

# Advanced Microeconomics

## Comparative statics and duality theory

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## Part B. Household theory and theory of the firm

- 1 The household optimum
- 2 **Comparative statics and duality theory**
- 3 Production theory
- 4 Cost minimization and profit maximization

# Comparative statics and duality theory

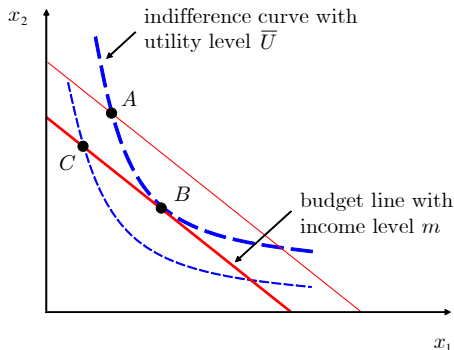
## Overview

- 1 The duality approach
- 2 Shephard's lemma
- 3 The Hicksian law of demand
- 4 Slutsky equations
- 5 Compensating and equivalent variations

# Maximization and minimization problem

- Maximization problem:  
Find the bundle that maximizes the utility for a given budget line.

- Minimization problem:  
Find the bundle that minimizes the expenditure needed to achieve a given utility level.



# The expenditure function and the Hicksian demand function I

Expenditure function:

$$e : \mathbb{R}^\ell \times \mathbb{R} \rightarrow \mathbb{R},$$
$$(p, \bar{U}) \mapsto e(p, \bar{U}) := \min_{\substack{x \text{ with} \\ U(x) \geq \bar{U}}} px$$

The solution to the minimization problem is called the Hicksian demand function:

$$\chi : \mathbb{R}^\ell \times \mathbb{R} \rightarrow \mathbb{R}_+^\ell,$$
$$(p, \bar{U}) \mapsto \chi(p, \bar{U}) := \arg \min_{\substack{x \text{ with} \\ U(x) \geq \bar{U}}} px$$

# The expenditure function and the Hicksian demand function II

## Problem

*Express*

- $e$  in terms of  $\chi$  and
- $V$  in terms of the household optima!

## Lemma

For any  $\alpha > 0$ :

$$\chi(\alpha p, \bar{U}) = \chi(p, \bar{U}) \text{ and } e(\alpha p, \bar{U}) = \alpha e(p, \bar{U}).$$

Obvious?!

# The expenditure function and the Hicksian demand function III

## Problem

Determine the expenditure function and the Hicksian demand function for the Cobb-Douglas utility function  $U(x_1, x_2) = x_1^a x_2^{1-a}$  with  $0 < a < 1$ !

Hint: We know that the indirect utility function  $V$  is given by:

$$\begin{aligned} V(p, m) &= U(x(p, m)) \\ &= \left( a \frac{m}{p_1} \right)^a \left( (1-a) \frac{m}{p_2} \right)^{1-a} \\ &= \left( \frac{a}{p_1} \right)^a \left( \frac{1-a}{p_2} \right)^{1-a} m. \end{aligned}$$

# Hicksian demand and the expenditure function

	utility maximization	expenditure minimization
objective function	utility	expenditure
parameters	prices $p$ , income $m$	prices $p$ , utility $\bar{U}$
notation for best bundle(s)	$x(p, m)$	$\chi(p, \bar{U})$
name of demand function	Marshallian	Hicksian
value of objective function	$V(p, m)$ $= U(x(p, m))$	$e(p, \bar{U})$ $= p \cdot \chi(p, \bar{U})$



# Applying the Lagrange method (recipe)

$$L(x, \mu) = \sum_{g=1}^{\ell} p_g x_g + \mu [\bar{U} - U(x)]$$

with:

- $U$  – strictly quasi-concave and strictly monotonic utility function;
- prices  $p \gg 0$ ;
- $\mu > 0$  – Lagrange multiplier - translates a utility surplus (in case of  $U(x) > \bar{U}$ ) into expenditure reduction
- Increasing consumption has
  - a positive direct effect on expenditure, but
  - a negative indirect effect via a utility surplus (scope for expenditure reduction) and  $\mu$

