Applied cooperative game theory:
Many games

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June 2009
Overview “Many games”

- Simple games
- Three non-simple games
- Cost-division games
- Endowment games
- Properties of coalition functions
**Simple games**

**Definition (monotonic game)**

A coalition function \( v \in \mathcal{V}_N \) is called monotonic if \( \emptyset \subseteq S \subseteq S' \) implies \( v(S) \leq v(S') \).

Thus, monotonicity means that the worth of a coalition cannot decrease if other players join. Simple games are a special subclass of monotonic games:

**Definition (simple game)**

A coalition function \( v \in \mathcal{V}_N \) is called simple if

- we have \( v(K) = 0 \) or \( v(K) = 1 \) for every coalition \( K \subseteq N \) and.
- \( v \) is monotonic.

Thus, if \( S' \) is a superset of \( S \) (or \( S \) a subset of \( S' \)), we cannot have \( v(S) = 1 \) and \( v(S') = 0 \).
Veto players and dictators

Definition (veto player, dictator)

Let \(v\) be a simple game. A player \(i \in N\) is called a veto player if

\[
v(N \setminus \{i\}) = 0
\]

holds. \(i\) is called a dictator if

\[
v(S) = \begin{cases} 
1, & i \in S \\
0, & \text{sonst}
\end{cases}
\]

holds for all \(S \subseteq N\).

Problem

- Can there be a coalition \(K\) such that \(v(K \setminus \{i\}) = 1\) for a veto player \(i\) or a dictator \(i\)?
- Is every veto player a dictator or every dictator a veto player?
### Definition (complement)

The set \( N \setminus K := \{ i \in N : i \notin K \} \) is called \( K \)'s complement (with respect to \( N \)).

### Definition (contradictory, decidable)

A simple game \( \nu \in \mathbb{V}_N \) is called non-contradictory if \( \nu(K) = 1 \) implies \( \nu(N \setminus K) = 0 \).

A simple game \( \nu \in \mathbb{V}_N \) is called decidable if \( \nu(K) = 0 \) implies \( \nu(N \setminus K) = 1 \).

### Problem

- *Show that a simple game with a veto player cannot be contradictory.*
- *A simple game with two veto players cannot be decidable.*
Unanimity games

Definition (unanimity game)

For any \( T \neq \emptyset \),

\[
u_T(K) = \begin{cases} 1, & K \supseteq T \\ 0, & \text{otherwise} \end{cases}
\]
defines a unanimity game.

- The players from \( T \) are the productive or powerful members of society.
  - Every player from \( T \) is a veto player and no player from \( N \setminus T \) is a veto player.
  - In a sense, the players from \( T \) exert common dictatorship.

- For example, each player \( i \in T \) possesses part of a treasure map.

Problem

*Find the core and the Shapley value for \( N = \{1, 2, 3, 4\} \) and \( u_{\{1,2\}} \).*
Apex games

**Definition (apex game)**

For \( i \in N \) with \( n \geq 2 \), the apex game \( h_i \) is defined by

\[
h_i(K) = \begin{cases} 
1, & i \in K \text{ and } K \setminus \{i\} \neq \emptyset \\
1, & K = N \setminus \{i\} \\
0, & \text{otherwise}
\end{cases}
\]

Player \( i \) is called the main, or apex, player of that game.

Generally, we work with apex games for \( n \geq 4 \).

**Problem**

- Consider \( h_1 \) for \( n = 2 \) and \( n = 3 \). How do these games look like?
- Is the apex player a veto player or a dictator?
- Show that the apex game is not contradictory and decidable.
- Find the Shapley value for the apex game \( h_1 \).
Weighted voting games

Definition (weighted voting game)

A voting game $v$ is specified by a quota $q$ and voting weights $g_i, i \in N$, and defined by

$$v(K) = \begin{cases} 
1, & \sum_{i \in K} g_i \geq q \\
0, & \sum_{i \in K} g_i < q 
\end{cases}$$

In that case, the voting game is also denoted by $[q; g_1, ..., g_n]$.

