Advanced Microeconomics
Pareto optimality in microeconomics

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Part D. Bargaining theory and Pareto optimality

1. Pareto optimality in microeconomics
2. Cooperative game theory
Pareto optimality in microeconomics

overview

1. Introduction: Pareto improvements
2. Identical marginal rates of substitution
3. Identical marginal rates of transformation
4. Equality between marginal rate of substitution and marginal rate of transformation
Pareto optimality in microeconomics

overview

- **Introduction: Pareto improvements**
- Identical marginal rates of substitution
- Identical marginal rates of transformation
- Equality between marginal rate of substitution and marginal rate of transformation
Introduction: Pareto improvements

- Judgements of economic situations
- Ordinal utility \(\approx\) comparison among different people
- Vilfredo Pareto, Italian sociologist, 1848-1923:

**Definition**

- Situation 1 is called Pareto superior to situation 2 (a Pareto improvement over situation 2) if no individual is worse off in the first than in the second while at least one individual is strictly better off.
- Situations are called Pareto efficient, Pareto optimal or just efficient if Pareto improvements are not possible.
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MRS $= \text{MRS}$

The Edgeworth box for two consumers

Francis Ysidro Edgeworth (1845-1926): “Mathematical Psychics”
MRS = MRS
The Edgeworth box for two consumers

Problem
Which bundles of goods does individual A prefer to his endowment?
Consider

\[(3 =) \left| \frac{dx_2^A}{dx_1^A} \right| = MRS^A < MRS^B = \left| \frac{dx_2^B}{dx_1^B} \right| (= 5)\]

If A gives up a small amount of good 1,
he needs \(MRS^A\) units of good 2 in order to stay on his indifference curve.

If individual B obtains a small amount of good 1,
she is prepared to give up \(MRS^B\) units of good 2.
\(\frac{MRS^A + MRS^B}{2}\) units of good 2 given to A by B leave both better off

Ergo: Pareto optimality requires \(MRS^A = MRS^B\)
Pareto optima in the Edgeworth box – contract curve aka exchange curve
Problem

Two consumers meet on an exchange market with two goods. Both have the utility function $U(x_1, x_2) = x_1x_2$. Consumer A’s endowment is $(10, 90)$, consumer B’s is $(90, 10)$.

a) Depict the endowments in the Edgeworth box!

b) Find the contract curve and draw it!

c) Find the best bundle that consumer B can achieve through exchange!

d) Draw the Pareto improvement (exchange lens) and the Pareto-efficient Pareto improvements!
**MRS = MRS**

The Edgeworth box for two consumers

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### Solution

b) \( x_A^1 = x_A^2 \),

c) \( (x_B^1, x_B^2) = (70, 70) \).

d) The exchange lens is dotted. The Pareto efficient Pareto improvements are represented by the contract curve within this lens.
The Exchange Edgeworth Box

the generalized Edgeworth box

Generalization

- \( n \) households, \( i \in N := \{1, 2, \ldots, n\} \)
- \( \ell \) goods, \( g = 1, \ldots, \ell \)
- \( \omega^i_g \) – \( i \)'s endowment of good \( g \)
- \( \omega^i := (\omega^i_1, \ldots, \omega^i_\ell) \) and \( \omega^g := (\omega^1_g, \ldots, \omega^n_g) \)
- \( \sum_{i=1}^n \omega^i \neq \sum_{g=1}^\ell \omega^g \)

Problem

Consider two goods and three households and explain \( \omega^3 \), \( \omega^1 \) and \( \omega \).
Definition

- Functions $N \rightarrow \mathbb{R}^\ell_+$, i.e. vectors $(x^i)_{i=1,\ldots,n}$ or $(x^i)_{i \in N}$ where $x^i$ is a bundle from $\mathbb{R}^\ell_+$ – allocations.
- Feasible allocations fulfill

$$\sum_{i=1}^{n} x^i \leq \sum_{i=1}^{n} \omega^i$$
MR(T)S = MR(T)S

The production Edgeworth box for two products

- Analogous to exchange Edgeworth box

\[ MRTS_1 = \left| \frac{dC_1}{dL_1} \right| \]

- Pareto efficiency

\[
\left| \frac{dC_1}{dL_1} \right| = MRTS_1 \neq MRTS_2 = \left| \frac{dC_2}{dL_2} \right|
\]
A firm that produces in one factory but supplies two markets 1 and 2.