# Applying the Lagrange method (recipe)

Differentiate  $L$  with respect to  $x_g$  :

$$\frac{\partial L(x_1, x_2, \dots, \mu)}{\partial x_g} = p_g - \mu \frac{\partial U(x_1, x_2, \dots, x_\ell)}{\partial x_g} \stackrel{!}{=} 0 \text{ or } \frac{\partial U(x_1, x_2, \dots, x_\ell)}{\partial x_g} \stackrel{!}{=} \frac{p_g}{\mu}$$

and hence, for two goods  $g$  and  $k$

$$\frac{\frac{\partial U(x_1, x_2, \dots, x_\ell)}{\partial x_g}}{\frac{\partial U(x_1, x_2, \dots, x_\ell)}{\partial x_k}} \stackrel{!}{=} \frac{p_g}{p_k} \text{ or } MRS \stackrel{!}{=} MOC$$

Note also:  $\frac{\partial L(x, \mu)}{\partial \mu} = \bar{U} - U(x) \stackrel{!}{=} 0$

# Applying the Lagrange method

## Comparing the Lagrange multipliers

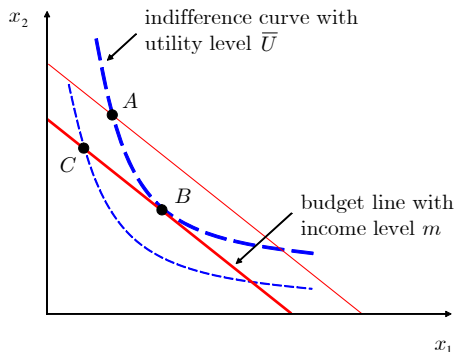
- $\lambda$  – the shadow price for utility maximization (translates additional income  $m$  into higher utility:  $\lambda = \frac{\partial V}{\partial m}$ );
- $\mu$  – the shadow price for expenditure minimization (translates additional utility  $\bar{U}$  into higher expenditure:  $\mu = \frac{\partial e(p, \bar{U})}{\partial \bar{U}}$ ).

We note without proof

$$\mu = \frac{1}{\lambda}$$

# The duality theorem

here, duality does work

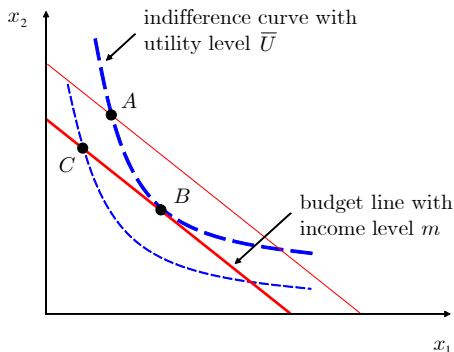


- At the budget line through point  $C$  (budget  $m$ ), the household optimum is at point  $B$  and the utility is  $U(B) = V(p, m)$ .
- The expenditure needed to obtain  $B$ 's utility level or  $A$ 's utility level is equal to  $m$ :

$$e(p, V(p, m)) = m$$

# The duality theorem

here, duality does work



- The minimal expenditure for the indifference curve passing through  $A$  (utility level  $\bar{U}$ ) is denoted by  $e(p, \bar{U})$  and achieved by bundle  $B$ .
- With income  $e(p, \bar{U})$  (at point  $C$ ), the highest achievable utility is  $V(p, e(p, \bar{U})) = \bar{U}$ .

