Applied Cooperative Game Theory
Topic 1: Basic Definitions

André Casajus, Martin Kohl and Maria Näther

University of Leipzig

October 2013
- Peleg, B. /Sudhölter P.: Introduction to the theory of cooperative games; Springer 2007
  overview of the field of cooperative game theory

  overview of the field of game theory, but main topic is noncooperative
  game theory

- Wiese, H.: Kooperative Spieltheorie; Oldenbourg 2005
  German text book

- Fischer, G.: Lineare Algebra- Eine Einführung für Studienanfänger
  for repetition of terms like linear mapping, linear independence, basis,...
methods of game theory have been used since the 19th century (Bertrand, Cournot), but no general theory, just applications to unique problems

first real theory bases on work of John Neumann (1928) and his book together with Oskar Morgenstern (Theory of Games and Economic Behavior, 1944)

8 game theorists won Nobel Memorial Prize in Economic Sciences: John Nash, John Harsanyi, Reinhard Selten in 1994
Robert Aumann and Thomas Scheling in 2005
Roger Myerson in 2007
Alvin Roth and Lloyd Shapley in 2012

Game theory consists of two main theories, non-cooperative game theory and cooperative game theory

mile stone of non-cooperative game theory: definition of Nash equilibrium by John Nash in 1950

this theory tries to find optimal actions and strategies, cooperation must be self-enforcing

in cooperative game theory, cooperation often is self-enforcing. Groups of players and not individual players compete
TU games

- cooperative games with transferable utility

Definition

A TU game is a pair \((N, v)\), where \(N\) is a no-empty and finite set— the \textbf{player set} — and
\[
v \in V(N) := \{ f : 2^N \to \mathbb{R} | f(\emptyset) = 0 \}
\]
— the \textbf{characteristic function} (or \textbf{coalition function}).

- players: persons, institutions, cost centers …
- subsets of \(N\)/ elements of \(2^N\): \textbf{coalitions}
- set of non-empty subsets: \(\mathcal{K}(N) := 2^N \setminus \{\emptyset\}\)
- set of all coalition functions on \(N\): \(V(N)\)
- coalition function assigns the \textbf{worth} \(v(K)\) to any coalition \(K\)
  - can be distributed among the members arbitrarily
  - note, more is not necessarily better, example costs
Cost Division #1

- doctors with a common secretary or commonly used facilities
- firms organized as a collection of profit-centers
- universities with computing facilities used by several departments or faculties

Definition

For a player set $N$, let $c : 2^N \to \mathbb{R}_+$ be a coalition function that is called a cost function. On the basis of $c$, the costs-saving game is defined by $v : 2^N \to \mathbb{R}$ where for each $K \subseteq N$

$$v(K) = \sum_{i \in K} c(\{i\}) - c(K).$$
Cost Division #2

- Two towns A and B plan a water-distribution system.
- Town A could build such a system for itself at a cost of 11 million Euro.
- Town B would need 7 million Euro for a system tailor-made to its needs.
- The cost for a common water-distribution system is 15 million Euro.
- The cost function is then given by

\[
c(\{A\}) = 11, \\
c(\{B\}) = 7, \\
c(\{A, B\}) = 15.
\]

- The associated cost-savings game is \( v : 2^{\{A, B\}} \rightarrow \mathbb{R} \) defined by

\[
v(\{A\}) = v(\{B\}) = 0, \\
v(\{A, B\}) = 11 + 7 - 15.
\]
Example: German Federal Parliament
(Deutscher Bundestag)

- distribution of seats (total 631)
  
1. CDU/CSU (Christian democrats): 311
2. SPD (social democrats): 193
3. DIE LINKE (left party): 64
4. BÜNDNIS 90/DIE GRÜNEN (green party): 63

