring is denoted by \( \theta_t \in [0, 1] \). For \( \theta_t > 0 \), the lifetime utility of unskilled parents raising skilled offspring must at least equal the lifetime utility of unskilled parents raising unskilled offspring. In light of the solution to the optimization problem of unskilled households (13)-(15) and (16)-(18), we can establish the following proposition

Proposition 1

The group of unskilled households raises both types of children, i.e. \( n_t^{u,s} > 0 \) and \( n_t^{u,u} > 0 \), if \( u_t^{u,u} = u_t^{u,s} \), such that\(^8\)

\[ \frac{w_t^{u,s}}{w_t^{u,u}} = \omega_t + 1 = \frac{z + \omega_t \phi}{z}. \]  
(19)

Relation (19) is an indifference condition, which will determine the fraction \( 0 \leq \theta_t \leq 1 \) of unskilled households in equilibrium that educate their offspring to skilled workers. As parents are altruistic towards their offspring with respect to the offspring’s aggregate potential labor income, the future wage differential between skilled and unskilled labor, \( \omega_{t+1} \), must at least compensate for the cost ratio between skilled and unskilled offspring. Moreover, as the right-hand side of (19) is greater than one, it follows that (19) holds only if \( w_t^{u,s} > w_t^{u,u} \).\(^9\) Comparing the respective fertility

\(^{8}\)The proof follows directly from (13)-(15) and (16)-(18) given that \( u_t^{u,u} = u_t^{u,s} \).

\(^{9}\)Similar to de la Croix and Doepke (2003) fertility differentials are generated by wage differentials. The difference is that de la Croix and Doepke consider a continuous wage distribution.
decisions yields

\[ n_t^{u,u} > n_t^{s,s} > n_t^{u,s}, \]  

given that (19) holds. The fertility of unskilled households raising unskilled offspring is the highest since they do not allocate resources to education, whereas the opposite is true for unskilled parents raising skilled offspring. Since the latter have to pay tuition fees in terms of \( w_t^s \), where \( w_t^u < w_t^s \), they trade a lower number of children against a higher income for their offspring. Hence, the fertility of unskilled parents raising skilled offspring is the lowest.

3 Production

3.1 Preliminary Remarks

The production side of the model builds on Acemoglu (1998,2002). Aggregate output, \( Y_t \), is composed of two intermediates, \( Y_t^s \) and \( Y_t^u \), stemming from two different production processes, one using skilled labor, and the other one using unskilled labor and a set of machines, \( x_t^s \) and \( x_t^u \), which are complementary to each kind of labor, respectively. The production of machines requires the existence of technological knowledge (a blueprint or design), labor and natural resources. Blueprints are the outcome of purposeful investments in research and development (R&D). Labor markets for skilled and unskilled labor are competitive and each kind of labor is assumed to move vertically within the production processes. This assumption takes different skill-intensities and a low inter-sectoral mobility of skilled and unskilled labor into account. Moreover, skilled labor is employed in production as well as in education.

3.2 Final Good Production

Final output is composed of intermediate goods, \( Y_t^s \) and \( Y_t^u \), both stemming from two different production processes. The elasticity of substitution between \( Y_t^s \) and \( Y_t^u \) is determined by \( \varepsilon \in (0, \infty) \) such that the production of final output is subject to the following nested CES-production function

\[ Y_t = \left[ \gamma(Y_t^u)^{\frac{1}{\varepsilon}} + (1 - \gamma)(Y_t^s)^{\frac{1}{\varepsilon}} \right]^{\frac{\varepsilon}{1-\varepsilon}}. \]  

(21)

The parameter \( \gamma \in (0, 1) \) is a distribution parameter determining how important the two goods are for aggregate output. In each period the price of final output, is normalized to 1 - that is \( p_t^Y \equiv 1 \) - where the prices of \( Y_t^s \) and \( Y_t^u \) are denoted by \( p_t^{Y,s} \) and \( p_t^{Y,u} \), such that

\[ \left[ \gamma^\varepsilon (p_t^{Y,u})^{1-\varepsilon} + (1 - \gamma)^\varepsilon (p_t^{Y,s})^{1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}} = p_t^Y = 1. \]  

(22)

\( ^{10} \)For details see Appendix B.1.
3.3 Production of Intermediates and Machines

The production of $Y_s^t$ and $Y_u^t$ requires skilled and unskilled labor, as well as a range of labor complementary machines, $x_s^t$ and $x_u^t$, respectively. In each period of time, $t$, there are $N_i^t$ ($i = u, s$) different types of machines available. Production functions of both intermediates read as

\[ Y_s^t = \frac{1}{1 - \beta} \int_0^{N_s^t} x_s^t(j)^{1-\beta} d(j(Y_s^t))^{\beta}, \]  
\[ Y_u^t = \frac{1}{1 - \beta} \int_0^{N_u^t} x_u^t(j)^{1-\beta} d(j(Y_u^t))^{\beta}, \]  

with $0 < \beta < 1$.

Machines, $x^i$, are manufactured with skilled or unskilled labor, $L_i^t$, respectively, and natural resources, $R_i^t$ ($i = u, s$), where labor as well as natural resources are considered to be essential for the production of machines. Moreover, we assume that current technological knowledge increases factor productivity in the machine producing sector. Therefore, production functions for a machine of type $j$ in sector $s$ or $u$ read as

\[ x_s^t(j) = AN_s^t(L_s^t(j))^{1-\alpha}(R_s^t(j))^{\alpha}, \]  
\[ x_u^t(j) = BN_u^t(L_u^t(j))^{1-\alpha}(R_u^t(j))^{\alpha}, \]  

with $A, B > 0$.

3.4 Research and Development

R&D constitutes the search for new designs (blueprints) of machines. To this end, research firms rent labor services, capital inputs and natural resources. It simplifies the analysis considerably though, if we assume that all the three of these factors combine to produce blueprints in exactly the same way that they combine to produce final output, i.e. we apply the so called lab-equipment approach. By doing so, we also take into account the criticism stressed by resource economists with respect to the so-called ”knowledge driven” specification, in which (skilled) labor is the only input to R&D and its therefore seemingly overly optimistic assumption that R&D could take place without natural resources. In addition we assume that blueprints are depreciated entirely after one period. We consider this to be quite realistic, since one period encompasses approximately 30 years and firms producing a steam engine must become obsolete at some point in time.\(^{12}\)

\(^{11}\)It is debateable whether this assumption induces an over-optimistic perspective with respect to non-reproducible capital into the model, but it is necessary to generate steady-state growth.

\(^{12}\)Without complete depreciation, the old generation would sell its assets on the capital market to the adult cohort. The latter would split up its savings between existing blueprints and investments in R&D. Under full depreciation the amount of savings is entirely allocated to R&D.
Both R&D sectors generate new blueprints according to the following scale invariant production function

\[ N_{t+1}^s = \frac{\eta^s}{(N_t^s)^\delta} (D_t^s)^\lambda \quad \text{and} \quad N_{t+1}^u = \frac{\eta^u}{(N_t^u)^\delta} (D_t^u)^\lambda \quad \lambda \in (0, 1], \quad (27) \]

where \( D_t^s \) and \( D_t^u \) are spending on R&D (in units of the final good) for skilled- and unskilled-labor complementary types of machines, respectively. The parameters \( \eta^s \) and \( \eta^u \) are productivity parameters that allow the costs of innovation to differ. As a non-critical but simplifying assumption with respect to the qualitative results of our analysis, we set \( \lambda = 1 \), that is lab-equipment linearly enters the production function of new blueprints. Finally, we will distinguish the cases \( \delta > 0 \) and \( \delta < 0 \). In the former, technological advances are partially hedged out by diminishing technological opportunities (Evenson, 1984; Kortum, 1993; Jones and Williams, 2000), when it may become more and more complicated to achieve productivity gains. In the case of \( \delta < 0 \), there are intertemporal knowledge spillovers; a case which is labeled as "standing on the shoulders of giants" in the literature. In either case, atomistic R&D firms take the level of technological knowledge in the current period as given and neglect the impact of their own R&D efforts on the future level of technological knowledge. For \( \lambda = 1 \) and \( \delta \neq 0 \), our R&D specifications read as

\[ N_{t+1}^s = \frac{\eta^s D_t^s}{(N_t^s)^\delta} \quad \text{and} \quad N_{t+1}^u = \frac{\eta^u D_t^u}{(N_t^u)^\delta}. \quad (28) \]

4 Equilibrium

4.1 Preliminary Remarks

In this section, we specify the equilibria on the markets for intermediate goods and machines, the labor market and the market for natural resources. The properties of these equilibria determine marginal production costs in the machine producing sector and the distribution of aggregate savings between the two R&D-sectors. With the information at hand, the rates of technological progress for skilled and unskilled-labor complementary innovations are known, which allows us to determine the dynamics of the depletion rate of natural resources.

4.2 Production, Labor, and Natural Resources

Intermediate goods markets are competitive, such that market clearing determines factor prices, \( p_t Y_t^s \) and \( p_t Y_t^u \), according to marginal productivities of \( Y_t^s \) and \( Y_t^u \) in the production of final output, \( Y_t \), such that

\[ p_t = p_t \frac{Y_t^s}{Y_t^u} = \frac{1 - \gamma}{\gamma} \left( \frac{Y_t^s}{Y_t^u} \right)^{-\frac{1}{\gamma}}. \quad (29) \]
The usual characteristics of a symmetric equilibrium imply equal prices and quantities (produced and demanded) for any type of machine within the $s$ and the $u$-sector, such that $p^s_t(j) = p^u_t$, $x^s_t(j) = x^u_t$, and $x^s_t(j) = x^u_t(j)$, which implicates that the demand for machines is equally spread over all types of machines

$$x^s_t = \left( \frac{p^Y_s}{p^s_t} \right)^{\frac{1}{\beta}} L^Y_s$$ and $$x^u_t = \left( \frac{p^Y_u}{p^u_t} \right)^{\frac{1}{\beta}} L^Y_u.$$ (30)

Intuitively, a higher price $p^Y_i$ increases the value marginal product of all factors employed in the production of $Y_i$, encouraging firms to employ more machines. A higher level of labor employment similarly increases the demand for machines as more workers are using them (market size effect). Conversely, demand for machines is inversely related to the rental price of machines, $p^s_t$ and $p^u_t$. Since machine producers have to cover R&D expenses for respective blueprints, machines are sold under monopolistic competition, facing demands as described in (30). Hence, the profits of machine producers are obtained as

$$\pi^s_t = (p^s_t - c^s_t)x^s_t$$ and $$\pi^u_t = (p^u_t - c^u_t)x^u_t$$ (31)

with

$$c^s_t(w^s_t, p^R_t) = \frac{(w^s_t)^{1-\alpha}(p^R_t)\alpha_{AN}^s(1-\alpha)^{1-\alpha}}{\alpha_{AN}^s}$$ and $$c^u_t(w^u_t, p^R_t) = \frac{(w^u_t)^{1-\alpha}(p^R_t)\alpha_{BN}^u}{\alpha_{BN}^u(1-\alpha)^{1-\alpha}}.$$ (32)

representing the marginal costs of machine producing firms\(^\text{13}\), opening out into a markup over marginal production costs, which is determined by the elasticity of substitution between different types of machines

$$p^s_t = \frac{c^s_t(w^s_t, p^R_t)}{1-\beta}$$ and $$p^u_t = \frac{c^u_t(w^u_t, p^R_t)}{1-\beta}.$$ (33)

In conclusion, the profits of machine producers (31) write as

$$\pi^s_t = \beta(1-\beta)^{\frac{1-\beta}{\beta}} (p^s_t L^s_t(c^s_t)^{\frac{\beta-1}{\beta}}),$$ (34)

and

$$\pi^u_t = \beta(1-\beta)^{\frac{1-\beta}{\beta}} (p^u_t L^u_t(c^u_t)^{\frac{\beta-1}{\beta}}).$$ (35)

Because of the market size effect, the profits of technology owners increase with the employment of the type of labor that is complementary to the respective type of machines, the value marginal products of intermediates in final good production, $p^Y_i, i = u,s$, since this increases machine demand (see (30)) and decreasing in marginal production costs of machines, $c^s_i, i = u,s$, as $\beta < 1$, where production costs are increasing functions in the factor prices for labor and natural

\(^{13}\)For details see Appendix B.2.
resources. Given the structure of this symmetric equilibrium, the levels of $Y_t^s$ and $Y_t^u$ are obtained from (30) together with (23) and (24) as

$$ Y_t^s = (1 - \beta) \frac{1-2\beta}{\beta} N_t^s \left( \frac{p_t Y_t^s}{c_t^s} \right) \frac{1-\beta}{\beta} L_t^s, $$

and

$$ Y_t^u = (1 - \beta) \frac{1-2\beta}{\beta} N_t^u \left( \frac{p_t Y_t^u}{c_t^u} \right) \frac{1-\beta}{\beta} L_t^u. $$

Consequently, the relative price for the two intermediates is identified by (29) together with (36) and (37), such that

$$ p_t = \left( \frac{1 - \gamma}{\gamma} \right) \frac{\partial}{\partial p_t} \left( \frac{N_t^s \frac{L_t^s}{N_t^u} \frac{L_t^u}{Y_t^u}}{\frac{L_t^s}{c_t^s} \frac{L_t^u}{c_t^u}} \right) \frac{1-\beta}{\beta} - \frac{1-\beta}{\beta} $$

with $\sigma \equiv \epsilon - (\epsilon - 1)(1 - \beta)$.

Given that labor markets for skilled and unskilled labor are competitive, the skilled wage premium $\omega_t = \frac{w_t^s}{w_t^u}$ is given by $p_t \left( \frac{\partial Y_t^s}{\partial L_t^s} / \frac{\partial Y_t^u}{\partial L_t^u} \right)$. Hence, we can write the wage differential in light of (38) together with (36) and (37) as follows

$$ \omega_t = \left( 1 - \gamma \right) \frac{\epsilon \beta}{\sigma \xi^x} \left( \frac{N_t^s}{N_t^u} \right) \left( \frac{\frac{L_t^s}{c_t^s} \frac{L_t^u}{Y_t^u}}{\frac{L_t^s}{c_t^s} \frac{L_t^u}{Y_t^u}} \right) \frac{1-\beta}{\beta} $$

with $\xi^L = \beta + (1 - \alpha)(1 - \beta)$.

If $\sigma > 1$, then the skilled wage premium of the current period is increasing in the present blueprint ratio $\frac{N_t^s}{N_t^u}$ because the two intermediates are gross substitutes in final good production. The relative factor reward of skilled labor declines, however, in the current skilled-unskilled employment ratio, $\frac{L_t^s}{L_t^u}$ for a given blueprint ratio, $\frac{N_t^s}{N_t^u}$, as labor is subject to diminishing marginal returns. In the next section, we will see that the evolution of the blueprint ratio depends likewise on $\sigma \gtrsim 1$. More specifically, an expected increase in the employment ratio of skilled and unskilled labor $\frac{L_t^s}{L_t^u}$ will increase the profitability of skilled-labor complementary innovations by means of the market size effect and will bias technological progress towards the $s-$sector, whenever $\sigma > 1$.