# The duality theorem

## Conditions for duality

### Theorem

Let  $U : \mathbb{R}_+^\ell \rightarrow \mathbb{R}$  be a continuous utility function that obeys local nonsatiation and let  $p \gg 0$  be a price vector.  $\Rightarrow$

- If  $x(p, m)$  is the household optimum for  $m > 0$ :

$$\chi(p, V(p, m)) = x(p, m)$$

$$e(p, V(p, m)) = m.$$

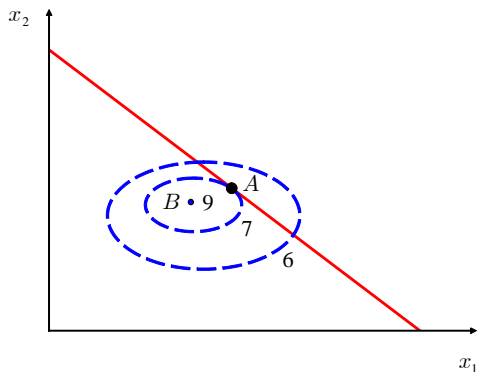
- If  $\chi(p, \bar{U})$  is the expenditure-minimizing bundle for  $\bar{U} > U(0)$ :

$$x(p, e(p, \bar{U})) = \chi(p, \bar{U})$$

$$V(p, e(p, \bar{U})) = \bar{U}.$$

# The duality theorem

here, duality does not work



- At  $(p, m)$ , the household optimum is at the bliss point with  $V(p, m) = 9$ .
- The expenditure needed to obtain this utility level is smaller than  $m$ :

$$e(p, V(p, m)) < m.$$

Here, local nonsatiation is violated.

## Theorem

Consider a household with a continuous utility function  $U$ , Hicksian demand function  $\chi$  and expenditure function  $e$ .

- *Shephard's lemma: The price increase of good  $g$  by one small unit increases the expenditure necessary to uphold the utility level by  $\chi_g$ .*
- *Roy's identity: A price increase of good  $g$  by one small unit decreases the budget available for the other goods by  $\chi_g$  and indirect utility by the product of the marginal utility of income  $\frac{\partial V}{\partial m}$  and  $\chi_g$ .*
- *Hicksian law of demand: If the price of a good  $g$  increases, the Hicksian demand  $\chi_g$  does not increase.*
- *The Hicksian cross demands are symmetric:  $\frac{\partial \chi_g(p, \bar{U})}{\partial p_k} = \frac{\partial \chi_k(p, \bar{U})}{\partial p_g}$ .*
- *Slutsky equations see next slide*



## Theorem

- *Money-budget Slutsky equation:*

$$\frac{\partial x_g}{\partial p_g} = \frac{\partial \chi_g}{\partial p_g} - \frac{\partial x_g}{\partial m} \chi_g.$$

- *Endowment Slutsky equation:*

$$\frac{\partial x_g^{\text{endowment}}}{\partial p_g} = \frac{\partial \chi_g}{\partial p_g} + \frac{\partial x_g^{\text{money}}}{\partial m} (\omega_g - \chi_g).$$

# Shephard's lemma

## Overview

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- 2 **Shephard's lemma**
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# Shephard's lemma

- Assume a price increase for a good  $g$  by one small unit.
- To achieve the same utility level, expenditure must be increased by at most

$$\frac{\partial e}{\partial p_g} \leq \chi_g$$

- Shephard's Lemma (see manuscript):

$$\frac{\partial e}{\partial p_g} = \chi_g$$

# Roy's identity

Duality equation:

$$\bar{U} = V(p, e(p, \bar{U})).$$

Differentiating with respect to  $p_g$ :

$$\begin{aligned} 0 &= \frac{\partial V}{\partial p_g} + \frac{\partial V}{\partial m} \frac{\partial e}{\partial p_g} \\ &= \frac{\partial V}{\partial p_g} + \frac{\partial V}{\partial m} \chi_g \quad (\text{Shephard's lemma}) \end{aligned}$$

Roy's identity:  $\frac{\partial V}{\partial p_g} = \frac{\partial V}{\partial m} (-\chi_g)$ .