- simple majority rule: 316 votes required
- player set: $N = \{1, 2, 3, 4\}$
- coalition function:

$$\nu(K) = \begin{cases} 
1, & \text{parties in } K \text{ form a majority,} \\
0, & \text{otherwise,} 
\end{cases} \quad K \subseteq N$$
Example: Weighted voting games

- given by: \([q, w_1, \ldots, w_n]\), \(q, w_1, \ldots, w_n \in \mathbb{R}^+_0\)
- player set: \(N = \{1, \ldots, n\}\)
- coalition function:

\[
    v(K) = \begin{cases} 
        1, & \sum_{i \in K} w_i \geq q, \\
        0, & \sum_{i \in K} w_i < q,
    \end{cases} \quad K \subseteq N
\]

- winning coalitions: \(v(K) = 1\)
- loosing coalitions: \(v(K) = 0\)
Example: Multiple linear regression

■ exotic example
■ observations of the dependent variable: \((y_1, \ldots, y_T)\)
■ set of independent variables \(N\)
■ observations of independent variables: \((x^i_1, \ldots, x^i_T), i \in N\)
■ perform linear regression using the independent variables in \(K \subseteq N\)
■ coalition function using the coefficient of determination \(R^2\)

\[\nu(K) = \begin{cases} R^2(K), & K \neq \emptyset, \\ 0, & K = \emptyset, \end{cases}\quad K \subseteq N\]
Example: Strategic form games

- strategic form game: \((N, (S_i)_{i \in N}, (u_i)_{i \in N})\)
  - players set \(N\), non-empty, finite
  - strategy sets \(S_i, i \in N\), non-empty, finite
  - \(K \subseteq N, S_K := \prod_{i \in K} S_i, S := S_N\)
  - payoff functions \(u_i : S \to \mathbb{R}, i \in N\)
  - coalition function:

\[
    v(K) = \max_{s_K \in S_K} \min_{s_{N \setminus K} \in S_{N \setminus K}} \sum_{i \in K} u_i\left(s_K, s_{N \setminus K}\right), \quad K \subseteq N
\]
Null game, standard games, unanimity games

- **Null game**: \( 0 \in \mathbb{V}(N) \), \( 0(K) = 0 \) for all \( K \subseteq N \)
- **standard game**: \( e_T \in \mathbb{V}(N) \), \( T \in 2^N \setminus \{\emptyset\} \)
  
  \[
e_T(K) = \begin{cases} 
  1, & K = T, \\
  0, & K \neq T,
\end{cases} \quad K \subseteq N
\]

- **unanimity games**: \( u_T \in \mathbb{V}(N) \), \( T \in 2^N \setminus \{\emptyset\} \)
  
  \[
u_T(K) = \begin{cases} 
  1, & T \subseteq K, \\
  0, & T \not\subseteq K,
\end{cases} \quad K \subseteq N
\]
The vector space of coalition functions on $N$

- $\mathcal{V}(N)$ is a finite-dimensional real vector space
  - addition: $v + w \in \mathcal{V}(N) : (v + w)(K) = v(K) + w(K)$ for all $K \subseteq N$, $v, w \in \mathcal{V}(N)$
  - scalar multiplication: $\alpha \cdot v \in \mathcal{V}(N) : (\alpha \cdot v)(K) = \alpha \cdot v(K)$ for all $K \subseteq N$, $v \in \mathcal{V}(N), \alpha \in \mathbb{R}$

- dimension: $|2^N \setminus \{\emptyset\}| = 2^{|N|} - 1$

- bases:
  - standard basis: $(e_T)_{T \in \mathcal{K}(N)}$ (obviously: $v = \sum_{T \in \mathcal{K}(N)} v(T) \cdot e_T$)
  - unanimity basis: $(u_T)_{T \in \mathcal{K}(N)}$ (to be shown!)
  - of course, there are many more, among them the following (interesting) one:

$$w_T(K) = \begin{cases} \binom{|K|}{|T|}^{-1}, & T \subseteq K, \\ 0, & T \nsubseteq K, \end{cases} \quad K \subseteq N, \ T \in 2^N \setminus \{\emptyset\}$$
Unanimity basis (1)—Harsanyi dividends