Marginal revenue $MR = \frac{dR}{dy_i}$ can be seen as the monetary marginal willingness to pay for selling one extra unit of good $i$.

- Denominator good $\rightarrow$ good 1 or 2, respectively
- Nominator good $\rightarrow$ “money” (revenue).

Profit maximization by a firm selling on two markets 1 and 2 implies

$$\left| \frac{dR}{dy_1} \right| = MR_1 \quad ; \quad \frac{dR}{dy_2} = MR_2$$
The monetary marginal willingness to pay for producing \textit{and} selling one extra unit of good $y$ is a marginal rate of substitution.

Two firms in a cartel maximize

$$\Pi_{1,2}(y_1, y_2) = \Pi_1(y_1, y_2) + \Pi_2(y_1, y_2)$$

with FOCs

$$\frac{\partial \Pi_{1,2}}{\partial y_1} \neq 0 \neq \frac{\partial \Pi_{1,2}}{\partial y_2}$$

If $\frac{\partial \Pi_{1,2}}{\partial y_2}$ were higher than $\frac{\partial \Pi_{1,2}}{\partial y_1}$ ...

How about the Cournot duopoly with FOCs

$$\frac{\partial \Pi_1}{\partial y_1} \neq 0 \neq \frac{\partial \Pi_2}{\partial y_2}?$$
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Marginal cost $MC = \frac{dC}{dy}$ is a monetary marginal opportunity cost of production

$$MRT = \left| \frac{dy_2}{dy_1} \right|$$

Transformation curve

One firm with two factories or a cartel in case of homogeneous goods:

$$MC_1 = MC_2.$$ 

Pareto improvements (optimality) have to be defined relative to a specific group of agents!
David Ricardo (1772–1823)
“comparative cost advantage”, for example

\[ 4 = MRT^P = \left| \frac{dW}{dCl} \right|^P > \left| \frac{dW}{dCl} \right|^E = MRT^E = 2 \]

**Lemma**

*Assume that \( f \) is a differentiable transformation function \( y_1 \mapsto y_2 \). Assume also that the cost function \( C(y_1, y_2) \) is differentiable. Then, the marginal rate of transformation between good 1 and good 2 can be obtained by*

\[ MRT(x_1) = \left| \frac{df(y_1)}{dy_1} \right| = \frac{MC_1}{MC_2}. \]
Proof.

- Assume a given volume of factor endowments and given factor prices. Then, the overall cost for the production of goods 1 and 2 are constant along the transformation curve:

\[ C(y_1, y_2) = C(y_1, f(y_1)) = \text{constant}. \]

- Forming the derivative yields

\[ \frac{\partial C}{\partial y_1} + \frac{\partial C}{\partial y_2} \frac{df(y_1)}{dy_1} = 0. \]

- Solving for the marginal rate of transformation yields

\[ MRT = -\frac{df(y_1)}{dy_1} = \frac{MC_1}{MC_2}. \]
Before Ricardo:
England exports cloth and imports wine if

\[ MC^E_{CI} < MC^P_{Cl} \text{ and } MC^E_{W} > MC^P_{W} \]

hold.

Ricardo:

\[ \frac{MC^E_{CI}}{MC^E_{W}} < \frac{MC^P_{Cl}}{MC^P_{W}} \]

suffices for profitable international trade.
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Assume

\[ MRS = \left| \frac{dy_2}{dy_1} \right| \quad \text{indifference curve} \quad < \quad \left| \frac{dy_2}{dy_1} \right| \quad \text{transformation curve} = MRT \]

- If the producer reduces the production of good 1 by one unit ...
- Inequality points to a Pareto-inefficient situation
- Pareto-efficiency requires

\[ MRS \overset{!}{=} MRT \]
MRS = MRT

Perfect competition - output space

- FOC output space
  \[ p = MC \]

- Let good 2 be money with price 1
- \textit{MRS} is
  - consumer’s monetary marginal willingness to pay for one additional unit of good 1
  - equal to \( p \) for marginal consumer
- \textit{MRT} is the amount of money one has to forgo for producing one additional unit of good 1, i.e., the marginal cost
- Thus,
  \[
  \text{price} = \text{marginal willingness to pay} = \text{marginal cost}
  \]
  which is also fulfilled by first-degree price discrimination.
FOC input space

\[ MVP = p \frac{dy}{dx} = w \]

where

- the marginal value product \( MVP \) is the monetary marginal willingness to pay for the factor use and