# The Hicksian law of demand

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# Compensated (Hicksian) law of demand

Assume:

- $p$  and  $p'$  – price vectors from  $\mathbb{R}^\ell$ ;
- $\chi(p, \bar{U}) \in \mathbb{R}_+^\ell$  and  $\chi(p', \bar{U}) \in \mathbb{R}_+^\ell$  – expenditure-minimizing bundles necessary to achieve a utility of at least  $\bar{U}$ .

$$p' \cdot \underbrace{\chi(\bar{U}, p')}_{\substack{\text{expenditure-} \\ \text{minimizing} \\ \text{bundle at } p'}} \leq p' \cdot \underbrace{\chi(\bar{U}, p)}_{\substack{\text{expenditure-} \\ \text{minimizing} \\ \text{bundle at } p}} .$$

... (see the manuscript)

⇒ If the price of one good increases, the Hicksian demand for that good cannot increase:  $\frac{\partial \chi_g}{\partial p_g} \leq 0$ .

# Concavity and the Hesse matrix

concavity

## Definition

Let  $f : M \rightarrow \mathbb{R}$  be a function on a convex domain  $M \subseteq \mathbb{R}^\ell$ .

$\Rightarrow$

- $f$  is concave if

$$f(kx + (1 - k)y) \geq kf(x) + (1 - k)f(y)$$

for all  $x, y \in M$  and for all  $k \in [0, 1]$  (for  $\leq$  - convex).

- $f$  is strictly concave if

$$f(kx + (1 - k)y) > kf(x) + (1 - k)f(y)$$

holds for all  $x, y \in M$  with  $x \neq y$  and for all  $k \in (0, 1)$  (for  $<$  - strictly convex).

## Definition

Let  $f : \mathbb{R}^\ell \rightarrow \mathbb{R}$  be a function.

The second-order partial derivative of  $f$  with respect to  $x_i$  and  $x_j$  (if it exists) is given by

$$f_{ij}(x) := \frac{\partial \frac{\partial f(x)}{\partial x_i}}{\partial x_j} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}.$$

If all the second-order partial derivatives exist, the Hesse matrix of  $f$  is given by

$$H_f(x) = \begin{pmatrix} f_{11}(x) & f_{12}(x) & \dots & f_{1\ell}(x) \\ f_{21}(x) & f_{22}(x) & \dots & \dots \\ \vdots & \vdots & \ddots & \vdots \\ f_{\ell 1}(x) & f_{\ell 2}(x) & \dots & f_{\ell\ell}(x) \end{pmatrix}.$$

## Problem

Determine the Hesse matrix for  $f(x, y) = x^2y + y^2$ .



Believe me (or check in the script) that

## Lemma (diagonal entries)

**If** a function  $f : \mathbb{R}^\ell \rightarrow \mathbb{R}$  is concave (strictly concave), the diagonal entries of its Hesse matrix are non-positive (negative).

Remember: For  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f$  is concave **iff**  $f''(x) \leq 0$  for all  $x \in \mathbb{R}$  (chapter on von Neumann-Morgenstern utility)

## Lemma (symmetry)

*If all the second-order partial derivatives of  $f : \mathbb{R}^\ell \rightarrow \mathbb{R}$  exist and are continuous, then*

$$f_{ij}(x) = f_{ji}(x) \text{ for all } i, j = 1, \dots, \ell$$

## Lemma (expenditure fct. concave)

*The expenditure function is concave.*

# Hesse matrix of expenditure function

$$H_e(p, \bar{U}) = \begin{pmatrix} \frac{\partial^2 e(p, \bar{U})}{(\partial p_1)^2} & \frac{\partial^2 e(p, \bar{U})}{\partial p_1 \partial p_2} & \frac{\partial^2 e(p, \bar{U})}{\partial p_1 \partial p_\ell} \\ \frac{\partial^2 e(p, \bar{U})}{\partial p_2 \partial p_1} & \frac{\partial^2 e(p, \bar{U})}{(\partial p_2)^2} & \\ \frac{\partial^2 e(p, \bar{U})}{\partial p_\ell \partial p_1} & \frac{\partial^2 e(p, \bar{U})}{\partial p_\ell \partial p_2} & \frac{\partial^2 e(p, \bar{U})}{(\partial p_\ell)^2} \end{pmatrix}.$$