The apex game $h_1$ for $n$ players can be considered a weighted voting game given by

$$\left[n - 1; n - \frac{3}{2}, 1, ..., 1\right].$$

Problem

Consider the unanimity game $u_T$ given by $t < n$ and $T = \{1, ..., t\}$. Can you express it as a weighted voting game?
UN Security Council

- 5 permanent members: China, France, Russian Federation, the United Kingdom and the United States
- 10 non-permanent members
- For substantive matters, the voting rule can be described by

\[ [39; 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1] \]

where the weights 7 accrue to the five permanent and the weights 1 to the non-permanent members.

Problem

- Show that every permanent member is a veto player.
- Show also that the five permanent members need the additional support of four non-permanent ones.
- Is the Security Council’s voting rule non-contradictory and decidable?
For the fifteen members of the Security Council, we have

\[ 15! = 1.307.674.368.000 \]

different rank orders.

The Shapley values are

- 0.19627 for each permanent member
- 0.00186 für each non-permanent member.
Buying a car I

- Andreas (A) has a used car he wants to sell, Frank (F) and Tobias (T) are potential buyers with willingness to buy of 700 and 500, respectively.
- Coalition function:

\[
\begin{align*}
v(A) &= v(F) = v(T) = 0, \\
v(A, F) &= 700, \\
v(A, T) &= 500, \\
v(F, T) &= 0 \text{ and } \\
v(A, F, T) &= 700.
\end{align*}
\]
Buying a car II

The core is the set of those payoff vectors \((x_A, x_F, x_T)\) that fulfill

\[ x_A + x_F + x_T = 700 \]

and

\[ x_A \geq 0, x_F \geq 0, x_T \geq 0, \]
\[ x_A + x_F \geq 700, \]
\[ x_A + x_T \geq 500 \text{ and} \]
\[ x_F + x_T \geq 0. \]
Buying a car III

- Tobias obtains

\[ x_T = 700 - (x_A + x_F) \text{ (efficiency)} \]
\[ \leq 700 - 700 \text{ (by } x_A + x_F \geq 700) \]
\[ = 0 \]

- and hence zero, \( x_T = 0 \).
- By \( x_A + x_T \geq 500 \), the seller Andreas can obtain at least 500.
- The core is the set of vectors \((x_A, x_F, x_T)\) obeying

\[
500 \leq x_A \leq 700,
\]
\[
x_F = 700 - x_A \text{ and}
\]
\[
x_T = 0.
\]

- Therefore, the car sells for a price between 500 and 700.
The Maschler game

Coalition function:

\[ v(K) = \begin{cases} 
0, & |K| = 1 \\
60, & |K| = 2 \\
72, & |K| = 3 
\end{cases} \]

Core:
- Efficiency:
  \[ x_1 + x_2 + x_3 = 72 \]
- and non-blockability:
  \[ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, \]
  \[ x_1 + x_2 \geq 60, x_1 + x_3 \geq 60 \text{ and } x_2 + x_3 \geq 60. \]

- Summing the last three inequalities yields
  \[ 2x_1 + 2x_2 + 2x_3 \geq 3 \cdot 60 = 180 \]
  and hence a contradiction to efficiency.
- The core is empty!
The gloves game, once again I

- Gloves game with minimal scarcity:

  \[ L = \{1, 2, \ldots, 100\} \]
  \[ R = \{101, \ldots, 199\} . \]

- Are the right-hand glove owners much better off?

- If

  \[ x = (x_1, \ldots, x_{100}, x_{101}, \ldots, x_{199}) \in \text{core} \left( v_{L,R} \right) \]

  then, by efficiency,

  \[ \sum_{i=1}^{199} x_i = 99. \]
We now pick any left-glove holder \( j \in \{1, 2, \ldots, 100\} \). We find

\[
\nu (L \setminus \{j\} \cup R) = 99
\]

and hence

\[
x_j = 99 - \sum_{i=1, i \neq j}^{199} x_i \quad \text{(efficiency)}
\]

\[
\leq 99 - 99 \quad \text{(blockade by coalition } L \setminus \{j\} \cup R) \]

\[
= 0.
\]

Therefore, we have \( x_j = 0 \) for every \( j \in L \).

