Advanced Microeconomics

Pareto optimality in microeconomics

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1. **Pareto optimality in microeconomics**
2. Cooperative game theory

Pareto optimality in microeconomics

overview

1. Introduction: Pareto improvements
2. Identical marginal rates of substitution
3. Identical marginal rates of transformation
4. Equality between marginal rate of substitution and marginal rate of transformation

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- **Introduction: Pareto improvements**
- Identical marginal rates of substitution
- Identical marginal rates of transformation
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Introduction: Pareto improvements

- Judgements of economic situations
- Ordinal utility $\leftrightsquigarrow$ comparison among different people
- Vilfredo Pareto, Italian sociologist, 1848-1923:

Definition

- Situation 1 is called Pareto superior to situation 2 (a Pareto improvement over situation 2) if no individual is worse off in the first than in the second while at least one individual is strictly better off.
- Situations are called Pareto efficient, Pareto optimal or just efficient if Pareto improvements are not possible.

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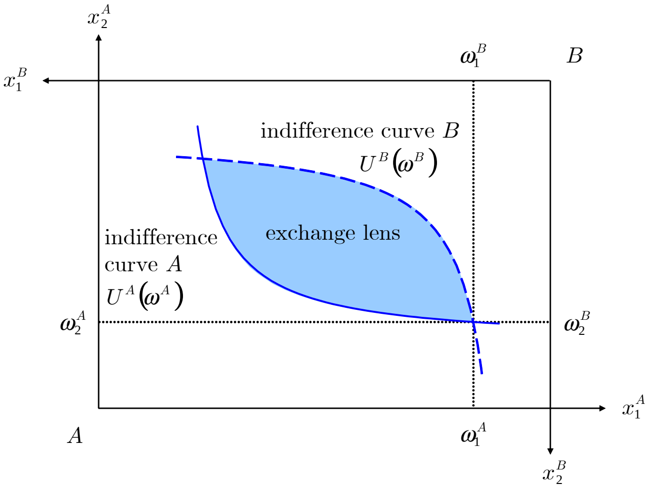
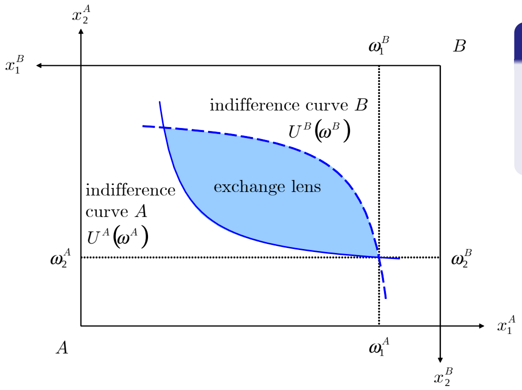
Francis Ysidro Edgeworth (1845-1926): "Mathematical Psychics"





Problem

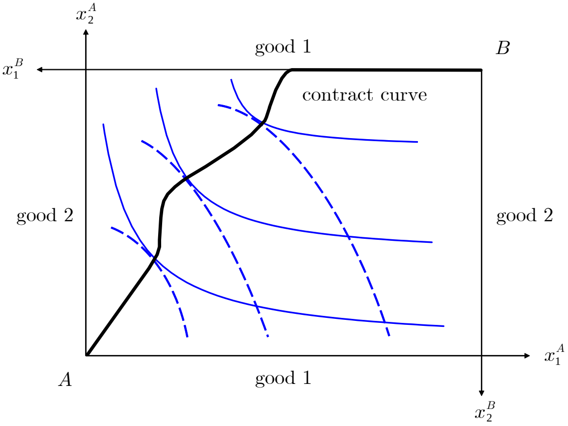
Which bundles of goods does individual A prefer to his endowment?