We now turn to the employment structure in equilibrium. Skilled labor is employed in production and education. Each kind of labor is allowed to move vertically within the $s$ and the $u$-sector. By doing so, we take into account differing skill intensities and a low mobility of labor between the skilled and unskilled labor intensive sector. Obviously, full employment of labor requires

$$ L_t^s + L_t^x + L_t^E^s = L_t^s \quad \text{and} \quad L_t^u + L_t^x = L_t^u. $$

Therewith, competitive labor markets generate an employment structure as follows\(^{14}\)

$$ L_t^s = \frac{\beta}{\xi^L} (L_t^s - L_t^E^s), \quad L_t^x = \frac{\xi^L}{\xi^E^s}(L_t^s - L_t^E^s), $$

$$ L_t^u = \frac{\beta}{\xi^L} L_t^u, \quad L_t^x = \frac{\xi^L}{\xi^L} L_t^u, $$

\(^{14}\)For details see Appendix B.3.
with $\xi^{tx} = (1 - \alpha)(1 - \beta)$ and $\xi^L = \beta + (1 - \alpha)(1 - \beta)$.

Further, demand for labor in the education sector is determined by the total number of children sent to school and the structural parameter $\phi$ - the student-teacher ratio, such that

$$L^E_t = \phi(n^{s,s} L^s_t + \theta L^u_t n^{u,s} L^u_t).$$

Consequently, employment ratios in production are specified as follows

$$\frac{L^Y_s}{L^x_u} = \frac{L^x_s}{L^x_u} = (1 - \phi n^{s,s}) \frac{L^s_t}{L^u_t} - \phi \theta L^u_t n^{u,s}.$$  

Natural resources are mobile within the machine producing sectors and an efficient use of natural resources requires that resource demand of the machine producing sectors equals the overall amount of natural resources extracted. Hence,

$$R_t^{xu} + R_t^{xs} = R_t.$$  

Moreover, perfect competition on the market for natural resources results in\(^{15}\)

$$R_t^{xu} = \frac{\phi \gamma^u}{1 + \frac{\phi \gamma^u}{\gamma} \left( \frac{N^u}{N^x} \right)^{\frac{1-\alpha}{\sigma}} \left( \frac{L^x_s}{L^x_u} \right)^{\frac{1-\alpha}{\sigma}} \left( \frac{c^u_x}{c^x_u} \right)^{\frac{(\sigma-1)(1-\beta)}{\sigma \beta}} R_t} \quad (46)$$

$$R_t^{xs} = \frac{\phi \gamma^s}{1 + \frac{\phi \gamma^s}{\gamma} \left( \frac{N^s}{N^x} \right)^{\frac{1-\alpha}{\sigma}} \left( \frac{L^x_s}{L^x_u} \right)^{\frac{1-\alpha}{\sigma}} \left( \frac{c^u_x}{c^x_u} \right)^{\frac{(\sigma-1)(1-\beta)}{\sigma \beta}} R_t} \quad (47)$$

If $\sigma = 1$, which necessarily implies that $\varepsilon = 1$, then the shares of extracted natural resources, $\phi^{xu}$ and $\phi^{xs}$, depend only on the relative importance of intermediates $Y^s_t$ and $Y^u_t$ for final good production $Y_t$, i.e. $\phi^{xu} = \gamma$ and $\phi^{xs} = 1 - \gamma$. If $\sigma > 1$ the market size for machines plays a crucial role, i.e. relative profitability and demand for machines increases in accordance with that type of labor which is complementary to the respective type of machine. In conclusion, demand shares for natural resources depend not only on $\gamma$, but also on the blueprint ratio, $\frac{N^s}{N^x}$, the employment ratio in intermediate production, $\frac{L^Y_s}{L^x_u}$, and the marginal production cost ratio,

$$\frac{c^u_x}{c^x_u} = A \frac{N^s}{N^x} \frac{Y_t}{\omega_t^{\alpha-1}}.$$  

Equation (48) states a positive link between the skilled wage premium and relative marginal production costs of skilled labor complementary innovations. In light of (48), (47), (46) and (39), we obtain (for details see Appendix A.1.):

\(^{15}\)For details see Appendix B.4.
Proposition 2

An increase in the skilled-unskilled employment ratio in intermediate production, \( \frac{L^s_t}{L^u_t} \), and/or an increase in the blueprint ratio, \( \frac{N^s_t}{N^u_t} \), raise machine demand and production in the s-sector relative to the u-sector. Therefore, the share of extracted natural resources allocated to the u-sector, \( \varphi^x_{t} \), must decline and \( \varphi^y_{t} \) must increase, given that \( \sigma > 1 \).

Having established the equilibria on the markets for intermediate goods and machines, the labor market and the market for natural resources, we now turn to the capital market and the dynamics of resource extraction, i.e. the depletion rate of natural resources.

4.3 Man-made Capital and Depletion of Natural Resources

Free entry in R&D drives profits down to zero in both R&D-sectors. The value of each blueprint equals the discounted profit stream (i.e. \( \frac{\pi^x_t}{1 + r_{t+1}} \)) generated by patent owners, that is, machine producers. On the other hand, Eq. (28) implies that the marginal productivity of one unit of final output allocated to R&D equals \( \eta^i(N^i_t)^{-\delta} \). As \( p^Y_t \) has been normalized to 1, in equilibrium this yields

\[
\frac{\pi^x_t}{1 + r_{t+1}} \eta^i(N^i_t)^{-\delta} = 1. \tag{49}
\]

Consequently, in equilibrium the following non-arbitrage condition must hold

\[
\pi^x_{t+1} \eta^s(N^s_t)^{-\delta} = \pi^x_{t+1} \eta^u(N^u_t)^{-\delta}, \tag{50}
\]

and therefore

\[
\frac{\pi^x_{t+1} \eta^s(N^s_t)^{-\delta}}{\pi^x_{t+1} \eta^u(N^u_t)^{-\delta}} = \frac{\eta^u}{\eta^s} \left( \frac{N^u_t}{N^s_t} \right)^{-\delta}. \tag{51}
\]

This technology market clearing condition can be solved for the blueprint ratio in the subsequent period by making use of (34), (35) and (38)\(^16\)

\[
\frac{N^s_{t+1}}{N^u_{t+1}} = \left[ \left( \frac{\eta^s}{\eta^u} \right) \left( \frac{N^s_t}{N^u_t} \right)^{\beta + \xi_R} \right] \left( \frac{\beta + (\sigma - 1) \xi_L}{\beta - \xi_R} \right) \left( \frac{1 - \gamma}{\gamma} \right)^{\frac{1}{\beta - \xi_R}} \left( \frac{\sigma - 1}{\sigma} \right)^{\frac{1}{\beta - \xi_R}} \left( \frac{A}{B} \right)^{\frac{1}{\beta - \xi_R}}, \tag{52}
\]

with \( \xi_R = \alpha(1 - \beta) \) and \( \beta - \xi_R(\sigma - 1) > 0 \) within the range of plausible parameter values.\(^17\)

Since intermediates \( Y^s_t \) and \( Y^u_t \) are gross substitutes, meaning that \( \sigma > 1 \), the blueprint ratio in

\(^16\)For details see Appendix B.5.

\(^17\)Note that: \( \beta \approx 0.65 \) and \( \xi_R \) around 0.04 - see Section 5.2 for more details, such that \( \sigma > 17.25 \) would be necessary, in order to violate \( \beta - \xi_R(\sigma - 1) > 0 \).
$t + 1$ will be biased towards skilled labor complementary innovations if it is expected that the skilled-unskilled labor ratio, $\frac{L^s_{t+1}}{L^u_{t+1}}$, increases in the intermediate sector. An increase in $\frac{1}{\rho}$ enhances the efficiency of labor and natural resources in the machine production of sector $u$, such that marginal production costs, $\frac{C_u^{w}}{C_u^{f}}$, increase. If $\delta > 0$, the number of existing blueprints dampens the speed of innovations in the future because it is comparatively difficult to innovate skill-complementary machines for the next period. For $\delta < 0$, in contrast, the existing level of blueprints speeds up innovations due to positive intertemporal knowledge spillovers.

Outlays for R&D are financed by aggregate savings $S_t$ which are not invested in ownership of natural resources implying that the amount of aggregate savings can be written as\textsuperscript{18}

$$S_t = p_t^Y Y_t^u \left\{ \frac{\rho}{1+\gamma + \rho} \xi L \left[ 1 + \omega_t \frac{L^s_t}{L^u_t} \right] - \frac{\xi R}{\varphi^u_t} \frac{1 - \tau_t}{\tau_t} \right\}. \quad (53)$$

Clearly, a prerequisite not only for sustained economic growth, but also for a non-trivial interior solution of the model is, $S_t > 0$ implied by $Z_{1t} > 0$, i.e. $\frac{\rho}{1+\gamma + \rho} \xi L \left[ 1 + \omega_t \frac{L^s_t}{L^u_t} \right] > \frac{\xi R}{\varphi^u_t} \frac{1 - \tau_t}{\tau_t}$. Therefore, it is not sufficient that the labor income share exceeds the natural resource income share, i.e. $\xi^L > \xi^R$. A necessary condition for $S_t > 0$ and sustained economic growth therefore is a sufficiently high expenditure share for old-age consumption $\frac{\rho}{1+\gamma + \rho}$. Everything else being constant, the size of the wage differential and the skilled-unskilled population ratio can also have a positive impact on aggregate savings. As we will see further below, however, the size of $Z_{1t}$ governs not only current saving, but also the dynamics of resource extraction. Thus, the interaction between the two terms contained in $Z_{1t}$ is also crucial for the depletion of natural resources during the transition and in the long-run. We come back to this point further below.

Free entry in R&D and perfect capital markets imply equality between future aggregate profits and revenues of aggregate savings allocated to the R&D-sector $u$ or $s$, such that $N_{t+1}^{\delta} \pi_{t+1}^{\delta} = (1 + r_{t+1})D_t^\delta$. Thus, we obtain from the technology market clearing condition (51), given that $D_t^s + D_t^u = S_t$:

$$D_t^u = \frac{1}{1 + \Omega_t} S_t \quad \text{and} \quad D_t^s = \frac{\Omega_t}{1 + \Omega_t} S_t, \quad (54)$$

with $\Omega_t = \frac{N_{t+1}^{\delta} \eta^u}{N_{t+1}^{\delta} \eta^s} \left( \frac{N_t^s}{N_t^u} \right)^\delta$ and $\frac{N_{t+1}^{\delta}}{N_{t+1}^{\delta}}$ given by (52).

Bearing in mind that the price per unit of natural resources equals its value marginal product in intermediate production, the evolution of the natural resource price, $g_{t+1}^R$, is given by the change in the value marginal product of natural resources.\textsuperscript{19} On the other hand, in equilibrium, the change of this factor price is tied to the interest factor by Hotelling’s rule $(1 + r_{t+1} = g_{t+1}^R)$, where the technology market clearing condition $(1 + r_{t+1} = \pi_{t+1}^i \eta^i (N_t^i)^{1-\delta}$, with $i = u, s$) links the

\textsuperscript{18}For details see Appendix B.6.

\textsuperscript{19}We denote the gross growth rate of $x$, $\frac{x_{t+1}}{x_t}$, by $g_{t+1}^R$. 

16
interest rate to the profitability of future innovations. Thus, the dynamics of the depletion rate of natural resources $\tau$ is reduced to the following difference equation\footnote{For details see Appendix B.7(a).}

$$ \tau_{t+1} = \frac{Z_{1t}}{Z_{2t}} \frac{\tau_t}{1 - \tau_t} \left( g_{t+1}^{R^u} \right)^{-1}, \quad (55) $$

with $Z_{1t} = \frac{\rho}{1 + \nu + \rho} L_t \left( 1 + \omega_t \frac{L_t^s}{\nu_t} \right) - \frac{\xi}{\nu_t} \frac{1 - \eta_t}{\eta_t}, \quad Z_{2t} = \beta(1 - \beta)(1 + \Omega_t)$.

As we remarked before, a necessary condition for aggregate savings being positive is $Z_{1t} > 0$. Otherwise, savings, demand for machines and the depletion rate would jump to zero implying zero output in the machine and the intermediate goods producing sectors. It is precisely the behavior of $\frac{Z_{1t}}{Z_{2t}}$ which steers natural resource depletion during the transition and in the long-run. During the transition, an increase in aggregate savings caused by an increase in $Z_{1t}$ as compared to $Z_{2t}$ - which contains the aggregate relative profits of future innovations (see (54)) - increases the depletion rate of natural resources. In this sense, higher aggregate savings in the current period caused for example by an increase in $Z_{1t}$, increases future resource depletion via a higher demand for machines. We will now clarify the mechanism behind difference equation (55) by starting from a different point of origin. In equilibrium, demand for machines in sector $i = u, s$ has to match its supply\footnote{For details see Appendix B.7(b).}

$$ X_t^i = N_t^i x_t^i = N_t^i \left( \frac{g_t^{Y^i}}{c_t^i} \right)^{\frac{1}{\beta}} L_t^i = N_t^i G \left( L_t^{xi} \right)^{1-\alpha} \left( \varphi_t^i R_t \right)^{\alpha}, \quad (56) $$

with $R_t = \tau_t M_{t-1}$ and $G = A \lor B$, if $i = s \lor u$.

Accordingly, depletion of natural resources must increase whenever demand for machines in one or both sectors augments in such a way that factor reallocation (changes in $\varphi_t^i$) alone is not sufficient, and a higher extraction of natural resources becomes necessary. In light of (56), changes in demand for and the output of machines in sector $u$ are given by

$$ \left( \frac{g_t^{Y^u}}{g_t^{R^u}} \right)^{\frac{1}{\beta}} g_{t+1}^u = \left( g_{t+1}^u \right)^{1-\alpha} \left( g_{t+1}^{\varphi^u} R_{t+1} \right)^{\alpha}, \quad (57) $$

such that, after some manipulations,

$$ \frac{Z_{1t}}{Z_{2t}} \left( g_{t+1}^{R^u} \right)^{-1} = g_{t+1}^R. \quad (58) $$

As $g_{t+1}^R = \frac{\tau_{t+1}}{\tau_t} (1 - \tau_t)$, the last equation equals exactly (55).