Every right-glove owner can claim at least 1 because he can point to coalitions where he is joined by at least one left-glove owner.

Therefore, every right-glove owner obtains the payoff 1 and every left-glove owner the payoff zero.
Cost division games I

- doctors with a common secretary or commonly used facilities
- firms organized as a collection of profit-centers
- universities with computing facilities used by several departments or faculties

Definition (cost-division game)

For a player set $N$, let $c : 2^N \to \mathbb{R}_+$ be a coalition function that is called a cost function. On the basis of $c$, the cost-savings game is defined by $v : 2^N \to \mathbb{R}$ and

$$v(K) = \sum_{i \in K} c(\{i\}) - c(K), \quad K \subseteq N.$$

The idea behind this definition is that cost savings can be realized if players pool their resources so that $\sum_{i \in K} c(\{i\})$ is greater than $c(K)$ and $v(K)$ is positive.
Two towns $A$ and $B$ plan a water-distribution system.

Cost:
- Town $A$ could build such a system for itself at a cost of 11 million Euro and
- town $B$ would need 7 million Euro for a system tailor-made to its needs.
- The cost for a common water-distribution system is 15 million Euro.

The cost function is given by

$$c (\{A\}) = 11, \quad c (\{B\}) = 7 \quad \text{and} \quad c (\{A, B\}) = 15.$$ 

The associated cost-savings game is $\nu : 2^{\{A, B\}} \rightarrow \mathbb{R}$ defined by

$$\nu (\{A\}) = 0, \quad c (\{B\}) = 0 \quad \text{and} \quad \nu (\{A, B\}) = 7 + 11 - 15 = 3.$$
Cost division games III

- ν’s core is obviously given by

\[ \{(x_A, x_B) \in \mathbb{R}^2_+ : x_A + x_B = 3\} \].

- The cost savings of $3 = 11 + 7 - 15$ can be allotted to the towns such that no town is worse off compared to going alone. Thus, the set of undominated cost allocations is

\[ \{(c_A, c_B) \in \mathbb{R}^2 : c_A + c_B = 15, c_A \leq 11, c_B \leq 7\} \].
Problem

Calculate the Shapley values for c and v! Comment!
Definition (endowment economy)

An endowment economy is a tuple

$$\mathcal{E} = \left( N, \mathcal{L}, (\omega^i)_{i \in N}, f \right)$$

consisting of

- the set of agents $N = \{1, 2, \ldots, n\}$,
- the finite set of goods $\mathcal{L} = \{1, \ldots, \ell\}$,
- for every agent $i \in N$, an endowment $\omega^i = (\omega^i_1, \ldots, \omega^i_\ell) \in \mathbb{R}_+^\ell$ where

$$\omega := \sum_{i \in N} \omega^i = \left( \sum_{i \in N} \omega^i_1, \ldots, \sum_{i \in N} \omega^i_\ell \right)$$

is the economy’s total endowment, and ...
...and

- an aggregation function \( f : \mathbb{R}^\ell \rightarrow \mathbb{R} \).

The aggregation function aggregates the different goods' amounts into a specific real number in the same way as the min-operator does in the gloves game.