# Compensated (Hicksian) law of demand

By Shephard's lemma:

$$\frac{\partial e(p, \bar{U})}{\partial p_g} = \chi_g(p, \bar{U})$$

Forming the derivative of the Hicksian demand, we find

$$\frac{\partial \chi_g(p, \bar{U})}{\partial p_k} = \frac{\partial \frac{\partial e(p, \bar{U})}{\partial p_g}}{\partial p_k} = \frac{\partial^2 e(p, \bar{U})}{\partial p_g \partial p_k}$$

with two interesting conclusions:

1.  $g = k$

Hicksian law of demand (by lemma “expenditure fct. concave” and lemma “diagonal entries”):

$$\frac{\partial \chi_g(p, \bar{U})}{\partial p_g} \leq 0$$

# Substitutes and complements (the Hicksian definition)

2.  $g \neq k$  :

If the off-diagonal entries in the expenditure function's Hesse matrix are continuous, lemma "symmetry" implies

$$\frac{\partial \chi_g(p, \bar{U})}{\partial p_k} = \frac{\partial \chi_k(p, \bar{U})}{\partial p_g}.$$

## Definition

Goods  $g$  and  $k$  are

- substitutes if

$$\frac{\partial \chi_g(p, \bar{U})}{\partial p_k} = \frac{\partial \chi_k(p, \bar{U})}{\partial p_g} \geq 0;$$

- complements if

$$\frac{\partial \chi_g(p, \bar{U})}{\partial p_k} = \frac{\partial \chi_k(p, \bar{U})}{\partial p_g} \leq 0.$$

# Every good has at least one substitute.

For all  $\alpha > 0$  :

$$\chi_g(\alpha p, \bar{U}) = \chi_g(p, \bar{U}).$$

Differentiating with the adding rule (chapter on preferences) yields

$$\sum_{k=1}^{\ell} \frac{\partial \chi_g(\alpha p, \bar{U})}{\partial p_k} \cdot p_k = \frac{\partial \chi_g(\alpha p, \bar{U})}{\partial \alpha} = \frac{\partial \chi_g(p, \bar{U})}{\partial \alpha} = 0$$

$\Rightarrow$

$$\sum_{\substack{k=1, \\ k \neq g}}^{\ell} \frac{\partial \chi_g(\alpha p, \bar{U})}{\partial p_k} \cdot p_k = - \underbrace{\frac{\partial \chi_g(\alpha p, \bar{U})}{\partial p_g}}_{\leq 0} \cdot p_g \geq 0.$$

$\Rightarrow$

## Lemma

Assume  $\ell \geq 2$  and  $p \gg 0$ . Every good has at least one substitute.

# Slutsky equations

## Overview

- 1 The duality approach
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# Three effects of a price increase

$$\frac{\partial \chi_1(p, \bar{U})}{\partial p_1} \stackrel{!}{\leq} 0 \text{ and } \frac{\partial x_1(p, m)}{\partial p_1} \stackrel{?}{\leq} 0$$