In sum, the evolution of the economy is governed by a four-dimensional system of difference equations containing the laws of motion for the population ratio, $L_s^s$, the blueprint ratio, $N_s^s$, the
depletion rate, $\tau$, and the fraction of unskilled households raising skilled offspring, $\theta$:

$$\frac{L^s_{t+1}}{L^u_{t+1}} = \frac{\theta_t n^{u,s}_t + n^{s,s} L^s_t}{n^{u,u}_t},$$

(59)

$$\frac{N^s_{t+1}}{N^u_{t+1}} = \left[\left(\frac{\eta^s}{\eta^u}\right) \left(\frac{N^s_t}{N^u_t}\right) \frac{c + (\sigma - 1) L^s_t}{\beta - \xi^u(\sigma - 1)}\right]^{\frac{1}{\beta - \xi^u(\sigma - 1)}},$$

(60)

$$\tau_{t+1} = \frac{Z_{1t}}{Z_{2t}} \frac{\tau_t}{1 - \tau_t} \left(\frac{\phi}{\gamma}\right)^{-1},$$

(61)

with $\frac{L^s_{t+1}}{L^u_{t+1}} = (1 - \phi n^{s,s}) \frac{L^s_{t+1}}{L^u_{t+1}} - \phi \theta_{t+1} n^{u,s}_{t+1}$ and indifference condition (19)

$$\omega_{t+1} = \frac{Z_{1t}}{Z_{2t}} \frac{\tau_t}{1 - \tau_t} \left(\frac{\phi}{\gamma}\right)^{-1},$$

(62)

which determines $\theta_{t+1}$ implicitly via (39).

### 4.4 Balanced Growth Properties

Imposing steady state conditions, and denoting by subscript ‘$^*$’ stationary values, the long-run equilibrium is characterized by (for details see Appendix A.2):

**Proposition 3**

Along the unique balanced growth path, $(\frac{L^s}{L^u})^*$, $(\frac{N^s}{N^u})^*$, $\theta^*$ and $\tau^*$ are constant, such that $\omega^*$, $p^*_s$, $p^*_u$ are constant as well.

(i) From (59), the skilled-unskilled population ratio is identified as

$$\left(\frac{L^s}{L^u}\right)^* = \frac{\theta^* n^{u,s}_*}{(1 - \theta^*) n^{u,u} - n^{s,s}_*},$$

(63)

Employment ratios in production are therefore given by (see (44)): $\left(\frac{L^s}{L^u}\right)^* = \left(\frac{L^s}{L^u}\right)_* = (1 - \phi n^{s,s}) \left(\frac{L^s}{L^u}\right)_* - \phi \theta^* n^{u,s}_*.$

(ii) In steady state, the share of unskilled households educating skilled offspring $\theta^*$ is determined by

$$\omega^* = \frac{Z_{1t}}{Z_{2t}} \frac{\tau_t}{1 - \tau_t} \left(\frac{\phi}{\gamma}\right)^{-1},$$

(64)

where $\omega^* = \left[\left(\frac{\theta^*}{\eta^u}\right)^{(\sigma - 1)} \left(\frac{1 - \gamma}{\gamma}\right) \left(\frac{L^s}{L^u}\right)^* \left(\frac{\phi}{\beta}\right)^{(\sigma - 1)(1 - \beta)(1 + \delta)}\right]^{\frac{1}{\beta}}$. 

18
(iii) The blueprint ratio and the depletion rate satisfy:

\[
\left( \frac{N^s}{N^u} \right)_* = \left[ \left( \frac{\eta^s}{\eta^u} \right)^{\beta+(\sigma-1)\xi^L} \left( \frac{1-\gamma}{\gamma} \right)^{\varepsilon_\beta} \left( \frac{LY^s}{LY^u} \right)^{(\sigma-1)\xi^L} \left( \frac{A}{B} \right)^{(\sigma-1)(1-\beta)} \right]^{\frac{1}{\psi}},
\]

where \( \psi = \beta - \xi^R(\sigma - 1) + \delta(\beta + (\sigma - 1)\xi^L) > 0 \) in the range of plausible parameters.

(iv) Skilled and unskilled-labor complementary innovations evolve along the balanced growth path in compliance with

\[
g_* = g^N_* = \left[ n^*_u (1 - \tau^*_s) \right]^{\frac{1}{\psi}}, \quad i = u, s,
\]

where \( n_* = (1 - \theta_*)n^u_* = \theta_* \left( \frac{L^u}{L^s} \right) n^u_* + n^{s,*} \) represents the average number of children in steady state.

In light of Item (iv) of Proposition 3, it follows that the long-run growth rate of the population has a positive impact on the growth rate of innovations as long as \( \delta > 0 \). Whenever, long-run population growth stabilizes below or at the reproduction level, \( n_* = 1 \), the long-run growth rate of productivity \( (g_* - 1) \) becomes negative, given \( \delta > 0 \). In this sense, skill-biased technological change would dig its own grave via the negative feedback effect induced by the decline in fertility because \( \frac{\partial n_*}{\partial \theta_*} < 0 \). If one considers positive population growth as not feasible due to space restrictions and/or making allowance for United Nations’ long-run population projections that predict a stationary world population after the year 2150, the steady state exhibits no population growth, such that \( n_* = 1 \) should be a reasonable prediction for the future. Under these circumstances the maximum productivity growth rate would be 0 (see (67)) for \( \tau_* = 0 \) and \( \delta > 0 \). Hence for any interior solution \( 0 < \tau_* < 1 \), the economy would exhibit negative productivity growth and a negative growth rate of the wage rate in both sectors (see Corollary 1 below). In other words, a stationary population confronted with non-renewable natural resources is not able to generate sustained economic growth in our setting, when the productivity of R&D is negatively related to the level of technology \( (\delta > 0) \). A prerequisite for long-run growth in the face of stationary or shrinking populations therefore includes positive external returns in R&D with respect to the existing stock of blueprints, i.e. \( \delta < 0 \) in our specification.

The following proposition characterizes the behavior of the share of unskilled households raising skilled offspring in the long-run, \( \theta_* \), and the depletion rate of natural resources, \( \tau_* \), with respect to variations in the teacher-student ratio \( \phi \) and the productivity ratio in R&D, \( \frac{\eta^s}{\eta^u} \) (for details see Appendix A.3):
Proposition 4
Along the balanced growth path:

(i) The skilled wage premium is specified as \( \omega_s = \frac{z}{s - \phi} \), such that \( \frac{\partial \omega_s}{\partial \phi} > 0 \) and \( \frac{\partial \omega_s}{\partial n_s} < 0 \).

(ii) The reaction of the long-run depletion rate \( \tau_s \) w.r.t. changes in \( \phi \) and \( \frac{n^*}{\eta^*} \) is ambiguous, i.e. \( \frac{\partial \tau_s}{\partial \phi}, \frac{\partial \tau_s}{\partial n_s} \gtrless 0 \).

Item (i) of Proposition 4 follows from Item (ii) of Proposition 3. An increase in \( \phi \) raises the right-hand side of (19), such that the skilled-unskilled population ratio must increase which requires an increase in the share of unskilled households raising skilled offspring \( \theta_s \), hence \( \frac{\partial \theta_s}{\partial \phi} > 0 \). Conversely, an increase in \( \frac{n^*}{\eta^*} \) raises the left-hand side of (19) where the right-hand side remains constant. Thus, the skilled-unskilled population ratio must decline in the long-run, such that \( \theta_s \) declines and \( \frac{\partial \theta_s}{\partial n_s} < 0 \). The sign of \( \frac{\partial \tau_s}{\partial \phi} \) and \( \frac{\partial \tau_s}{\partial n_s} \) in turn is ambiguous (Item (ii) of Proposition 4). Whether \( \frac{\partial \tau_s}{\partial \phi} \) is positive or negative depends essentially on \( \frac{\partial \tau_s}{\partial \phi} \leq \frac{\partial \tau_s}{\partial n_s} \) (for details see Appendix A.3). Intuitively, whenever \( \frac{\partial \tau_s}{\partial \phi} > \frac{\partial \tau_s}{\partial n_s} \), the reaction of relative profitability of skill-complementary innovations with respect to variations in \( \phi \) is larger than the reaction of aggregate savings, such that the depletion rate of natural resources must increase, i.e. \( \frac{\partial \tau_s}{\partial \phi} > 0 \) and vice versa. An analogous argument applies for \( \frac{\partial \tau_s}{\partial n_s} \). Since an increase in \( \phi \) or a reduction in \( \frac{n^*}{\eta^*} \) cause an increase in \( \theta_s \), the average growth rate of the population \( n_s \) must decline. The latter in turn causes a negative impact on the long-run growth rate of productivity, if \( \delta > 0 \), and is conducive to economic growth whenever positive spillover effects exist with respect to existing technological knowledge, i.e. \( \delta < 0 \).

The levels of per-capita consumption \( c_t^i \) and \( c_{t+1}^i \) depend on wage incomes \( w_t^i, i = u, s \), such that the evolution of per-capita consumption levels is tied to the evolution of wages.

Corollary 1
In light of Proposition 3 we obtain for the evolution of wages, \( g_s^w \), with \( i = u, s \), in steady state (for details see Appendix A.4):

\[
g_s^w = \left[ \frac{g_s^{N^t} (s - \phi)}{g_s^{R^t}} \right] \left[ \frac{\xi^L}{\xi^R} \right] = \left[ \frac{\xi^L}{n_s^*(1 - \tau_s^*)} \right] \left[ \frac{1 - \tau_s^*}{n_s^*} \right] \xi^R.
\]

(68)

The economy moves along a sustainable growth path, whenever \( g_s^w \geq 1 \), which implies at least non-declining per-capita consumption levels in the long-run. From (68), sustainable development in its weakest form requires that productivity growth must at least compensate for the increase in the natural resource price caused by an increasing shortage of natural resources.

\footnote{The impact of the change in \( \tau_s \) on \( g_s^w \) with respect to variations in \( \phi \) or \( \frac{n^*}{\eta^*} \) may work in the opposite direction however. Since reasonable parameter ranges imply \( \xi^L >> \xi^R \), the change in \( n_s^* \) always dominates the change in \( \tau_s \), such that we can expect under realistic parameter constellations \( \frac{\partial \tau_s}{\partial \phi} \gtrless 0 \) and \( \frac{\partial \tau_s}{\partial n_s} \gtrless 0 \) for \( \delta \leq 0 \).}
5 Numerics

5.1 Preliminary Remarks

We now turn to three numerical experiments in order to illustrate the behavior of the economy under consideration with respect to: an exogenous increase in the skilled-unskilled population ratio $\frac{L_s}{L_u}$, a decline in relative research costs $\frac{\eta_s}{\eta_u}$ and an increase in schooling quality $\phi$. The first experiment generates only transitory effects since the steady state remains unaffected by this shock. The second experiment shifts the long-run values of all endogenous variables except the skilled wage premium which corresponds to $\frac{\omega}{z-\phi}$ in the long-run. Therefore a non-monotone transition of $\omega$ in the first and the second experiment is the result. The third experiment obviously also alters the long-run level of the skilled wage premium.

5.2 Calibration

Since one period encompasses approximately thirty years, we choose for the discount factor of future consumption, $\rho$, a value that is standard in real-business-cycle literature: 0.99 per quarter, i.e. $\rho = 0.99^{1/40}$ in our context. The parameter $\beta$ represents the labor share in intermediate goods production and is set to 0.65. In the U.S., energy expenditures as a share of GDP amounted to 8.8% in 2006 with a maximum close to 14% at the beginning of the eighties (see Energy Information Administration, 2009). Hence 8.8% constitutes an upper limit for the resource income share of non-renewables in our model. We therefore set $\alpha = 0.08$, which implies $\xi_R = \alpha(1-\beta) = 0.028$ in each intermediate sector. The parameter $\phi$ reflects the teacher-student ratio and is set to 1/20. Moreover, child-rearing is subject to forgone consumption possibilities and losses in potential lifetime earnings which amount to 13% for highly educated women and higher if women drop out of the labor market completely (Dankmeyer, 1996). The direct time cost for parents raising a child to adulthood amounts to 50% of parents’ time endowment (see de la Croix and Doepke, 2003), which would imply $z = 0.075$ as a lower limit. Taking losses in lifetime earnings into account, we set $z = 0.15$, which implies a skilled wage premium $\omega_\ast = 1.5$ which matches U.S. data (Acemoglu, 2002). The weight $\nu$ of children in the utility function drives the growth rate of the population. We choose a value of $\nu = 0.26$, which generates approximately zero population growth along the balanced growth path ($-0.0007 \approx 0$ per year). Now there are five parameters left: the elasticity of substitution between intermediate goods in final good production, $\varepsilon$, the weight of $Y_t^u$ in final good production, $\gamma$, the ratio of the productivity parameters in R&D, $\eta = \frac{\eta_s}{\eta_u}$, the ratio of productivity parameters in machine production $\frac{A}{B}$, and the externality parameter in R&D, $\delta$. We calibrate these parameters so that they match a long-run productivity growth rate of 2.4%, an investment share $I_t/Y_t$ in the vicinity of 14.43%, fitting the 10 year average of US
private fixed capital formation as a share of GDP\textsuperscript{23} (OECD Economic Outlook Database), an employment share in education of around 2\%, and a long-run decline in the natural resource stock of 2.4\% per year, which causes, via Hotelling’s rule a long-run interest rate close to 4\%.

The remaining parameters are therefore fixed as follows: \( \varepsilon = 2.4, \gamma = 0.655, \frac{\alpha_s}{\eta_s} = 2.2, \delta = -0.045, \) and \( \frac{A}{B} = 8.\textsuperscript{24} \)

**Figure 2 about here**

### 5.3 Comparative Statics and Equilibrium Dynamics

The behavior of the depletion rate with respect to changes in \( \phi \) and \( \frac{\nu^s}{\eta^s} \) is presented in Figure 2. Indeed, \( \tau^*_t \) follows a u-shaped relation with respect to changes in \( \phi \), where the increasing arm is not relevant since the implied \( \theta^*_t \) is close to 1 for \( \phi > 0.14 \) and a realistic range of \( \phi \) falls within 0.045 and 0.055, such that \( \frac{\partial \tau^*_t}{\partial \phi} < 0 \). Similarly, in order to match current data \( \frac{\nu^s}{\eta^s} \) should fall between 2 and 2.5, such that \( \frac{\partial \tau^*_t}{\partial \frac{\nu^s}{\eta^s}} > 0 \).

**Experiment 1:** Increase in the skilled-unskilled population ratio \( \frac{L^s}{L^u} \)

In our first experiment, we analyze transitory effects of an exogenous increase in the skilled-unskilled population ratio with respect to the interaction between skill-biased innovations, population dynamics and the depletion of natural resources.\textsuperscript{25} In our experiment, the unanticipated shock occurs in period \( t = 0 \), while our point of reference is the steady state determined by the parameter constellation mentioned above. The results are presented in Figure 3.