- Consider

$$(3 =) \left| \frac{dx_2^A}{dx_1^A} \right| = MRS^A < MRS^B = \left| \frac{dx_2^B}{dx_1^B} \right| (= 5)$$

- If A gives up a small amount of good 1,
- he needs MRS^A units of good 2 in order to stay on his indifference curve.
- If individual B obtains a small amount of good 1,
- she is prepared to give up MRS^B units of good 2.
- $\frac{MRS^A + MRS^B}{2}$ units of good 2 given to A by B leave both better off
- Ergo: Pareto optimality requires $MRS^A = MRS^B$

Pareto optima in the Edgeworth box
– contract curve aka exchange curve

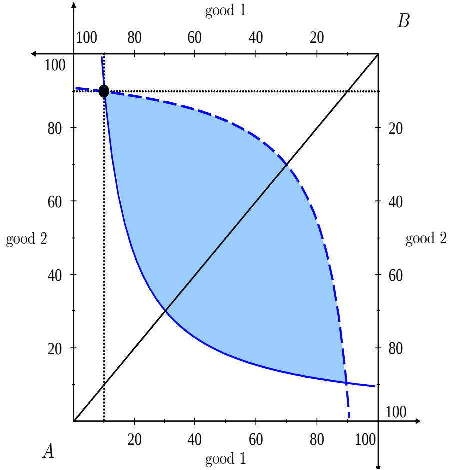


Problem

Two consumers meet on an exchange market with two goods. Both have the utility function $U(x_1, x_2) = x_1 x_2$. *Consumer A's endowment is* $(10, 90)$, *consumer B's is* $(90, 10)$.

a) Depict the endowments in the Edgeworth box!
b) Find the contract curve and draw it!
c) Find the best bundle that consumer B can achieve through exchange!
d) Draw the Pareto improvement (exchange lens) and the Pareto-efficient Pareto improvements!

a)



Solution

b) $x_1^A = x_2^A$,
c) $\left(x_1^B, x_2^B\right) = (70, 70)$.
d) *The exchange lens is dotted. The Pareto efficient Pareto improvements are represented by the contract curve within this lens.*

Generalization

- n households, $i \in N := \{1, 2, .., n\}$
- ℓ goods, $g = 1, ..., \ell$
- ω_g^i – i's endowment of good g
- $\omega^i := \left(\omega_1^i, ..., \omega_\ell^i\right)$ and $\omega_g := \left(\omega_g^1, ..., \omega_g^n\right)$
- $\sum_{i=1}^n \omega^i \neq \sum_{g=1}^\ell \omega_g$

Problem

Consider two goods and three households and explain ω^3, ω_1 and ω.

Definition

- Functions $N \longrightarrow \mathbb{R}_+^\ell$, i.e. vectors $(x^i)_{i=1,\ldots,n}$ or $\left(x^i\right)_{i\in N}$ where x^i is a bundle from $\mathbb{R}_+^\ell$– allocations.
- Feasible allocations fulfill

$$\sum_{i=1}^{n} x^i \leq \sum_{i=1}^{n} \omega^i$$

- Analogous to exchange Edgeworth box
- $MRTS_1 = \left|\frac{dC_1}{dL_1}\right|$
- Pareto efficiency

$$\left|\frac{dC_1}{dL_1}\right| = MRTS_1 \overset{!}{=} MRTS_2 = \left|\frac{dC_2}{dL_2}\right|$$

MRS = MRS

Two markets – one factory

- A firm that produces in one factory but supplies two markets 1 and 2.
- Marginal revenue $MR = \frac{dR}{dy_i}$ can be seen as the monetary marginal willingness to pay for selling one extra unit of good i.
    - Denominator good —> good 1 or 2, respectively
    - Nominator good —> "money" (revenue).
- Profit maximization by a firm selling on two markets 1 and 2 implies

$$\left|\frac{dR}{dy_1}\right| = MR_1 \overset{!}{=} MR_2 = \left|\frac{dR}{dy_2}\right|$$

MRS = MRS

Two firms in a cartel

- The monetary marginal willingness to pay for producing *and* selling one extra unit of good y is a marginal rate of substitution.
- Two firms in a cartel maximize

$$\Pi_{1,2}(y_1, y_2) = \Pi_1(y_1, y_2) + \Pi_2(y_1, y_2)$$

with FOCs

$$\frac{\partial \Pi_{1,2}}{\partial y_1} \overset{!}{=} 0 \overset{!}{=} \frac{\partial \Pi_{1,2}}{\partial y_2}$$

- If $\frac{\partial \Pi_{1,2}}{\partial y_2}$ were higher than $\frac{\partial \Pi_{1,2}}{\partial y_1}$ ...