**Definition (endowment game)**

Consider an endowment economy \( \mathcal{E} \). An endowment game \( \nu^\mathcal{E} : 2^N \rightarrow \mathbb{R} \) is defined by

\[
\nu^\mathcal{E} (K) := f \left( \sum_{i \in K} \omega_i^1, \ldots, \sum_{i \in K} \omega_i^\ell \right).
\]
Definition (summing of endowment games)

For a player set $N$, consider two endowment economies $E$ and $F$ with endowments $\omega_E$ and $\omega_F$ and the derived endowment games $v_E$ and $v_F$. The endowment-based sum of these games is denoted by $v_E \oplus v_F$ and defined by

$$\omega^i_j = (\omega^E)_j^i + (\omega^F)_j^i$$

and

$$(v_E \oplus v_F)(K) : = f \left( \sum_{i \in K} \omega^i_1, ..., \sum_{i \in K} \omega^i_\ell \right).$$

This summation operation is not (!) the summation defined in the vector space of coalition functions!
Definition (gains from trade)

For a player set $N$, consider two endowment economies $\mathcal{E}$ and $\mathcal{F}$. The gains from trade are defined by

$$GfT(\mathcal{E}, \mathcal{F}) = (v_\mathcal{E} \oplus v_\mathcal{F})(N) - v_\mathcal{E}(N) - v_\mathcal{F}(N).$$

Problem

Show that the gains from trade are zero for any gloves game $v_\mathcal{E} := v_{L,R}$ and $v_\mathcal{F} := v_\mathcal{E}$. 
Superadditivity

Nearly all the coalition functions we work with in this book are superadditive. Roughly, superadditivity means that cooperation pays.

**Definition (superadditivity)**

A coalition function $v \in \mathcal{V}_N$ is called superadditive if for any two coalitions $R$ and $S$

$$R \cap S = \emptyset$$

implies

$$v(R) + v(S) \leq v(R \cup S).$$

$v(R \cup S) - (v(R) + v(S)) \geq 0$ is called the gain from cooperation.

**Problem**

Are gloves games superadditive? How about the apex game and unanimity games?
Definition (convexity)

A coalition function \( v \in \mathcal{V}_N \) is called convex if for any two coalitions \( S \) and \( S' \) with \( S \subseteq S' \) and for all players \( i \in N\setminus S' \), we have

\[
v(S \cup \{i\}) - v(S) \leq v(S' \cup \{i\}) - v(S').
\]

\( v \) is called strictly convex if the inequality is strict.

Problem

Is the unanimity game \( u_T \) convex? Is \( u_T \) strictly convex? Hint: Distinguish between \( i \in T \) and \( i \notin T \).
Convexity: Illustration of the term

\[ v(\{1,2,3\}) = f(3) \]
\[ = f(2) + [f(3) - f(2)] \]

\[ v(\{1,2\}) = f(2) \]
\[ = f(1) + [f(2) - f(1)] \]

\[ v(\{1\}) = f(1) = f(1) - f(0) \]

\[ v(\emptyset) = f(0) \]

number of players

0 1 2 3
Theorem (criterion for convexity)

A coalition function \( v \) is convex if and only if for all coalitions \( R \) and \( S \), we have

\[
v(R \cup S) + v(R \cap S) \geq v(R) + v(S).
\]

\( v \) is strictly convex if and only if

\[
v(R \cup S) + v(R \cap S) > v(R) + v(S)
\]

holds for all coalitions \( R \) and \( S \) with \( R \setminus S \neq \emptyset \) and \( S \setminus R \neq \emptyset \).

Problem

*Is the Maschler game convex? Is it superadditive?*

Theorem

*Every convex coalition function is superadditive.*
The Shapley value and the core

The Shapley value need not be in the core even if the core is nonempty.

Problem

Consider the coalition function given by \( N = \{1, 2, 3\} \) and

\[
\begin{align*}
v(K) &= \begin{cases} 
0, & |K| = 1 \\
\frac{1}{2}, & K = \{1, 3\} \text{ or } K = \{2, 3\} \\
\frac{8}{10}, & K = \{1, 2\} \\
1, & K = \{1, 2, 3\}
\end{cases}
\end{align*}
\]

Show that \( \left( \frac{4}{10}, \frac{4}{10}, \frac{2}{10} \right) \) belongs to the core but that the Shapley value does not.

Theorem

If a coalition function \( v \) is convex, the Shapley value \( Sh(v) \) lies in the core.