Given the blueprint ratio \( \frac{N^s}{N^u} \) in the current period, an increase in the skilled-unskilled population ratio, \( \frac{L^s}{L^u} \), leads to a decline in productivity of skilled labor relative to unskilled labor. Therefore, the current wage ratio \( \omega_t \) is reduced such that it is more affordable for unskilled households to raise skilled offspring, i.e. \( n^u,s_t \) increases (see Eq. 17), provided that \( \theta_t > 0 \). Recall now, that the skilled-unskilled population ratio also adjusts by means of changes in the fraction of unskilled households raising skilled offspring, \( \theta_t \).

\textsuperscript{23}Note that: \[
\frac{L}{L_t^s} = \frac{\left[ \frac{\gamma + (1-\gamma)\alpha(p_t)^{1-\varepsilon}}{\gamma + (1-\gamma)(Y^s_t/Y^u_t)^{\varepsilon}} \right]^{\varepsilon-1}_{\varepsilon} \gamma^* Z_{1t}, \]
for details see Appendix B.9.

\textsuperscript{24}The numeric method is described in Appendix B.10. Given the parameter constellation the dynamic system exhibits two positive eigenvalues within the unit circle and two outside the unit circle, such that the economy is subject to saddle-point stability. This constellation is robust for a very large range of parameters, except for \( \varepsilon < 2 \). Note also that the transitory behavior is independent from \( \delta \).

\textsuperscript{25}Evoked by the Vietnam war in the 1960s, the US experienced an exogenous increase in the skilled-unskilled labor ratio, see Acemoglu (1998,2002).
As the skilled-unskilled population ratio is above its long-run value, $\theta_t$ must adjust in compliance with (19) from below to its long-run value which ensures a decline of $\frac{L_t^s}{L_t^u}$. The growth rate of the unskilled labor force, $g_{L_t^u}$, is now above its long-run value and the growth rate of the skilled labor force, $g_{L_t^s}$, is below it. Since we know, moreover, that: $n_t^{u,u} > n_t^{s,s} > n_t^{u,s}$, the growth rate of the population, $n_t$, must be above its long-run value as well and population growth is essentially driven by an expansion of the unskilled labor force. The drop in the number of children sent to school caused by the decline of $\theta_t$, lowers the fraction of skilled labor, $\frac{L_t^s}{L_t}$, allocated to the education sector. The latter and the initial increase in the skilled-unskilled population ratio induce an increase in skilled-unskilled employment ratios in both production sectors which again is compatible with the observed decline in $\omega_t$. Since the market size for skilled-labor complementary machines has increased, given the current blueprint ratio $\frac{N_t^s}{N_t^u}$, relative aggregate demand for skilled-labor complementary machines $\frac{X_t^s}{X_t^u}$ increases, such that the share of extracted natural resources allocated to the s-sector, $\phi_t^{xs}$, must increase. The increased relative market size for skilled labor complementary machines reflects also higher employment of labor in the s-sector such that the depletion rate of natural resources $\tau_t$ is reduced. Hence, the current working cohort invests more in natural resources. At the same time, the investment share of GDP, $\frac{I_t}{Y_t}$, declines.

In the subsequent period, the skilled-unskilled population ratio shrinks according to the fertility decisions of the previous period ($\theta_t$ declined in the previous period). The new level of the skilled-unskilled population ratio is still above its long-run value and has been anticipated by the allocation of savings to both R&D-sectors in the previous period, i.e. $\frac{D_t^s}{D_t^u}$ increased in the previous period. Therefore, technological progress is biased towards skilled-labor complementary innovations, such that $\frac{N_t^s}{N_t^u}$ is now above its long-run value as well (see Eq. (52)). Accordingly, the skilled-wage premium increases and overshoots its long-run level, which makes education for unskilled households less affordable, such that $n_t^{u,s}$ decreases to below its long-run value. At the same time, more unskilled households wish to train their offspring to skilled workers. Consequently, $\theta_t$ rises in compliance with indifference condition (19), which results in a higher demand for skilled labor in the education sector. As the skilled-unskilled population ratio shrinks and population growth is mainly created by an expansion of the unskilled population group, the increase in $\theta_t$ leads to an increase in demand for skilled labor in the education sector relative to the overall working force, i.e. $\frac{L_t^s}{L_t}$ increases. Again, with regard to adjustments in natural resource use in production, we have to consider two effects: First, the adjustment in the share of extracted natural resources allocated to the s- or u-sector, $\phi_t^i$, $i = u, s$, and second, the adjustment in the extraction rate of natural resources, i.e. $\tau_t$. The aforementioned decline in the skilled-unskilled population ratio as compared to the previous period accompanied by an increase in $\theta$ and $\frac{L_t^s}{L_t}$

\[ 26 \text{Note that } \frac{L_t^s}{L_t} = \frac{L_t^u}{L_t} \text{ due to symmetry assumptions (see also Eqs. (41) and (42)).} \]
during the transition leads to a drop in skilled-labor employment ratios in production. Thus, it appears that the skill bias of innovations is declining and the share of natural resources allocated to the s- (u-) sector is diminishing (growing). The u—sector in turn, is subject to declining population growth as θ increases towards its long-run value. Consequently, the depletion rate τ increases in order to match aggregate machine demand and follows the movement of θ. As τ increases, households shift their investments from natural capital to the capital market, such that the investment share of GDP is increasing during the transition. The increase in θ, in turn, is responsible for a decline in population growth nt. Hence, the model suggests an inverse relationship between τ and nt after an increase in \( \frac{L_s}{L_t} \).

**Experiment 2:** Reduction in the relative research productivity \( \frac{n_s}{n_u} \)

In our second experiment we reduce the relative research productivity/research costs, \( \frac{n_s}{n_u} \), by 1%, where the unanticipated shock occurs again in period \( t = 0 \). As a point of reference, we use again the steady state mentioned above. The results are presented in Figure 4.

(a) Long-run

Indifference condition (19) implies that the long-run wage differential is determined by \( \omega = \frac{x}{z-\phi} \) which remains unaffected by a change in \( \frac{n_s}{n_u} \). Hence, we only observe transitory changes in \( \omega_t \) caused by adjustments of \( \frac{L_s}{L_t} \) and \( \frac{N_s}{N_t} \) to their new long-run values. Since relative research costs \( \frac{n_s}{n_u} \) are reduced, the long-run skilled-unskilled population ratio, \( \left( \frac{L_s}{L_u} \right)_s \), must increase in order to match (19), supported by a rise in skill-biased technological change, \( \left( \frac{N_s}{N_u} \right)_s \), that comes along with an increase in relative R&D expenditures \( \left( \frac{D_s}{D_u} \right)_s \), relative aggregate machine demand \( \left( \frac{X_s}{X_u} \right)_s \) and the share of extracted natural resources \( \varphi_s^{x_s} \), allocated to the s-sector. An increase in \( \left( \frac{L_s}{L_u} \right)_s \) evoked by a higher fraction of unskilled households raising skilled offspring represented by \( \theta_s^* \), induces a decline in the long-run growth rate of the population \( n_s \). In light of Proposition 4, the increase in \( \theta_s^* \) is associated with a decline in the depletion rate of natural resources \( \tau_s \), mirrored by a higher investment share of GDP, \( \left( \frac{I}{Y} \right)_s \), and productivity growth increased to 2.64%.

(b) Transition

A reduction in \( \frac{n_s}{n_u} \) induces, via technology-market clearing condition (51), an increase in the relative profitability of skilled-labor complementary innovations, \( \frac{\pi_{x,s}^{e+1}}{\pi_{x,u}^{e+1}} \). This effect will be supported by an increase in the skilled-unskilled population ratio which causes an increase in next period’s blueprint ratio.
As the skilled unskilled population ratio is below its new long-run value after the shock, the share of unskilled households raising skilled offspring, $\theta_t$, must adjust - in compliance with (19) - from above to its new long-run value such that a smooth convergence of the skilled-unskilled population ratio is ensured. Accordingly, the growth rate of the skilled population group must be above its long-run value and the growth rate of the unskilled population group below it. Moreover, given differential fertility of the kind that: $n^{u,u}_n > n^{s,s}_n > n^{u,s}_t$, overall population growth $n_t$ must be below its new and old long-run value. The initial increase in $\theta_t$ after the shock advances the demand for skilled labor in the education sector relative to the total working force with the consequence of declining employment ratios of skilled labor in production. The skilled wage premium $\omega_t$ is therefore above its long-run value. Moreover, relative demand for skilled labor complementary machines is reduced causing a lower fraction of natural resources to be allocated to the $s$-sector.

Given the amount of unskilled labor in the current period, the increase in aggregate demand for machines produced in the $u$-sector creates an increase in demand for natural resources, such that $\tau_t$ rises. In the subsequent periods, the skilled-unskilled population ratio and the blueprint ratio increase. In the following periods, the monotone decline of $\theta_t$ generates a smooth convergence of the population ratio to its long-run value, which is responsible for skill-biased technical change mirrored by an increase in the blueprint ratio. Consequently, the wage ratio converges from below its long-run value to its actual long-run value. The decline in $\theta_t$ lowers the employment ratio of skilled labor, $\frac{L_{Es}}{L_t}$, in education. Since technological change and natural resources are directed towards that sector, which benefits from higher employment ratios, the decline in $\theta_t$ allows for a reduction of the depletion rate where the growth rate of the population is increasing.

Experiment 3: Increase in the teacher-student ratio $\phi$

In our third numerical exercise (see Figure 5), we increase the teacher-student ratio $\phi$ by 1% which may reflect an increase in the quality of schools.

(a) Long-run

Indifference condition (19) implies that the long-run wage differential adjusts to $\omega_* = \frac{\omega}{1-\phi}$. After an increase in $\phi$, it is more beneficial for the unskilled population group to raise skilled offspring.

---

27 Initially, the blueprint ratio of the subsequent period may fall as in the case presented here, i.e. the decline in $\frac{\eta^t}{\eta^{t+1}}$ overcompensates the increase in $\frac{\eta^{t+1}}{n^{t+1}_{s}}$, see also Eq.(52).
Consequently, $\theta_s$ increases. With a higher fraction of unskilled households raising skilled offspring, the long-run skilled-unskilled population ratio $\left( \frac{L_s}{L_u} \right)_s$ must increase as well. Consequently, relative demand for skilled-labor complementary machines $\left( \frac{X_s}{X_u} \right)_s$ increases which generates an increase in the ratio of R&D expenditures $\left( \frac{D_s}{D_u} \right)_s$ and the blueprint ratio $\left( \frac{N_s}{N_u} \right)_s$. Therefore, the share of extracted natural resources allocated to the $s$-sector, $\varphi_s^{x,s}$, must increase as well. With a higher fraction of unskilled households wishing to educate their offspring to skilled workers, the long-run growth rate of the population, $n_s$, must decline. Again, with a higher share of unskilled households raising skilled offspring, the depletion rate of natural resources must decline, whereas the investment share of GDP, $\left( \frac{I}{Y} \right)_s$, increases in the long-run and long-run productivity growth amounts to 2.62%.

(b) Transition

An increase in $\phi$ implies that the skilled-wage premium $\omega_t$ must increase to its new long-run value equal to $w_*=\frac{z}{z-\phi}$. Hence, the current wage differential is below its long-run value, as well as the skilled-unskilled population ratio. A smooth convergence of the latter to its new long-run value requires the share of unskilled households raising skilled offspring to adjust from above its long-run value to its actual long-run value. Hence, the growth rate of the skilled population group must be above its long-run value, too. The opposite is true for the unskilled population group. As $n_u^u > n_s^s > n_t^u,s$, the growth rate of the population is converging from below its long-run value to its actual long-run value.

The increase in $\theta_t$ after the shock raises the demand for skilled labor in the education sector such that $L_E^s_t$ increases at the expense of employment ratios of skilled labor in production. Consequently, the wage differential between skilled and unskilled labor increases. Since the (relative) market size for skilled-labor complementary innovations is currently reduced, the share of natural resources allocated to the $u$-sector must increase, i.e. $\varphi_t^{x,s}$ declines. At the same time, the depletion rate increases in order to satisfy aggregate demand for machines. The decline in the share of unskilled households raising skilled offspring during the transition reduces the demand for skilled labor in the education sector. As this process corresponds to an increase in the skilled-unskilled population ratio, the employment ratios of skilled labor in production must rise. Skill-biased technological change meets an increase in the skilled population group. The decline in $\theta_t$ and the increase in relative wages that is responsible for a decline in $n_t^{u,s}$ cause a decline in the growth rate of the skilled population group and an increase in the growth rate of the unskilled population group. With a lower demand for labor in the education sector and increasing employment ratios of skilled labor in production the depletion rate of natural resources declines during the transition.
while the growth rate of the population is increasing.

6 Summary and Conclusions

This paper integrates the features of skill-biased technological change, fertility decline and natural resource use into a comprehensive framework. More in detail, we consider an overlapping generations economy populated by skilled and unskilled households. The educational choices of parents for their offspring generate differential fertility between skilled and unskilled households, in the spirit of Galor and Mountford (2006), and de la Croix and Doepke (2003). The production side of the model is characterized by a scale invariant two-sector growth model with multiple production stages. Moreover, the direction of technological change is determined by the relative supply of skilled and unskilled labor (Acemoglu 1998; 2002) resulting from the educational choices of parents for their offspring. In addition, production is subject to a non-renewable natural resource as an essential input.

Our analysis reveals with respect to the long-run, the following results: First, an increase in the skilled population group relative to the unskilled population group reduces the growth rate of the population and lowers the depletion rate of natural resources. Second, taking account for a stationary or even shrinking population, sustained economic growth can only be achieved by intertemporal knowledge spillovers in research and development. Otherwise, skill-biased productivity growth would dig its own grave through the induced negative feedback-effect of fertility decline.

As regards the transition, we obtain the following results: During the transition to the steady state, the depletion rate of natural resources follows the evolution of the share of unskilled households raising skilled offspring. As the latter is inversely related to the growth rate of the population, the model suggests an inverse relation between population growth and the depletion rate of natural resources during the transition to the long-run equilibrium (steady state). Generally speaking, the last result hinges on the demand for skilled labor in the education sector triggered by the share of unskilled households raising skilled offspring. An increase in the latter reduces population growth and advances demand for natural resources in production in order to compensate for the reallocation of skilled labor from production to education.

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References


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and

Consequently, \( \varphi^u_t \) (\( \varphi^s_t \)) is declining (increasing) in \( \frac{N^s}{N^u} \) and \( \frac{Y^s}{L^u} \), if \( \sigma > 1 \).
A.2 Proof of Proposition 3

Item (i),(ii) and (iii):

Along the balanced growth path, the blueprint ratio, employment ratios of labor and the wage differential of skilled labor are constant, we therefore know that

\[
\left(\frac{N^s}{N^u}\right)_* = \left(\frac{\eta^s}{\eta^u}\right)^{\mu^s + \sigma^u} \left(1 - \frac{\gamma}{\gamma^s}\right) \left(\frac{L^s}{L^u}\right)_* \left(\frac{c^u}{c^s}\right)_*,
\]  

(A.5)

Substitution for \(\frac{c^u}{c^s}\) yields

\[
\left(\frac{N^s}{N^u}\right)_* = \left(\frac{\eta^s}{\eta^u}\right)^{\frac{\sigma^u}{\psi_1}} \left(1 - \frac{\gamma}{\gamma^s}\right) \left(\frac{L^s}{L^u}\right)_* \left(\frac{A}{B}\right)^{(\sigma - 1)(1 - \beta)} \psi_1,
\]  

(A.6)

with \(\psi_1 = \beta(1 + \delta \sigma) - (\sigma - 1)(1 - \beta)\).