How about the Cournot duopoly with FOCs

$$\frac{\partial \Pi_1}{\partial y_1} \overset{!}{=} 0 \overset{!}{=} \frac{\partial \Pi_2}{\partial y_2}?$$

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MRT = MRT

Two factories – one market

- Marginal cost $MC = \frac{dC}{dy}$ is a monetary marginal opportunity cost of production

$$MRT = \left| \frac{dy_2}{dy_1} \right|^{\text{transformation curve}}$$

- One firm with two factories or a cartel in case of homogeneous goods:

$$MC_1 \overset{!}{=} MC_2.$$

- Pareto improvements (optimality) have to be defined relative to a specific group of agents!

David Ricardo (1772–1823)
"comparative cost advantage", for example

$$4 = MRT^P = \left|\frac{dW}{dCl}\right|^P > \left|\frac{dW}{dCl}\right|^E = MRT^E = 2$$

Lemma

Assume that f is a differentiable transformation function $y_1 \mapsto y_2$. Assume also that the cost function $C(y_1, y_2)$ is differentiable. Then, the marginal rate of transformation between good 1 and good 2 can be obtained by

$$MRT = \left|\frac{df(y_1)}{dy_1}\right| = \frac{MC_1}{MC_2}.$$

Proof.

- Assume a given volume of factor endowments and given factor prices. Then, the overall cost for the production of goods 1 and 2 are constant along the transformation curve:

$$C\left(y_1, y_2\right) = C\left(y_1, f\left(y_1\right)\right) = \text{constant}.$$

- Forming the derivative yields

$$\frac{\partial C}{\partial y_1} + \frac{\partial C}{\partial y_2}\frac{df\left(y_1\right)}{dy_1} = 0.$$

- Solving for the marginal rate of transformation yields

$$MRT = -\frac{df\left(y_1\right)}{dy_1} = \frac{MC_1}{MC_2}.$$

- Before Ricardo:
  England exports cloth and imports wine if

$$\begin{aligned} MC^E_{Cl} &< MC^P_{Cl} \text{ and} \\ MC^E_W &> MC^P_W \end{aligned}$$

  hold.
- Ricardo:

$$\frac{MC^E_{Cl}}{MC^E_W} < \frac{MC^P_{Cl}}{MC^P_W}$$

  suffices for profitable international trade.

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MRS = MRT

Base case

- Assume

$$MRS = \left| \frac{dy_2}{dy_1} \right|^{\text{indifference curve}} < \left| \frac{dy_2}{dy_1} \right|^{\text{transformation curve}} = MRT$$

- If the producer reduces the production of good 1 by one unit ...
- Inequality points to a Pareto-inefficient situation
- Pareto-efficiency requires

$$MRS \stackrel{!}{=} MRT$$

- FOC output space

$$p \overset{!}{=} MC$$

- Let good 2 be money with price 1
- MRS is
  - consumer's monetary marginal willingness to pay for one additional unit of good 1
  - equal to p for marginal consumer
- MRT is the amount of money one has to forgo for producing one additional unit of good 1, i.e., the marginal cost
- Thus,

$$\text{price} = \text{marginal willingness to pay} \overset{!}{=} \text{marginal cost}$$

which is also fulfilled by first-degree price discrimination.

FOC input space

$$MVP = p\frac{dy}{dx} \overset{!}{=} w$$

where

- the marginal value product MVP is the monetary marginal willingness to pay for the factor use and
- w, the factor price, is the monetary marginal opportunity cost of employing the factor.