- \( w \), the factor price, is the monetary marginal opportunity cost of employing the factor.
For the Cournot monopolist, the $MRS = MRT$ can be rephrased as the equality between

- the monetary marginal willingness to pay for selling – this is the marginal revenue $MR = \frac{dR}{dy}$ – and

- the monetary marginal opportunity cost of production, the marginal cost $MC = \frac{dC}{dy}$
MRS = MRT

Household optimum

Consuming household “produces” goods by using his income to buy them,
\[ m = p_1 x_1 + p_2 x_2, \]
which can be expressed with the transformation function
\[ x_2 = f (x_1) = \frac{m}{p_2} - \frac{p_1}{p_2} x_1. \]

Hence,
\[ MRS = MRT = MOC = \frac{p_1}{p_2}. \]
**Sum of MRS = MRT**

**Public goods**

- **Definition:** non-rivalry in consumption
- **Setup:**
  - $A$ and $B$ consume a private good $x$ ($x^A$ and $x^B$)
  - and a public good $G$
- **Optimality condition**

\[
\frac{dX^A}{dG} \text{ indifference curve} + \frac{dX^B}{dG} \text{ indifference curve} = \frac{d(x^A + x^B)}{dG} \text{ transformation curve} = \text{MRT}
\]

- Assume $MRS^A + MRS^B < MRT$. Produce one additional unit of the public good ...

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Sum of MRS = MRT

Public goods

- Good $x$ as the numéraire good (money with price 1)
- Then, the optimality condition simplifies: sum of the marginal willingness’ to pay equals the marginal cost of the good.
Problem

In a small town, there live 200 people $i = 1, \ldots, 200$ with identical preferences. Person $i$’s utility function is $U_i(x_i, G) = x_i + \sqrt{G}$, where $x_i$ is the quantity of the private good and $G$ the quantity of the public good. The prices are $p_x = 1$ and $p_G = 10$, respectively. Find the Pareto-optimal quantity of the public good.

Solution

- $MRT = \left| \frac{d\left(\sum_{i=1}^{200} x_i\right)}{dG} \right|$ equals $\frac{p_G}{p_x} = \frac{10}{1} = 10$.
- $MRS$ for inhabitant $i$ is $\left| \frac{dx_i}{dG} \right|$ indifference curve $\frac{MU_G}{MU_{x_i}} = \frac{1}{2\sqrt{G}} \cdot 1 = \frac{1}{2\sqrt{G}}$.
- Hence: $200 \cdot \frac{1}{2\sqrt{G}} = 10$ and $G = 100$. 

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Advanced Microeconomics
Agent $A$ has preferences on $(x_1, x_2)$, that can be represented by $u^A(x_1^A, x_2^A) = x_1^A$. Agent $B$ has preferences, which are represented by the utility function $u^B(x_1^B, x_2^B) = x_2^B$. Agent $A$ starts with $\omega_1^A = \omega_2^A = 5$, and $B$ has the initial endowment $\omega_1^B = 4, \omega_2^B = 6$.

(a) Draw the Edgeworth box, including
- $\omega$,
- an indifference curve for each agent through $\omega$!

(b) Is $(x_1^A, x_2^A, x_1^B, x_2^B) = (6, 0, 3, 11)$ a Pareto-improvement compared to the initial allocation?

(c) Find the contract curve!
Further exercises: Problem 2

Consider the player set $N = \{1, \ldots, n\}$. Player $i \in N$ has 24 hours to spend on leisure or work, $24 = l_i + t_i$ where $l_i$ denotes $i$’s leisure time and $t_i$ the number of hours that $i$ contributes to the production of a good that is equally distributed among the group. In particular, we assume the utility functions $u_i (t_1, \ldots, t_n) = l_i + \frac{1}{n} \sum_{j \in N} \lambda t_j$, $i \in N$. Assume $1 < \lambda$ and $\lambda < n$.

(a) Find the Nash equilibrium!

(b) Is the Nash equilibrium Pareto-efficient?