- $(u_T)_{T \in 2^N \setminus \{\emptyset\}}$ is a basis $V(N)$
- any $v \in V(N)$ has a unique representation

$$v = \sum_{T \in \mathcal{K}(N)} \lambda_T(v) \cdot u_T, \quad \lambda_T(v) \in \mathbb{R}$$

- define the so-called **Harsanyi dividends** $\lambda_T(v)$ inductively

  $$\lambda_T(v) = \begin{cases} v(T), & |T| = 1 \\ v(T) - \sum_{S \in \mathcal{K}(T) \setminus \{T\}} \lambda_S(v), & |T| > 1 \end{cases}$$

- or explicitly

  $$\lambda_T(v) = \sum_{S \in \mathcal{K}(T)} (-1)^{|T| - |S|} v(S)$$
Harsanyi dividends: Equivalence #1

- both approaches uniquely define the $\lambda_T(v)$

- **Induction basis:** for $|T| = 1$:
  $$\lambda_T(v) = v(T) = (-1)^{|T|-1} v(T) = \sum_{S \in \mathcal{K}(T)} (-1)^{|T|-|S|} v(S)$$

- **Induction hypothesis (H):**
  $$\lambda_T(v) = \sum_{S \in \mathcal{K}(T)} (-1)^{|T|-|S|} v(S)$$
  for all $T \in \mathcal{K}(N)$, $|T| \leq t$

- **Induction step:** $|T| = t + 1$

\[
\begin{align*}
\lambda_T(v) & = v(T) - \sum_{S \in \mathcal{K}(T) \setminus \{T\}} \lambda_S(v) \\
& \overset{H}{=} v(T) - \sum_{S \in \mathcal{K}(T) \setminus \{T\}} \sum_{K \in \mathcal{K}(S)} (-1)^{|S|-|K|} v(K) \\
& = v(T) - \sum_{K \in \mathcal{K}(T) \setminus \{T\}} v(K) \sum_{s=|K|}^{|T|-1} \left( \begin{array}{c} |T| - |K| \\ s - |K| \end{array} \right) (-1)^{s-|K|} \\
& = v(T) - \sum_{K \in \mathcal{K}(T) \setminus \{T\}} v(K) \sum_{s=0}^{|T|-|K|-1} \left( \begin{array}{c} |T| - |K| \\ s \end{array} \right) (-1)^s
\end{align*}
\]
Harsanyi dividends: Equivalence #2

- the binomial formula implies

$$0 = (1 - 1)^t = \sum_{k=0}^{t} \binom{t}{k} (-1)^k$$

- this gives

$$\lambda_T (v) = v(T) - \sum_{K \in \mathcal{K}(T) \backslash \{T\}} v(K) \left( -\left( \frac{|T|}{|T| - |K|} \right) \right) (-1)^{|T| - |K|}$$

$$= v(T) + \sum_{K \in \mathcal{K}(T) \backslash \{T\}} (-1)^{|T| - |K|} v(K)$$

$$= \sum_{K \in \mathcal{K}(T)} (-1)^{|T| - |K|} v(K)$$
Unanimity basis #2: System of generators

- if \( w = \sum_{T \in \mathcal{K}(N)} \lambda_T(v) \cdot u_T \), for all \( K \subseteq N \) we have

\[
\begin{align*}
    w(K) &= \sum_{T \in \mathcal{K}(N)} \lambda_T(v) \cdot u_T(K) = \sum_{T \in \mathcal{K}(K)} \lambda_T(v) \\
    &= \lambda_K(v) + \sum_{T \in \mathcal{K}(K) \backslash \{K\}} \lambda_T(v) \\
    &= v(K) - \sum_{T \in \mathcal{K}(K) \backslash \{K\}} \lambda_T(v) + \sum_{T \in \mathcal{K}(K) \backslash \{K\}} \lambda_T(v) \\
    &= v(K)
\end{align*}
\]