Proceeding in a similar fashion we obtain for the skilled wage premium

\[
\omega_* = \left(1 - \frac{\gamma}{\gamma^s}\right) \left(\frac{\eta^s}{\eta^u}\right)^{\frac{\sigma^u}{\psi_2}} \left(\frac{L^s}{L^u}\right)_* \left(\frac{A}{B}\right)^{(\sigma - 1)(1 - \beta)} \psi_2,
\]  

(A.7)

with \(\psi_2 = \sigma \beta + (1 - \alpha)(1 - \beta)(\sigma - 1)\).

Combining (A.6) and (A.7) yields

\[
\omega_* = \left[\left(\frac{\eta^s}{\eta^u}\right)^{\frac{\sigma^u}{\psi_1}} \left(1 - \frac{\gamma}{\gamma^s}\right) \left(\frac{L^s}{L^u}\right)_* \left(\frac{A}{B}\right)^{(\sigma - 1)(1 - \beta)(1 + \delta)} \right] \frac{1}{\psi_1},
\]  

(A.8)

\[
\left(\frac{N^s}{N^u}\right)_* = \left[\left(\frac{\eta^s}{\eta^u}\right)^{\frac{\sigma^u}{\psi_2}} \left(1 - \frac{\gamma}{\gamma^s}\right) \left(\frac{L^s}{L^u}\right)_* \left(\frac{A}{B}\right)^{(\sigma - 1)(1 - \beta)} \right] \frac{1}{\psi_2},
\]  

(A.9)

with \(\psi = \beta - \xi R(\sigma - 1) + \delta(\beta + (\sigma - 1)\xi L)\).

Since \(\frac{L^s}{L^u}\)_* = \(\frac{\theta_{u,n^u,s}}{(1 - \theta_{s,n^u})n^u_u - n^u_s} \) \((L^s/L^u)_* = (1 - \phi n^s,s)(L^s/L^u)_* - \phi \theta_{s,n^u,s}\) and \(\omega_* = \frac{z}{z - \phi}\) with \(\omega_*\) specified by (A.7), we yield

\[
(1 - \phi n^s,s) \left(\frac{L^s}{L^u}\right)_* - \phi \theta_{s,n^u,s} = \Phi,
\]

\[
\Phi = \left[\left(\frac{z}{z - \phi}\right) \left(\frac{\eta^s}{\eta^u}\right)^{1 - \sigma} \left(1 - \frac{\gamma}{\gamma^s}\right) \left(\frac{A}{B}\right)^{(1 - \sigma)(1 - \beta)(1 + \delta)} \right] \frac{1}{\sigma - 1 - \beta(1 + \delta)},
\]

such that

\[
\theta^2_* + \theta_* \frac{\Phi_2}{\Phi_1} - \frac{\Phi_3}{\Phi_1} = 0,
\]  

(A.10)
with $\Phi_1 = \phi n^u u^u_n u^s_n$, $\Phi_2 = (1 - \phi n^s s_n) n^u_n + \Phi n^u_n - (n^u_n - n^s_n) \phi n^u_n$; $\Phi_3 = \Phi (n^u_n - n^s_n)$. With one positive and one negative root, the solution to (A.10) is unique. Hence, $(L_s Y_s L_u Y_u)^*, (L_s Y_t Y_u)^*$ and $(N_s N_u)^*$ are unique and constant as well which implies that $p_s^{Y_s}$ and $p_u^{Y_u}$ are constant and unique as well (see Appendix B.8). Since, $\tau_s$ is also defined in a unique way (see Proof of Proposition 4), there exists a unique balanced growth path.

**Item (iv):**

In the presence of a symmetric equilibrium, the production function of intermediate $Y^i, i = u, s$ writes as

$$Y^i_t = N^i_t (L^x_t)^{(1-\alpha)(1-\beta)} (R^i_t)^{\alpha(1-\beta)} (L^Y_t)^{\beta},$$  

(A.11)

such that the steady state gross-growth rate of $Y^i_t, i = u, s$ is given by

$$g^{Y^i} = g^{N^i} (g^{L^L})^{\xi^L} (g^{R^R})^{\xi^R}.$$  

(A.12)

Since innovations evolve according to (28) and are fueled by aggregate savings, we obtain in steady state $g^{D_i} = g^{Y^i}$. Moreover, in steady state, we yield from (28)

$$1 = \frac{g^{D_i}}{(g^{N^i})^{1+\delta}} = \frac{g^{Y^i}}{(g^{N^i})^{1+\delta}}.$$  

(A.13)

Hence,

$$(g^{N^i})^{1+\delta} = g^{Y^i} = g^{N^i} (g^{L^L})^{\xi^L} (g^{R^R})^{\xi^R}$$  

(A.14)

and

$$g^{N^i} = \left[ n^L (1 - \tau^s) \xi^R \right]^{\frac{1}{\delta}},$$  

(A.15)

since $g^{R^R} = (1 - \tau^s) \frac{\xi^{R+1}}{\tau^s}$ and $\tau_s = \tau_t = \tau_{t+1}$. In Appendix B.8, we exploit the general equilibrium structure more in detail, in order to verify $g^{N^i}, i = u, s$.

**A.3 Proof of Proposition 4**

**Item (ii):**

From (A.3) we know that

$$\varphi^{x^u} = \frac{1}{1 + \Psi},$$  

(A.16)
In light of the Implicit function theorem it follows immediately that

$$\Psi_s = \left( \frac{1 + \psi}{\tau} \right) \frac{\phi \left( N^s \right)_s}{N^s} \frac{d}{d \tau_s} \left( A \left( \frac{N^s}{N} \right)_s \right) \omega^a - 1 \right) \left( \sigma - 1 \right) (1 - \beta).$$

Hence, in view of (66), the long-run depletion rate is specified by

$$\tau_s = 1 - \frac{Z_1}{Z_2}, \quad (A.17)$$

$$\tau_s = 1 - \frac{\rho + \rho + 1}{\beta(1 - \beta)} \frac{\xi L^L \left[ 1 + \frac{\phi}{L} \left( \frac{L^L}{L} \right)_s \right] + \xi R (1 + \Psi) \left[ 1 + \frac{\eta^u}{\eta^U} \left( \frac{N^s}{N} \right)_s \right]}{1 + \frac{\xi R (1 + \Psi) \left( N^s \right)_s}{Z_2}}, \quad (A.18)$$

Therefore,

$$\tau_s^2 + \left( \frac{\rho + \rho + 1}{\beta(1 - \beta)} \frac{\xi L^L \left[ 1 + \frac{\phi}{L} \left( \frac{L^L}{L} \right)_s \right] + \xi R (1 + \Psi) \left[ 1 + \frac{\eta^u}{\eta^U} \left( \frac{N^s}{N} \right)_s \right]}{Z_2} \right) \tau_s - \frac{\xi R (1 + \Psi) \left( N^s \right)_s}{Z_2} = 0, \quad (A.19)$$

with one positive and one negative root, such that the only economic meaningful solution is given by

$$\tau_s = \sqrt{\frac{B + A - Z_2}{2Z_2}} + \frac{1}{4} \left( \frac{B + A - Z_2}{Z_2} \right)^2 + \frac{B}{Z_2}, \quad (A.20)$$

where the discriminant is always positive, since $\frac{B}{Z_2} > 0$.

In order to detect the behavior of $\tau_s$ in response to shifts in $\theta_s$, we define the following implicit function, $F$, with

$$F = 1 - \frac{Z_1}{Z_2} - \tau_s, \quad (A.21)$$

where

$$Z_1 = \frac{\rho - \rho + \rho}{1 + \beta} \xi L^L \left[ 1 + \frac{\phi}{L} \left( \frac{L^L}{L} \right)_s \right] - \xi R (1 + \Psi) \left[ 1 + \frac{\eta^u}{\eta^U} \left( \frac{N^s}{N} \right)_s \right], \quad (A.22)$$

$$Z_2 = \beta(1 - \beta) \left[ 1 + \frac{\eta^u}{\eta^U} \left( \frac{N^s}{N} \right)_s \right]^{1+\delta}, \quad (A.23)$$

and $\frac{\eta^u}{\eta^U} \left( \frac{N^s}{N} \right)_s^{1+\delta}$ equals aggregate relative profits of future innovation $(\frac{N^s}{N})_s (\frac{\pi^s}{\pi^u})_s$.

In light of the Implicit function theorem it follows immediately that

$$\frac{\partial \tau_s}{\partial \phi} \approx \frac{d \tau_s}{d \phi} = -\frac{F_\phi}{F_{\tau_s}}, \quad (A.24)$$

where

$$F_{\tau_s} = -\xi R (1 + \Psi) \left( \frac{Z_1}{Z_2} \right) - 1 < 0, \quad (A.25)$$
and

\[ F_\phi = -\frac{\partial Z_1^* Z_2^* - \partial Z_2^* Z_1^*}{(Z_2^*)^2}, \tag{A.26} \]

with

\[ \frac{\partial Z_1^*}{\partial \phi} = \frac{\rho}{1 + \nu + \rho} \xi L \left[ \frac{\partial \omega_s (L^u)}{\partial \phi \tau_s} \right] + \omega_s \frac{\partial (L^u)}{\partial \phi \tau_s} - \xi R \frac{1 - \tau_s}{\tau_s} \partial \beta \tau_s \frac{\partial (\frac{Y_s^u}{\beta})}{\partial \phi \tau_s}, \tag{A.27} \]

\[ \frac{\partial Z_2^*}{\partial \phi} = \beta (1 - \beta)(1 + \delta) \frac{\eta^u}{\eta^u} \left( \frac{N^s}{N^u} \right) \frac{\delta (\frac{N^s}{\beta})}{\partial \phi \tau_s} - \xi R \frac{1 - \tau_s}{\tau_s} \partial \beta \tau_s \frac{\partial (\frac{Y_s^u}{\beta})}{\partial \phi \tau_s}. \tag{A.28} \]

Hence, \( \frac{\partial \tau_s}{\partial \phi} = -\frac{F_\phi}{F_{\tau_s}} \leq 0 \), if

\[ \frac{\partial Z_1^*}{\partial \phi} Z_2^* - \frac{\partial Z_2^*}{\partial \phi} Z_1^* \geq 0, \tag{A.29} \]

as \( Z_1^*, Z_2^*; \frac{\partial Z_1^*}{\partial \phi}, \frac{\partial Z_2^*}{\partial \phi} > 0 \) for economic meaningful solutions.

Moreover for interior solutions, i.e. \( 0 < \tau_s < 1 \): \( Z_1^* < Z_2^* \), such that \( \frac{\partial Z_1^*}{\partial \phi} > \frac{\partial Z_2^*}{\partial \phi} \) is a necessary condition for \( \frac{\partial \tau_s}{\partial \phi} > 0 \).

As regards \( \frac{\partial \tau_s}{\partial \phi^*} \), we obtain

\[ \frac{\partial \tau_s}{\partial \phi^*} = -\frac{F_{\phi^*}}{F_{\tau_s^*}}, \tag{A.30} \]

where

\[ F_{\phi^*} = -\frac{\frac{\partial Z_1^*}{\partial \phi} Z_2^* - \frac{\partial Z_2^*}{\partial \phi} Z_1^*}{(Z_2^*)^2}, \tag{A.31} \]

with

\[ \frac{\partial Z_1^*}{\partial \phi^*} = \frac{\rho}{1 + \nu + \rho} \xi L \left[ \frac{\partial \omega_s (L^u)}{\partial \phi \tau_s} \right] - \xi R \frac{1 - \tau_s}{\tau_s} \partial \beta \tau_s \frac{\partial (\frac{Y_s^u}{\beta})}{\partial \phi \tau_s}, \tag{A.32} \]

\[ \frac{\partial Z_2^*}{\partial \phi^*} = \frac{\partial Z_2^*}{\partial \phi^*} \left( (\frac{N^s}{\eta^u})^{1+\delta} + (1 + \delta) (\frac{\eta^u}{\eta^u})^{1+\delta} \left( \frac{N^s}{\eta^u} \right) \frac{\delta (\frac{N^s}{\beta})}{\partial \phi \tau_s} \right), \tag{A.33} \]

and \( Z_1^*, Z_2^* > 0; \frac{\partial Z_1^*}{\partial \phi^*}, \frac{\partial Z_2^*}{\partial \phi^*} < 0 \) for economic meaningful solutions. Therefore, \( \frac{\partial \tau_s}{\partial \phi^*} > 0 \), if

\[ \left| \frac{\partial Z_2^*}{\partial \phi^*} \right| < \left| \frac{\partial Z_1^*}{\partial \phi^*} \right|. \]

### A.4 Proof of Corollary 1

Since,

\[ N_{t+1}^u = \frac{\eta^u D_t^u}{(N_t^u)^\delta}, \tag{A.34} \]

\[ N_{t+1}^u = \frac{\eta^u Z_t^u}{(1 + \Omega_t)(N_t^u)^\delta} \left[ (1 - \beta) \frac{1 - 2\beta}{\beta - 1} N_t^u (p_t^u) \frac{1}{\beta} (c_t^u)^{\frac{\beta - 1}{\beta}} L_t^u \right], \tag{A.35} \]

\[ g_{t+1}^u = \frac{N_{t+1}^u - N_t^u}{N_t^u} = \frac{\eta^u Z_t^u}{(1 + \Omega_t)(N_t^u)^\delta} \left[ (1 - \beta) \frac{1 - 2\beta}{\beta - 1} (p_t^u) \frac{1}{\beta} (c_t^u)^{\frac{\beta - 1}{\beta}} L_t^u \right], \tag{A.36} \]
and \( \{ \Omega_t, Z_{2t}, p^Y_t, g^N_u \} = \text{const.} \) in steady state, we obtain
\[
\frac{g_{t+1}^N}{g_t^N} = \frac{g_t^N}{g^N_u} = 1 = (g^N_u)^{-\delta} (g^N_u)^{\delta-1} n_*. 
\]
(A.37)

As moreover
\[
c_t^u = \frac{(p_t^R) \alpha (w^u_t)^{1-\alpha}}{N_t^\alpha (1-\alpha)^{1-\alpha}},
\]
(A.38)

it follows
\[
g^u_s = (g^N_u)^{-1} (g^R_s)^{\alpha} (g^u_s)^{1-\alpha}.
\]
(A.39)

Therefore it remains to specify \( g^u_s \) and \( g^R_s \). Since
\[
w^u_t = \left[ (1-\beta)^{1-2\beta} N_t^\alpha \left( \frac{p_t^R \alpha}{B \alpha (1-\alpha)^{1-\alpha}} \right)^\beta \right]^{\frac{1}{1-\gamma}},
\]
(A.40)

we obtain
\[
g^u_s = \left[ g^N_u \left( g^R_s \right)^{\xi R} \right]^{\frac{1}{1-\gamma}}.
\]
(A.41)

As moreover \( g^N_u = g^N_s \): \( g^u_s = g^u_s \).