For the Cournot monopolist, the $MRS \overset{!}{=} MRT$ can be rephrased as the equality between

- the monetary marginal willingness to pay for selling – this is the marginal revenue $MR = \frac{dR}{dy}$ – and
- the monetary marginal opportunity cost of production, the marginal cost $MC = \frac{dC}{dy}$

MRS = MRT

Household optimum

Consuming household "produces" goods by using his income to buy them, $m = p_1 x_1 + p_2 x_2$, which can be expressed with the transformation function

$$x_2 = f(x_1) = \frac{m}{p_2} - \frac{p_1}{p_2} x_1.$$

Hence,

$$MRS \stackrel{!}{=} MRT = MOC = \frac{p_1}{p_2}$$

Sum of MRS = MRT

Public goods

- Definition: non-rivalry in consumption
- Setup:
  - A and B consume a private good x (x^A and x^B)
  - and a public good G
- Optimality condition

$$\begin{aligned} & MRS^A + MRS^B \\ = & \left|\frac{dx^A}{dG}\right|^{\text{indifference curve}} + \left|\frac{dx^B}{dG}\right|^{\text{indifference curve}} \\ \overset{!}{=} & \left|\frac{d\left(x^A + x^B\right)}{dG}\right|^{\text{transformation curve}} = MRT \end{aligned}$$

- Assume $MRS^A + MRS^B < MRT$. Produce one additional unit of the public good ...

Sum of MRS = MRT

- Good x as the numéraire good (money with price 1)
- Then, the optimality condition simplifies: sum of the marginal willingness' to pay equals the marginal cost of the good.

Problem

In a small town, there live 200 people $i = 1, ..., 200$ with identical preferences. Person i's utility function is $U_i(x_i, G) = x_i + \sqrt{G}$, where x_i is the quantity of the private good and G the quantity of the public good. The prices are $p_x = 1$ and $p_G = 10$, respectively. Find the Pareto-optimal quantity of the public good.

Solution

- $MRT = \left| \frac{d\left(\sum_{i=1}^{200} x_i\right)}{dG} \right|$ *equals* $\frac{p_G}{p_x} = \frac{10}{1} = 10.$
- *MRS for inhabitant i is* $\left| \frac{dx^i}{dG} \right|^{\text{indifference curve}} = \frac{MU_G}{MU_{x^i}} = \frac{\frac{1}{2\sqrt{G}}}{1} = \frac{1}{2\sqrt{G}}.$
- *Hence:* $200 \cdot \frac{1}{2\sqrt{G}} \overset{!}{=} 10$ *and* $G = 100.$

Further exercises: Problem 1

Agent A has preferences on (x_1, x_2), that can be represented by $u^A(x_1^A, x_2^A) = x_1^A$. Agent B has preferences, which are represented by the utility function $u^B(x_1^B, x_2^B) = x_2^B$. Agent A starts with $\omega_1^A = \omega_2^A = 5$, and B has the initial endowment $\omega_1^B = 4, \omega_2^B = 6$.

(a) Draw the Edgeworth box, including
- ω,
- an indifference curve for each agent through ω!

(b) Is $(x_1^A, x_2^A, x_1^B, x_2^B) = (6, 0, 3, 11)$ a Pareto-improvement compared to the initial allocation?

(c) Find the contract curve!

Further exercises: Problem 2

Consider the player set $N = \{1, ..., n\}$. Player $i \in N$ has 24 hours to spend on leisure or work, $24 = l_i + t_i$ where l_i denotes i's leisure time and t_i the number of hours that i contributes to the production of a good that is equally distributed among the group. In particular, we assume the utility functions $u_i(t_1, ..., t_n) = l_i + \frac{1}{n} \sum_{j \in N} \lambda t_j$, $i \in N$. Assume $1 < \lambda$ and $\lambda < n$.

(a) Find the Nash equilibrium!

(b) Is the Nash equilibrium Pareto-efficient?