- indeed, \((u_T)_{T \in \mathcal{K}(N)}\) generates \( \mathcal{V}(N) \)
**Unanimity basis #3: Linear independence**

- $\mathbb{V}(N)$ has dimension $2^{|N|} - 1$ and the generating system $(u_T)_{T \in \mathcal{K}(N)}$ also has $2^{|N|} - 1$ elements $\Rightarrow (u_T)_{T \in \mathcal{K}(N)}$ is a basis of $\mathbb{V}(N)$

- Direct proof: $(u_T)_{T \in \mathcal{K}(N)}$ is linearly independent

- To show: $0 = \sum_{T \in \mathcal{K}(N)} \alpha_T \cdot u_T$, $\alpha_T \in \mathbb{R}$ implies $\alpha_T = 0$ for all $T \in \mathcal{K}(N)$

- Induction on $|T|$
  
  - **Induction basis**: for $|T| = 1$, i.e., $T = \{i\}$ for some $i \in N$ we have
    
    $$0 = 0(\{i\}) = \sum_{T' \in \mathcal{K}(N)} \alpha_{T'} \cdot u_{T'}(\{i\}) = \sum_{T' \in \mathcal{K}(\{i\})} \alpha_{T'} = \alpha_{\{i\}}$$

  - **Induction hypothesis (H)**: $\alpha_T = 0$ for all $T \in 2^N \setminus \{\emptyset\}$, $|T| \leq t$

  - **Induction step**: $|T| = t + 1$
    
    $$0 = 0(T) = \sum_{T' \in \mathcal{K}(N)} \alpha_{T'} \cdot u_{T'}(T) = \sum_{T' \in \mathcal{K}(T)} \alpha_{T'} \overset{H}{=} \alpha_T$$
Properties of TU games #1

- **Simplicity**: \( v(K) \in \{0, 1\} \) for all \( K \subseteq N \)
- **Non-negativity**: \( v(K) \geq 0 \) for all \( K \subseteq N \)
- **Monotonicity**: \( v(K) \geq v(S) \) for all \( K, S \subseteq N, S \subseteq K \)
  - monotonicity \( \Rightarrow \) non-negativity
- **0-normedness**: \( v(\{i\}) = 0 \) for all \( i \in N \)
- **0-normalization**: for \( v \in \mathbb{V}(N) \) define \( v^0 \in \mathbb{V}(N) \),
  \[
  v^0(K) = v(K) - \sum_{i \in K} v(\{i\}), \quad K \subseteq N
  \]
  \[
  v^0 = v - \sum_{i \in N} \lambda_{\{i\}}(v) \cdot u_{\{i\}}
  \]
Properties of TU games #2

- **Superadditivity**: $v(S \cup K) \geq v(S) + v(K)$ for all $K, S \subseteq N$, $S \cap K = \emptyset$
- **Subadditivity**: $v(S \cup K) \leq v(S) + v(K)$ for all $K, S \subseteq N$, $S \cap K = \emptyset$
  - $v \in \mathcal{V}(N)$ it subadditive $\iff -v \in \mathcal{V}(N)$ is superadditive
- **Modularity**: $v \in \mathcal{V}(N)$ is superadditive and subadditive
  - $\iff$ for all $K \subseteq N$: $v(K) = \sum_{i \in K} v(\{i\})$
- **Symmetry**: $v(K) = v(S)$ for all $S, K \subseteq N$, $|S| = |K|$
  - exists $f : \mathbb{R} \to \mathbb{R}$ such that $v(K) = f(|K|)$ for all $K \subseteq N$
Convexity #1