**B Further Derivations - Not for Publication**

**B.1 Derivation of the price index**

For readers’ convenience we omit the time index. Aggregate output \( Y \) is composed out of two intermediates \( Y^u \) and \( Y^s \) and is subject to the following CES-production function:
\[
Y = \left[ \gamma(Y^u)^{\frac{1}{\epsilon}} + (1-\gamma)(Y^s)^{\frac{1}{\epsilon}} \right]^{\frac{\epsilon}{\epsilon-1}}.
\]
(B.1)

Competitive behavior implies the following optimality conditions for profit maximizing factor demand, given that \( p^Y \equiv 1 \)
\[
\frac{\partial Y}{\partial Y^u} = \left[ \gamma(Y^u)^{\frac{1}{\epsilon}} + (1-\gamma)(Y^s)^{\frac{1}{\epsilon}} \right]^{\frac{\epsilon}{\epsilon-1}} \gamma(Y^u)^{\frac{1}{\epsilon}} p^Y u = 0,
\]
(B.2)
\[
\frac{\partial Y}{\partial Y^s} = \left[ \gamma(Y^u)^{\frac{1}{\epsilon}} + (1-\gamma)(Y^s)^{\frac{1}{\epsilon}} \right]^{\frac{\epsilon}{\epsilon-1}} (1-\gamma)(Y^s)^{\frac{1}{\epsilon}} p^Y s = 0.
\]
(B.3)

Hence, relative factor prices for intermediates read
\[
\frac{p^Y s}{p^Y u} = \frac{1-\gamma}{\gamma} \left( \frac{Y^s}{Y^u} \right)^{\frac{1}{\epsilon}}.
\]
(B.4)
implying that
\[ Y^s = \left( \frac{1 - \gamma}{\gamma} \right)^\varepsilon \left( \frac{p^{Y_u}}{p^{Y_s}} \right)^\varepsilon Y^u. \] (B.5)

Moreover, production costs write as
\[ C = p^{Y_u}Y^u + p^{Y_s}Y^s, \] (B.6)
\[ C = p^{Y_u}Y^u + p^{Y_s} \left( \frac{1 - \gamma}{\gamma} \right)^\varepsilon \left( \frac{p^{Y_u}}{p^{Y_s}} \right)^\varepsilon Y^u, \] (B.7)
\[ C = Y^u \left[ p^{Y_u} + p^{Y_s} \left( \frac{1 - \gamma}{\gamma} \right)^\varepsilon \left( \frac{p^{Y_u}}{p^{Y_s}} \right)^\varepsilon \right]. \] (B.8)

In light of the last expression we are able to express factor demand for intermediates in final good production in terms of production costs, C, factor prices and parameters of the production function
\[ Y^u = \frac{C}{p^{Y_u} + p^{Y_s} \left( \frac{1 - \gamma}{\gamma} \right)^\varepsilon \left( \frac{p^{Y_u}}{p^{Y_s}} \right)^\varepsilon}, \] (B.9)
\[ Y^u = \frac{C(p^{Y_u})^{\gamma-\varepsilon}[(p^{Y_u})^{1-\varepsilon} + (1 - \gamma)^\varepsilon (p^{Y_s})^{1-\varepsilon}]}{(p^{Y_u})^{1-\varepsilon} \gamma^\varepsilon + (1 - \gamma)^\varepsilon (p^{Y_s})^{1-\varepsilon}}, \] (B.10)
\[ Y^u = \frac{C(p^{Y_u})^{-\varepsilon} \gamma^\varepsilon}{(p^{Y_u})^{1-\varepsilon} \gamma^\varepsilon + (1 - \gamma)^\varepsilon (p^{Y_s})^{1-\varepsilon}}. \] (B.11)

Manipulating terms, yields
\[ (Y^u)^{\varepsilon^{-1}} = \frac{C^{\varepsilon^{-1}}(p^{Y_u})^{1-\varepsilon} \gamma^{\varepsilon^{-1}}}{[(p^{Y_u})^{1-\varepsilon} \gamma^{\varepsilon} + (1 - \gamma)^\varepsilon (p^{Y_s})^{1-\varepsilon}]^{\varepsilon^{-1}}}, \] (B.12)
\[ \gamma(Y^u)^{\varepsilon^{-1}} = \frac{C^{\varepsilon^{-1}}(p^{Y_u})^{1-\varepsilon} \gamma^{\varepsilon}}{[(p^{Y_u})^{1-\varepsilon} \gamma^{\varepsilon} + (1 - \gamma)^\varepsilon (p^{Y_s})^{1-\varepsilon}]^{\varepsilon^{-1}}}, \] (B.13)
and similarly
\[ (1 - \gamma)(Y^s)^{\varepsilon^{-1}} = \frac{C^{\varepsilon^{-1}}(p^{Y_s})^{1-\varepsilon}(1 - \gamma)^\varepsilon}{[(p^{Y_u})^{1-\varepsilon} \gamma^{\varepsilon} + (1 - \gamma)^\varepsilon (p^{Y_s})^{1-\varepsilon}]^{\varepsilon^{-1}}}. \] (B.14)

Taking these two results together yields
\[ \gamma(Y^u)^{\varepsilon^{-1}} + (1 - \gamma)(Y^s)^{\varepsilon^{-1}} = \] (B.15)
\[ = \frac{C^{\varepsilon^{-1}}(p^{Y_u})^{1-\varepsilon} \gamma^{\varepsilon}}{[(p^{Y_u})^{1-\varepsilon} \gamma^{\varepsilon} + (1 - \gamma)^\varepsilon (p^{Y_s})^{1-\varepsilon}]^{\varepsilon^{-1}}} + \frac{C^{\varepsilon^{-1}}(p^{Y_s})^{1-\varepsilon}(1 - \gamma)^\varepsilon}{[(p^{Y_u})^{1-\varepsilon} \gamma^{\varepsilon} + (1 - \gamma)^\varepsilon (p^{Y_s})^{1-\varepsilon}]^{\varepsilon^{-1}}}, \] (B.16)
\[ = \frac{C^{\varepsilon^{-1}}}{[(p^{Y_u})^{1-\varepsilon} \gamma^{\varepsilon} + (1 - \gamma)^\varepsilon (p^{Y_s})^{1-\varepsilon}]^{\varepsilon^{-1}}} \left[ \gamma^\varepsilon (p^{Y_u})^{1-\varepsilon} + (1 - \gamma)^\varepsilon (p^{Y_s})^{1-\varepsilon} \right], \] (B.17)
\[ = \frac{C^{\varepsilon^{-1}}}{[(p^{Y_u})^{1-\varepsilon} \gamma^{\varepsilon} + (1 - \gamma)^\varepsilon (p^{Y_s})^{1-\varepsilon}]^{\varepsilon^{-1}}}. \] (B.18)
Since, \( Y = \left[ \gamma (Y^u)^{\frac{1}{\epsilon}} + (1 - \gamma)(Y^s)^{\frac{1}{\epsilon}} \right]^{\frac{1}{1-\epsilon}} = \left[ \frac{C^{\frac{1}{\epsilon}}}{(p^{Y^u})^{1-\epsilon} \gamma + (1-\gamma)^{\epsilon} (p^{Y^s})^{1-\epsilon}} \right]^{\frac{1}{1-\epsilon}}, \)

it follows immediately that
\[
Y = C[(p^{Y^u})^{1-\epsilon} \gamma^\epsilon + (1 - \gamma)^{\epsilon} (p^{Y^s})^{1-\epsilon}]^{\frac{1}{1-\epsilon}},
\]

(B.19)

and finally
\[
Y[(p^{Y^u})^{1-\epsilon} \gamma^\epsilon + (1 - \gamma)^{\epsilon} (p^{Y^s})^{1-\epsilon}]^{\frac{1}{1-\epsilon}} = C,
\]

(B.20)

with \( p^Y \equiv 1 = [(p^{Y^u})^{1-\epsilon} \gamma^\epsilon + (1 - \gamma)^{\epsilon} (p^{Y^s})^{1-\epsilon}]^{\frac{1}{1-\epsilon}} \) representing the price index of \( Y \).

### B.2 Marginal Production Costs of Machines

Aggregate output in the machine producing sector reads as
\[
X^i_t = GN^i_t \left( L^x_i \right)^{1-\alpha} \left( R^x_i \right)^{\alpha},
\]

(B.21)

with \( G = A \lor G = B \) if \( i = s \lor i = u \).

Perfect competition and profit maximizing behavior imply
\[
R^x_i = \alpha \frac{L^x_i w^i_t}{p^R_t}.
\]

(B.22)

Combining the last expression with the aggregate cost functions in the machine producing sectors yields
\[
C^i_t = w^i_t L^x_i + p^R_t R^x_i = w^i_t L^x_i + \frac{\alpha}{1-\alpha} \frac{L^x_i w^i_t}{p^R_t} = \frac{1}{1-\alpha} w^i_t L^x_i.
\]

Hence,
\[
L^x_i = \frac{(1-\alpha)C^i_t}{w^i_t},
\]

(B.24)

and
\[
R^x_i = \frac{\alpha C^i_t}{p^R_t}.
\]

(B.25)

Combining the last two expressions with the production function gives
\[
X^i_t = GN^i_t C^i_t (1-\alpha)^{1-\alpha} \left( w^i_t \right)^{\alpha-1} \alpha^\alpha \left( p^R_t \right)^{-\alpha}.
\]

(B.26)

Obviously,
\[
C^i_t = X^i_t \frac{\left( w^i_t \right)^{1-\alpha} \left( p^R_t \right)^{\alpha}}{GN^i_t (1-\alpha)^{1-\alpha} \alpha^\alpha},
\]

(B.27)

In conclusion the ratio of marginal production costs reads as
\[
\frac{c^u_t}{c^s_t} = \frac{A N^s_t}{B N^u_t w^i_t} \alpha^{\alpha-1}.
\]

(B.28)
B.3 Labor Markets

Perfect competition and profit maximizing behavior imply \( w_t^Y = w_t^x \) and

\[
p_t^x \frac{\partial Y^x_t}{\partial L^x_t} = p_t^y \frac{\partial Y^y_t}{\partial L^y_t}, \tag{B.29}
\]

Since, \( p_t^x = \frac{c_t^x}{1-\beta} \)

\[
p_t^x = \frac{(1-\alpha)X^x_t}{L^x_t}, \tag{B.30}
\]

Since aggregate production of machines in sector \( i = u, s \) is given by \( X^x_i = N^x_i x^x_i \) we yield in light of (30) and (36) or (37)

\[
L^x_i = \frac{(1-\alpha)(1-\beta)}{\beta} L^y_i. \tag{B.31}
\]

As moreover \( L^u_t = L^Y_u + L^x_u \) and \( L^s_t - L^E_s = L^Y_s + L^x_s \), it follows

\[
L^Y_s = \frac{\beta}{\xi^L}(L^x_s - L^E_s), \quad L^x_s = \frac{\xi^L}{\xi^L}(L^x_s - L^E_s), \tag{B.32}
\]

\[
L^Y_u = \frac{\beta}{\xi^L} L^u, \quad L^x_u = \frac{\xi^L}{\xi^L} L^s, \tag{B.33}
\]

with \( \xi^L = (1-\alpha)(1-\beta) \) and \( \xi^L = \beta + (1-\alpha)(1-\beta) \).

B.4 Allocation of Extracted Natural Resources

Prefect competition and profit maximizing behavior imply

\[
p_t^R = p_t^x \frac{X^s_t}{R^x_t} = p_t^y \frac{X^u_t}{R^x_t}. \tag{B.34}
\]

Since \( p_t^x = \frac{c_t^x}{1-\beta} \), we obtain

\[
\frac{c_t^x u}{c_t^x s} \frac{X^x_t}{X^u_t} = \frac{c_t^x u}{c_t^x s} \frac{N^x_t x^x_t}{N^u_t x^u_t} = \left( \frac{c_t^x u}{c_t^x s} \right) \frac{1-\beta}{\beta} p_t^1 \frac{N^s_t}{N^u_t} \frac{L^Y_s}{L^Y_u} = \frac{R^x_s}{R^x_u}. \tag{B.35}
\]

Substitution \( p_t = \frac{p_t^y}{p_t^x} \) yields

\[
\frac{R^x_s}{R^x_u} = \left( \frac{1-\gamma}{\gamma} \right)^{\frac{\sigma}{\rho}} \left( \frac{N^s_t}{N^u_t} \right)^{\frac{\sigma-1}{\rho}} \left( \frac{L^Y_s}{L^Y_u} \right)^{\frac{\sigma-1}{\rho}} \left( \frac{c_t^x s}{c_t^x u} \right)^{\frac{(\sigma-1)(1-\beta)}{\rho \beta}}. \tag{B.36}
\]
As an efficient use of extracted natural resources requires \( R_t^{xu} + R_t^{xs} = R_t \),

we finally arrive to

\[
R_t^{xu} = \frac{1}{1 + \left( \frac{1 - \gamma}{\gamma} \right) \left( \frac{N_t^s}{N_t^u} \right)^{\sigma - 1} \left( \frac{L_t^{ys}}{L_t^{yu}} \right)^{\sigma - 1} \left( \frac{c_{t+1}^u}{c_{t+1}^s} \right)^{(1+\sigma)(1-\beta)/\beta}} = \varphi_t^{xu} R_t, \tag{B.37}
\]

\[
R_t^{xs} = \frac{1}{1 + \left( \frac{1 - \gamma}{\gamma} \right) \left( \frac{N_t^s}{N_t^u} \right)^{\sigma - 1} \left( \frac{L_t^{ys}}{L_t^{yu}} \right)^{\sigma - 1} \left( \frac{c_{t+1}^u}{c_{t+1}^s} \right)^{(1+\sigma)(1-\beta)/\beta}} = \varphi_t^{xs} R_t. \tag{B.38}
\]

### B.5 Dynamics of the Blueprint Ratio

In light of (51) and (34), (35), we know that

\[
\frac{N_{t+1}^s}{N_{t+1}^u} = \left( \frac{\eta^s}{\eta^u} \right)^{\sigma} \left( \frac{1 - \gamma}{\gamma} \right) \left( \frac{L_{t+1}^{ys}}{L_{t+1}^{yu}} \right)^{\sigma - 1} \left( \frac{N_t^s}{N_t^u} \right)^{-\delta \sigma} \left( \frac{c_{t+1}^u}{c_{t+1}^s} \right)^{(1+\sigma)(1-\beta)/\beta}, \tag{B.39}
\]

such that substitution for \( \omega_{t+1} \) in the marginal cost ratio yields

\[
\frac{N_{t+1}^s}{N_{t+1}^u} = \left( \frac{\eta^s}{\eta^u} \right)^{\sigma} \left( \frac{1 - \gamma}{\gamma} \right) \left( \frac{L_{t+1}^{ys}}{L_{t+1}^{yu}} \right)^{\sigma - 1} \left( \frac{N_t^s}{N_t^u} \right)^{-\delta \sigma} \left( \frac{A N_{t+1}^s}{B N_{t+1}^u} \right)^{(1+\sigma)(1-\beta)/\beta}, \tag{B.40}
\]

\[
\frac{N_{t+1}^s}{N_{t+1}^u} = \left( \frac{\eta^s}{\eta^u} \right)^{\sigma} \left( \frac{1 - \gamma}{\gamma} \right) \left( \frac{L_{t+1}^{ys}}{L_{t+1}^{yu}} \right)^{\sigma - 1} \left( \frac{N_t^s}{N_t^u} \right)^{-\delta \sigma} \left[ \frac{A N_{t+1}^s}{B N_{t+1}^u} \right]^{(1+\sigma)(1-\beta)/\beta}, \tag{B.41}
\]

Collecting terms and solving for \( \frac{N_{t+1}^s}{N_{t+1}^u} \) yields

\[
\frac{N_{t+1}^s}{N_{t+1}^u} = \left[ \left( \frac{\eta^s}{\eta^u} \right)^{\sigma} \left( \frac{N_t^s}{N_t^u} \right)^{-\delta} \left( \frac{L_{t+1}^{ys}}{L_{t+1}^{yu}} \right)^{(1+\sigma)(1-\beta)/\beta} \right]^{\frac{1}{(1-\sigma)(1-\beta)}} \tag{B.42}
\]

\[
\left[ \left( \frac{1 - \gamma}{\gamma} \right)^{\sigma} \left( \frac{L_{t+1}^{ys}}{L_{t+1}^{yu}} \right)^{(1+\sigma)(1-\beta)/\beta} \left( \frac{A}{B} \right)^{(1+\sigma)(1-\beta)/\beta} \right]^{\frac{1}{\beta}} \tag{B.43}
\]

### B.6 Aggregate Savings

Aggregate savings are given by the sum of savings in both population groups

\[
S_t = (s_t^u - p_t^R m_t^u) L_t^u + (s_t^s - p_t^R m_t^s) L_t^s. \tag{B.43}
\]
Hence,

\[
S_t = \frac{\rho}{1 + \gamma + \rho} (w_t^u L_t^u + w_t^s L_t^s) - p_t^R M_t, \quad (B.44)
\]

\[
= \frac{\rho}{1 + \gamma + \rho} w_t^u L_t^u \left[ 1 + \omega L_t^u \right] - p_t^R M_t, \quad (B.45)
\]

where \( M_t \) represents the aggregate natural resource stock in period \( t \). Since

\[
w_t^u = p_t^Y Y_t^u \frac{\beta}{L_t^u}, \quad (B.46)
\]

\[
p_t^R = p_t^Y Y_t^u \frac{\xi R}{Y_t^u}, \quad (B.47)
\]

and

\[
L_t^Y = \frac{\beta}{\xi} L_t^u, \quad (B.48)
\]

\[
R_t = \tau_t M_{t-1}, \quad (B.49)
\]

\[
R_t^x = \varphi_t^x R_t, \quad (B.50)
\]

we obtain

\[
S_t = \frac{\rho}{1 + \gamma + \rho} \xi L_t^u p_t^Y \left[ 1 + \omega L_t^u \right] - p_t^Y \frac{\xi R}{R_t^x} Y_t^u M_t, \quad (B.51)
\]

\[
= \frac{\rho}{1 + \gamma + \rho} \xi L_t^u p_t^Y \left[ 1 + \omega L_t^u \right] - p_t^R \frac{\xi R}{\varphi_t^x} Y_t^u M_t. \quad (B.52)
\]

Since \( \frac{M_t}{M_{t-1}} = 1 - \tau_t \), it follow outright that

\[
S_t = p_t^Y Y_t^u \left\{ \frac{\rho}{1 + \gamma + \rho} \xi L_t^u \left[ 1 + \omega L_t^u \right] - \frac{\xi R}{\varphi_t^x} \frac{1 - \tau_t}{\tau_t} \right\}. \quad (B.53)
\]

### B.7 Depletion Rate of Natural Resources

(a)

The derivation of the dynamic equation of the depletion rate of natural resource takes account for the fact that the price per unit of natural resources equals its value marginal product in intermediate production where the evolution of the natural resource price, \( g_{t+1}^R \), is given by the change in the value marginal product of natural resources. On the other hand, in equilibrium the change of this factor price is tight to the interest factor by Hotelling’s rule \((1 + r_{t+1} = g_{t+1}^R)\). Moreover the technology market clearing condition \((1 + r_{t+1} = \pi_{t+1}^u \eta^u (N_t^u)^{-\delta})\) links the interest rate to the profitability of future innovations. To begin with we start with the free-entry condition in R&D

\[
1 + r_{t+1} = \pi_{t+1}^u \eta^u (N_t^u)^{-\delta}. \quad (B.54)
\]
In order to specify the law of motion for the depletion rate, it remains to determine the equilibrium factor price of natural resources in equilibrium is given by

\[ p_t^R = p_t^Y u \alpha (1 - \beta) N_t^u B (L_t^{xu})^{(1-\alpha)(1-\beta)} (R_t^{xu})^{\alpha(1-\beta)-1} (L_t^{Y u})^\beta. \] (B.56)

Taking account for labor demand in equilibrium yields in terms of gross growth rates:

\[ g_{t+1}^R = g_{t+1}^{Y u} N_t^u (g_{t+1}^{L u})^{\xi_L} \left( g_{t+1}^{x u} \right)^{\xi_R} \left( g_{t+1}^{R u} g_{t+1} \right)^{-\xi_R-1}. \] (B.57)

Since \( M_t = M_{t+1} + R_{t+1} \), it follows \( R_{t+1} = M_t \left[ 1 - \frac{M_{t+1}}{M_t} \right] \) and

\[ \frac{R_{t+1}}{R_t} = \frac{M_t}{M_{t-1}} \left[ 1 - \frac{M_{t+1}}{M_t} \right]. \] (B.58)

As additionally \( R_{t+1} = \tau_{t+1} M_t \), we yield

\[ \frac{M_{t+1}}{M_t} = (1 - \tau_{t+1}). \] (B.60)

Therefore,

\[ g_{t+1}^R = \frac{R_{t+1}}{R_t} = (1 - \tau_t) \left[ 1 - \frac{1 - (\tau_{t+1})}{1 - (\tau_t)} \right] = (1 - \tau_t) \frac{\tau_{t+1}}{\tau_t}, \] (B.61)

and

\[ g_{t+1}^{p R} = g_{t+1}^{Y u} N_t^u (g_{t+1}^{L u})^{\xi_L} \left( g_{t+1}^{x u} \right)^{\xi_R} \left( R_t \right)^{-\xi_R-1}. \] (B.62)

Since \( \pi_{t+1}^u \eta^u (N_t^u)^{-\delta} = g_{t+1}^{p R} \), we obtain

\[ \pi_{t+1}^u \eta^u (N_t^u)^{-\delta} = g_{t+1}^{Y u} N_t^u (g_{t+1}^{L u})^{\xi_L} \left( g_{t+1}^{x u} \right)^{\xi_R} \left( 1 - \tau_t \right) \frac{\tau_{t+1}}{\tau_t}. \] (B.63)

In order to specify the law of motion for the depletion rate, it remains to determine the equilibrium rate of technological progress \( g_{t+1}^{N u} \), which depends on aggregate savings.

Since \( p_t^Y u Y_t^u = w_t^u L_t^{Y u} + p_t^{x u} X_t^u \), it follows

\[ p_t^Y u Y_t^u = \beta p_t^{Y u} \frac{Y_t^u}{L_t^{Y u}} L_t^{Y u} + p_t^{x u} X_t^u, \] (B.64)

and \( p_t^Y u Y_t^u = \frac{p_t^{x u} X_t^u}{1-\beta} \).

Profits of a machine producing firm in sector \( u \) read as

\[ \pi_t^u = \pi_{t+1}^{x u} a_t^u - c_t^u x_t^u. \] (B.65)
Hence, $\pi_t^u + c_t^u x_t^u = p_t^u x_t^u$, where substitution for $\pi_t^u$ and $x_t^u$ yields

$$p_t^u x_t^u = \beta (1 - \beta)^{\frac{1-\beta}{\eta}} (p_t^u)^\frac{1}{\eta} (c_t^u)^{\frac{\beta+1}{\eta}} L_t^u + c_t^u \left( \frac{(1 - \beta)p_t^u}{c_t^u} \right)^\frac{1}{\eta} L_t^u,$$  \hspace{1cm} (B.66)

$$= (1 - \beta)^{\frac{1}{\eta}} (p_t^u)^\frac{1}{\eta} (c_t^u)^{\frac{\beta+1}{\eta}} L_t^u \left[ \frac{\beta}{1 - \beta} + 1 \right],$$  \hspace{1cm} (B.67)

$$= (1 - \beta)^{\frac{1-\delta}{\eta}} (p_t^u)^\frac{1}{\eta} (c_t^u)^{\frac{\beta-1}{\eta}} L_t^u = \pi_t^u.$$  \hspace{1cm} (B.68)

Therefore, we are allowed to specify

$$p_t^u Y_t^u = \frac{p_t^u X_t^u}{1 - \beta} = \frac{N_t^u \pi_t^u}{\beta(1 - \beta)}.$$  \hspace{1cm} (B.69)

Combining the last expression with aggregate savings (53), we yield

$$S_t = \frac{N_t^u \pi_t^u}{\beta(1 - \beta)} Z_{3t}.$$  \hspace{1cm} (B.70)

As $D_t^u = \frac{1}{1 + \Omega_t} S_t$ we yield together with (28)

$$N_{t+1}^u = \frac{\eta^u (N_t^u)^{1-\delta} \pi_t^u}{\beta(1 - \beta)} Z_{1t}.$$

Defining $Z_{2t} = \beta(1 - \beta)(1 + \Omega_t)$ and $Z_{3t} = \frac{Z_{2t}}{Z_{2t}^u}$ yields

$$N_{t+1}^u = \frac{\eta^u (N_t^u)^{1-\delta} \pi_t^u}{\beta(1 - \beta)} Z_{3t},$$  \hspace{1cm} (B.72)

$$g_{t+1}^u = \frac{\eta^u (N_t^u)^{-\delta} \pi_t^u Z_{3t}}{\beta(1 - \beta)}.$$  \hspace{1cm} (B.73)

Combining the last expression with

$$\pi_{t+1}^u \eta^u (N_t^u)^{-\delta} = g_{t+1}^Y u N_t^u (g_{t+1}^L)^{\xi_L} \left( \frac{\phi^x u}{g_{t+1}^L} \right)^{\xi_R-1} \left( 1 - \frac{\tau_{t+1}}{\tau_t} \right)^{\xi_R-1} \left( 1 - \frac{\tau_{t+1}}{\tau_t} \right)^{\xi_R-1},$$  \hspace{1cm} (B.74)

yields

$$\frac{\pi_{t+1}^u \eta^u}{(N_t^u)^{\delta}} = g_{t+1}^Y u \left( \frac{\eta^u (N_t^u)^{-\delta} \pi_t^u Z_{3t} (g_{t+1}^L)^{\xi_L} \left( \frac{\phi^x u}{g_{t+1}^L} \right)^{\xi_R-1} \left( 1 - \frac{\tau_{t+1}}{\tau_t} \right)^{\xi_R-1}}{(1 - \frac{\tau_{t+1}}{\tau_t})} \right)^{\xi_R-1},$$  \hspace{1cm} (B.75)

$$g_{t+1}^u = \frac{g_{t+1}^Y u Z_{3t} (g_{t+1}^L)^{\xi_L} \left( \frac{\phi^x u}{g_{t+1}^L} \right)^{\xi_R-1} \left( 1 - \frac{\tau_{t+1}}{\tau_t} \right)^{\xi_R-1}}{\left( 1 - \frac{\tau_{t+1}}{\tau_t} \right)},$$  \hspace{1cm} (B.76)

With $g_{t+1}^u = \left( \frac{g_{t+1}^Y u}{g_{t+1}^L} \right)^\frac{1-\delta}{\eta} (g_{t+1}^L)^{\frac{\beta+1}{\eta}} g_{t+1}^L$, we obtain further

$$\left( \frac{g_{t+1}^Y u}{g_{t+1}^L} \right)^\frac{1-\delta}{\eta} = Z_{3t} \left( g_{t+1}^L \right)^{\xi_L-1} \left( \frac{\phi^x u}{g_{t+1}^L} \right)^{\xi_R-1} \left( 1 - \frac{\tau_{t+1}}{\tau_t} \right)^{\xi_R-1}.$$

(B.77)
Since \( w_t^Y_u = w_t^x_u \), we know that \( p_t^Y u \beta Y_t^u \equiv p_t^x u (1-\alpha) X_t^u L_t^x \). Substituting for the respective output levels, yields

\[
P_t^Y u \beta \left( \frac{\xi L}{\xi L} \right)^{(1-\alpha)(1-\beta)} (\varphi_{\xi^u} R_t)^{\alpha(1-\beta)} \left( \frac{\beta}{\xi L} \right)^{\beta-1} L_t^u = \frac{c_t^u}{1-\beta} B N_t^u \left( \frac{\beta}{\xi L} \right)^{-\alpha} (\varphi_{\xi^u} R_t)^{\alpha(1-\beta)}.
\]

Hence,

\[
P_t^Y u \beta B^{1-\beta} \left( \frac{\xi L}{\xi L} \right)^{(1-\alpha)(1-\beta)} (\varphi_{\xi^u} R_t)^{\alpha(1-\beta)} \left( \frac{\beta}{\xi L} \right)^{\beta-1} L_t^u = \frac{c_t^u}{1-\beta} B \left( \frac{\xi L}{\xi L} \right)^{-\alpha} (\varphi_{\xi^u} R_t)^{\alpha(1-\beta)}.
\]

\[
= \frac{c_t^u}{1-\beta} (1-\alpha) B \left( \frac{\xi L}{\xi L} \right)^{-\alpha} (\varphi_{\xi^u} R_t)^{\alpha(1-\beta)}
\]

\[
\frac{p_t^Y u}{c_t^u} = 1 - \frac{\alpha}{\beta} B^{1-\beta} \left( \frac{\xi L}{\xi L} \right)^{1-\beta(1-\alpha)} \left( \frac{\beta}{\xi L} \right)^{\beta-1} \left( \frac{R_t}{L_t^u} \right)^{\alpha(1-\beta)},
\]

such that

\[
\left( \frac{g_{t+1}^y u}{g_{t+1}^x} \right)^{1-\beta} = \left( \frac{g_{t+1}^x R}{g_{t+1}^x} \right)^{\alpha(1-\beta)}.
\]