- **Convexity:**
  - Def (i): \( v( S \cup K ) + v( S \cap K ) \geq v( S ) + v( K ) \) for all \( K, S \subseteq N \)
  - Def (ii): \( v( K \cup \{ i \} ) - v( K ) \geq v( S \cup \{ i \} ) - v( S ) \) for all \( i \in N \setminus K \) and \( K, S \subseteq N, S \subseteq K \)

- We have to check that both definitions are equivalent:
  - (i) \( \Rightarrow \) (ii): \( S' \subseteq K' \), set in (i) \( K = K' \) and \( S = S' \cup \{ i \} \):
    \[
    v( S' \cup \{ i \} \cup K') + v( S' \cup \{ i \} \cap K') \geq v( S' \cup \{ i \} ) + v( K')
    
    v( \{ i \} \cup K') + v( S') \geq v( S' \cup \{ i \} ) + v( K')
    
    \]
  - (ii) \( \Rightarrow \) (i): \( S \setminus K = \{ i_1, \ldots, i_n \} \)
    
    \[
    v( K \cup \{ i_1 \} ) - v( K ) \geq v( ( S \cap K ) \cup \{ i_1 \} ) - v( S \cap K )
    
    v( K \cup \{ i_1, i_2 \} ) - v( K \cup \{ i_1 \} ) \geq v( ( S \cap K ) \cup \{ i_1, i_2 \} )
    
    \vdots
    
    v( K \cup S \setminus K ) - \ldots \geq v( ( S \cap K ) \cup S \setminus K ) - \ldots
    
    \text{sum up:} \quad v( K \cup S ) - v( K ) \geq v( S ) - v( S \cap K )
    \]
Convexity #2

- let $f : \mathbb{R} \to \mathbb{R}$ be convex, i.e.,
  $$f(\alpha \cdot x + (1 - \alpha) \cdot y) \leq \alpha \cdot f(x) + (1 - \alpha) \cdot f(y)$$
  for all $x, y \in \mathbb{R}$ and $\alpha \in [0, 1]$

- let $v \in \mathcal{V}(N)$, $v(K) = f(|K|)$, $K \subseteq N$, i.e., $v$ is symmetric

- then $v$ is convex

- to show: $v$ exhibits non-decreasing “marginal contributions”

- to see it suffices to establish
  $$f(x + 2) - f(x + 1) \geq f(x + 1) - f(x), \quad x \in \mathbb{R}$$

- proof:
  $$\frac{1}{2}f(x + 2) + \frac{1}{2}f(x) \geq f\left(\frac{1}{2}(x + 2) + \frac{1}{2}x\right) = f(x + 1)$$
  $$f(x + 2) + f(x) \geq 2 \cdot f(x + 1)$$
  $$f(x + 2) - f(x + 1) \geq f(x + 1) - f(x)$$
Problems

1. Show that \((w_T)_{T \in 2^N \setminus \{\emptyset\}}\),

\[
    w_T(K) = \begin{cases} 
    (|K|^{-1}, & T \subseteq K, \\
    0, & T \not\subseteq K, 
    \end{cases} \quad K \subseteq N, \ T \in 2^N \setminus \{\emptyset\},
\]

is a basis of \(V(N)\)!

2. For \(v \in V(N)\) and \(K \subseteq N\), express \(v(K)\) in terms of Harsanyi dividends!

3. Show the following statements are equivalent:
   1. \(v \in V(N)\) is modular.
   2. \(v(K) = \sum_{i \in K} v(\{i\})\) for all \(K \subseteq N\).
   3. \(v = \sum_{i \in N} x_i \cdot u_i\) for some \(x \in \mathbb{R}^N\).

4. Show that superadditivity does not imply monotonicity!

5. Show that superadditivity and non-negativity imply monotonicity!

6. Show that for any \(v \in V(N)\) one can find a modular game \(w \in V(N)\) such that \(v + w\) is monotonic!

7. Show that for any \(v \in V(N)\) one can find a convex game \(w \in V(N)\) such that \(v + w\) is convex!