Since on the other hand

\[
\left( \frac{g_{t+1}^y u}{g_{t+1}^x} \right)^{1-\beta} = Z_3 t \left( g_{t+1}^x \right)^{\xi L-1} \left( g_{t+1}^x \right)^{\xi R-1} \left( 1 - \frac{\tau_{t+1}}{\tau_t} \right) \xi R^{-1},
\]

it follows immediately that\(^{28}\)

\[
\left( \frac{g_{t+1}^x R}{g_{t+1}^x} \right)^{\alpha(1-\beta)} = Z_3 t \left( g_{t+1}^x \right)^{\xi L-1} \left( g_{t+1}^x \right)^{\xi R-1} \left( 1 - \frac{\tau_{t+1}}{\tau_t} \right) \xi R^{-1},
\]

\[
1 = Z_3 t \left( g_{t+1}^x \left( 1 - \frac{\tau_{t+1}}{\tau_t} \right)^{-1},
\]

and

\[
\tau_{t+1} = Z_3 t \frac{\tau_t}{1-\tau_t} \left( g_{t+1}^x \right)^{-1}.
\]

(b)

An alternative point of origin is the equilibrium on the market for machines which determines

---

\(^{28}\)Remember: \( \xi^R = \alpha (1-\beta) \) and \( \xi^L + \xi^R = 1. \)
equality between aggregate output and aggregate demand for machines in sector \( i = u, s \), such that for example

\[
N_t^{x,u} = N_t^{u} \left( \frac{p_t^{Y,u}}{C_t^{u}} \right)^\frac{1}{\beta} L_t^{Y,u} = B N_t^{u} \left( L_t^{x,u} \right)^{1-\alpha} (\varphi_t^{x,u} R_t)^\alpha, \quad (B.87)
\]

\[
x_t^{u} = \left( \frac{p_t^{Y,u}}{C_t^{u}} \right)^\frac{1}{\beta} L_t^{Y,u} = B \left( L_t^{x,u} \right)^{1-\alpha} (\varphi_t^{x,u} R_t)^\alpha. \quad (B.88)
\]

Since \( R_t = \tau_t M_{t-1} \), \( \tau_t \) increases whenever the left-hand side of the last expression increases by more than the right-hand side holding the depletion rate constant, that is increases in \( L_t^{x,u} \) and pure reallocation of natural resources (changes in \( \varphi_t^{x,u} \)) are not sufficient. Since the gross rate of growth of machine production in sector \( u \) is given by

\[
g_{t+1}^{x,u} = \left( \frac{L_t^{Y,u}}{g_{t+1}^{L,u}} \right)^{\frac{1}{\beta}} g_{t+1}^{L,u} = \left( g_{t+1}^{L,u} \right)^{1-\alpha} \left( g_{t+1}^{\varphi,x,u} R_{t+1} \right)^\alpha, \quad (B.89)
\]

and we know in light of (B.83) that

\[
\left( \frac{L_t^{Y,u}}{g_{t+1}^{L,u}} \right)^{\frac{1}{\beta}} g_{t+1}^{L,u} = Z_{3t} \left( g_{t+1}^{L,u} \right)^{\xi_{L}-1} \left( g_{t+1}^{\varphi,x,u} R_{t+1} \right)^{\xi_{R}-1}, \quad (B.90)
\]

it follows outright that

\[
(Z_{3t})^{\frac{1}{1-\beta}} \left( g_{t+1}^{L,u} \right)^{\xi_{L}-1} \left( g_{t+1}^{\varphi,x,u} R_{t+1} \right)^{\xi_{R}-1} = (g_{t+1}^{R,u})^{\frac{1}{1-\beta} - \alpha(1-\beta) + 1 - \xi_{R}}, \quad (B.91)
\]

such that again

\[
g_{t+1}^{R,u} = (1 - \tau_t) \left( g_{t+1}^{L,u} \right)^{\frac{1}{1-\beta} - \alpha(1-\beta) + 1 - \xi_{R}} = Z_{3t} \left( g_{t+1}^{\varphi,x,u} R_{t+1} \right)^{-1}, \quad (B.92)
\]

\[
\tau_{t+1} = \left( 1 - \tau_t \right) \frac{Z_{3t} \left( g_{t+1}^{\varphi,x,u} R_{t+1} \right)^{-1}}{Z_{3t}}. \quad (B.93)
\]

**B.8 Productivity Growth**

This proof exploits the general equilibrium structure more in detail in order to verify the growth rate of innovations \( g_{t}^{N,i} \).
As we are able to specify,
\[ g_c^u = (g_N^u)^{-1} \left( g_c^R \right)^\alpha (g_w^u)^{1-\alpha}, \]  
(B.94)
\[ g_c^u = (g_N^u)^{-1} \left( g_c^R \right)^\alpha \left( \frac{g_N^u}{g_p^R} \right)^{\frac{1-\alpha}{\xi^R}}, \]  
(B.95)
\[ g_c^u = \left( \frac{g_N^u}{g_p^R} \right)^{\frac{\alpha\beta}{\xi^F}}. \]  
(B.96)

As \( p_t^R = p_t^Y \xi R \), we yield
\[ g_{t+1}^R = g_t^p Y_{t+1}^u \xi_L \left( g_{t+1}^L \right) \xi_R^{1-1} \]  
(B.97)
with \( p_t^Y \) and \( \varphi_t^Y \) being constant in steady state, we obtain
\[ g_{t+1}^R = g_{t+1}^N u \xi_L^{1-1}. \]  
(B.98)
Combining the last expression with \( g_c^u \) gives
\[ g_c^u = n_s^R (1-\tau_s)^{1-1}. \]  
(B.99)

Since in steady state
\[ 1 = \left( g_N^u \right)^{1-\delta} \left( g_c^u \right)^{\delta n_s}, \]  
(B.100)
we yield
\[ g_N^u = n_s^R \left[ n_s (1-\tau_s)^{\frac{1}{\xi}} \right]^{1-\frac{1}{1-1}}, = \left[ n_s (1-\tau_s)^{\frac{1}{\xi}} \right]^{\frac{1}{1-1}}, \]  
(B.101)
since \( \xi^R + \xi^L = 1 \).

### B.9 Investment Share of GDP

Since we consider a closed economy: \( I_t = S_t \). Investment as a share of GDP, \( \frac{I_t}{Y_t} \), writes as
\[ \frac{I_t}{Y_t} = \frac{p_t^Y Y_{t+1} Z_{1t}}{Y_t}. \]  
(B.102)

Since \( p_t^Y \equiv 1 = \left[ \gamma^\varepsilon (p_t^Y)^{1-\varepsilon} + (1-\gamma)^\varepsilon (p_t^Y)^{1-\varepsilon} \right]^{\varepsilon}, \) we obtain
\[ 1 = p_t^Y [\gamma^\varepsilon (p_t^Y)^{1-\varepsilon} + (1-\gamma)^\varepsilon (p_t^Y)^{1-\varepsilon}]^{\varepsilon}, \]  
(B.103)
\[ p_t^Y = 1 = \left[ \gamma^\varepsilon (p_t^Y)^{1-\varepsilon} + (1-\gamma)^\varepsilon (p_t^Y)^{1-\varepsilon} \right]^{\varepsilon}, \]  
(B.104)
For \( \frac{Y^s}{Y_t} \), we yield

\[
\frac{Y^u}{Y_t} = \frac{Y^u}{\left[ \gamma \left( Y^u_t \right)^{\frac{1}{\gamma}} + (1 - \gamma) \left( Y^s_t \right)^{\frac{1}{\gamma}} \right]^{\frac{1}{1-\gamma}}},
\]

(B.105)

\[
\frac{Y^u}{Y_t} = \frac{1}{\left[ \gamma + (1 - \gamma) \left( Y^s_t \right)^{\frac{1}{\gamma - 1}} \right]^{\frac{1}{1-\gamma}}},
\]

(B.106)

with \( \frac{Y^s}{Y^u} = \frac{N^s}{N^u} \frac{1}{\gamma} \frac{1}{\gamma - 1} L \frac{Y^s}{L^u} \).

Hence,

\[
\frac{I_t}{Y_t} = \frac{p^u_t Y^u_t Z_{Y_t}}{Y_t},
\]

(B.107)

\[
\frac{I_t}{Y_t} = \frac{\left[ \gamma^\varepsilon + (1 - \gamma)^\varepsilon (p^u_t)^{1-\varepsilon} \right]^{\frac{1}{1-\gamma}}}{\left[ \gamma + (1 - \gamma) \left( Y^s_t \right)^{\frac{1}{\gamma - 1}} \right]^{\frac{1}{1-\gamma}}} Z_{Y_t},
\]

(B.108)

with \(( \frac{I}{Y} \)_s = \text{const..}

### B.10 Numeric Method

As described above, the dynamic behavior is fully determined by a four-dimensional system of difference equations - as given by (59)-(62) and (19) - involving two state variables: \( \frac{L^s}{Y^s} \) and \( \frac{N^s}{Y^u} \), as well as two jump variables: \( \theta \) and \( \tau \). The Jacobian of the dynamic system evaluated at the steady state, \( J_s \), is equal to

\[
J_s = \begin{vmatrix}
\frac{\partial L^u_{t+1}}{\partial \theta_t} & \frac{\partial L^u_{t+1}}{\partial \tau_t} & \frac{\partial L^u_{t+1}}{\partial N^u_{t+1}} & \frac{\partial L^u_{t+1}}{\partial \tau_{t+1}} \\
\frac{\partial \theta_{t+1}}{\partial \theta_t} & \frac{\partial \theta_{t+1}}{\partial \tau_t} & \frac{\partial \theta_{t+1}}{\partial N^u_{t+1}} & \frac{\partial \theta_{t+1}}{\partial \tau_{t+1}} \\
\frac{\partial N^u_{t+1}}{\partial \theta_t} & \frac{\partial N^u_{t+1}}{\partial \tau_t} & \frac{\partial N^u_{t+1}}{\partial N^u_{t+1}} & \frac{\partial N^u_{t+1}}{\partial \tau_{t+1}} \\
\frac{\partial \tau_{t+1}}{\partial \theta_t} & \frac{\partial \tau_{t+1}}{\partial \tau_t} & \frac{\partial \tau_{t+1}}{\partial N^u_{t+1}} & \frac{\partial \tau_{t+1}}{\partial \tau_{t+1}}
\end{vmatrix},
\]

(B.109)

The dynamic system exhibits numerically two unstable eigenvalues \( (\lambda_1, \lambda_4 > 1) \) and two stable eigenvalues \( (\lambda_2, \lambda_3 < 1) \), such that the dynamics of the economy is subject to saddle-point stability

\(^{29}\)Note that this verifies also that \( p^u_s \) is constant as \( p^u_s \) is constant. As \( p^u_s = p^u_s \), it follows outright that \( p^u_s = \text{const..} \)
along a two-dimensional manifold. The solution of the linearized system reads as

\[
\begin{bmatrix}
L^s_t \\
\theta_t \\
N^s_t \\
\tau_t
\end{bmatrix}
= P
\begin{bmatrix}
A_1 & \lambda_1^t & 0 \\
A_2 & \lambda_2^t & 0 \\
A_3 & \lambda_3^t & 0 \\
A_4 & \lambda_4^t & 0
\end{bmatrix}
+ \begin{bmatrix}
\frac{L^s}{L^u} \\
\frac{\theta}{\tau^*}
\end{bmatrix},
\]

(B.110)

where \(P\) contains the eigenvectors \(p_1, p_2, p_3, p_4\) and \(A_1, A_2, A_3, A_4\) represent arbitrary constants.

With \(\lambda_1, \lambda_4 > 1\), it follows immediately that \(A_1 = A_4 = 0\), such that

\[
\begin{align*}
L^s_t &= p_2,1 A_2 \lambda_2^t + p_3,1 A_3 \lambda_3^t + \left( \frac{L^s}{L^u} \right)_t^*, \quad \text{(B.111)} \\
\theta_t &= p_2,2 A_2 \lambda_2^t + p_3,2 A_3 \lambda_3^t + \theta^*, \quad \text{(B.112)} \\
N^s_t &= p_2,3 A_2 \lambda_2^t + p_3,3 A_3 \lambda_3^t + \left( \frac{N^s}{N^u} \right)_t^*, \quad \text{(B.113)} \\
\tau_t &= p_2,4 A_2 \lambda_2^t + p_3,4 A_3 \lambda_3^t + \tau^*. \quad \text{(B.114)}
\end{align*}
\]

With \(\frac{L^s}{L^u} = \left( \frac{L^s}{L^u} \right)_t^* > 0\) and \(\frac{N^s}{N^u} = \left( \frac{N^s}{N^u} \right)_t^* > 0\) given, the unknown constants \(A_2, A_3\) are determined by the solution of

\[
\begin{align*}
\frac{L^s}{L^u}_t^* &= p_2,1 A_2 + p_3,1 A_3 + \left( \frac{L^s}{L^u} \right)_t^*, \quad \text{(B.115)} \\
\frac{N^s}{N^u}_t^* &= p_2,3 A_2 + p_3,3 A_3 + \left( \frac{N^s}{N^u} \right)_t^*. \quad \text{(B.116)}
\end{align*}
\]

such that the initial values \(0 < \tau_0 < 1\) and \(0 \leq \theta_0 \leq 1\) are known.

\[\text{30}\]

In the second row of \(J\) we applied the Implicit function theorem numerically to (19). Moreover we plotted the characteristic polynomial with respect to changes in one parameter. It turned out that the constellation of eigenvalues is robust. For \(\varepsilon < 2\), the system becomes unstable.
C Figures

Figure 1: World depletion rate of crude-oil. Sources: Weil (2005) and own calculations.

Figure 2: Reaction of the long-run depletion rate of natural resources, $\tau_*$ with respect to changes in the teacher-student ratio, $\phi$ and relative research productivity $\eta = \frac{\eta^s}{\eta^u}$. 
Figure 3: Experiment 1 - impulse response functions with respect to an increase in the skilled-unskilled population ratio $\frac{L_s}{L_u}$. 
Figure 4: Experiment 2 - impulse response functions with respect to a reduction in relative research productivity $\frac{\eta^s}{\eta^u}$. 
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Figure 5: Experiment 3 - impulse response functions with respect to an increase in the teacher-student ratio, $\phi$. 