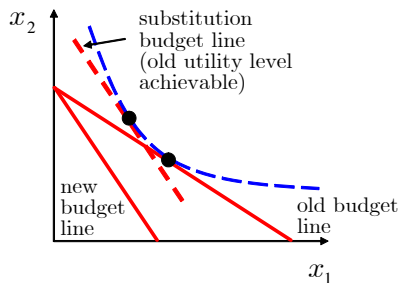
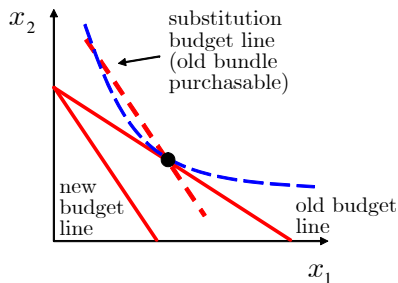
- 1 Substitution effect or opportunity-cost effect:  $p_1 \uparrow$ 
  - $\Rightarrow p_1 / p_2 \uparrow$
  - $\Rightarrow x_1 \downarrow$  and  $x_2 \uparrow$
- 2 Consumption-income effect:  $p_1 \uparrow$ 
  - $\Rightarrow$  overall consumption possibilities decrease
  - $\Rightarrow x_1 \downarrow$  if 1 is a normal good
- 3 Endowment-income effect:  $p_1 \uparrow$ 
  - $\Rightarrow$  value of endowment increases
  - $\Rightarrow x_1 \uparrow$  if 1 is a normal good

# Two different substitution effects

## Definitions

In response to a price change, there are two different ways to keep real income constant:

- Old-household-optimum substitution effect
- Old-utility-level substitution effect





# Two different substitution effects

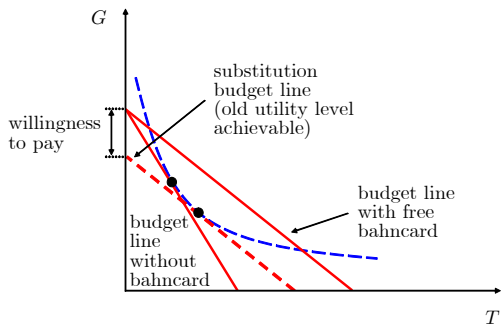
Bahncard 50

## Example

Two goods: train rides  $T$  and other goods  $G$

- $p_T = 0.2$  (per kilometer),  
 $p_G = 1$ .
- "Bahncard 50":  
 $p_T$  reduced to 0.1.

Willingness to pay for the "Bahncard 50"?



# Two different substitution effects

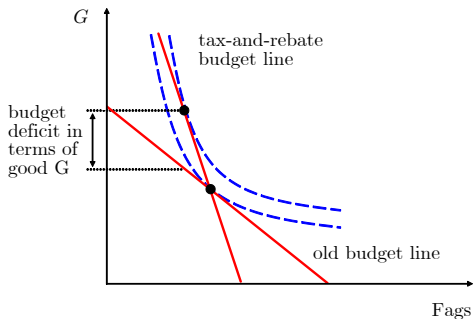
## Tax and rebate

### Example

You smoke 10 cigarettes per day. The government is concerned about your health.

- Quantity tax of 10 cents, but
- rebate of 1 Euro per day.

Budget deficit in terms of the other goods?



# The Slutsky equation for the money budget

## Derivation

Duality equation:  $\chi_g(p, \bar{U}) = x_g(p, e(p, \bar{U}))$

Differentiate with respect to  $p_k$

$$\begin{aligned}\frac{\partial \chi_g}{\partial p_k} &= \frac{\partial x_g}{\partial p_k} + \frac{\partial x_g}{\partial m} \frac{\partial e}{\partial p_k} \\ &= \frac{\partial x_g}{\partial p_k} + \frac{\partial x_g}{\partial m} \chi_k \quad (\text{Shephard's Lemma})\end{aligned}$$

The Slutsky equation ( $g = k$ ):

$$\frac{\partial x_g}{\partial p_g} = \underbrace{\frac{\partial \chi_g}{\partial p_g}}_{\leq 0} - \underbrace{\frac{\partial x_g}{\partial m}}_{> 0} \chi_g.$$

Hicksian law of demand      for normal goods

# The Slutsky equation for the money budget

## Implications

The Slutsky equation:

$$\frac{\partial x_g}{\partial p_g} = \underbrace{\frac{\partial \chi_g}{\partial p_g}}_{\leq 0} - \underbrace{\frac{\partial x_g}{\partial m}}_{> 0} x_g.$$

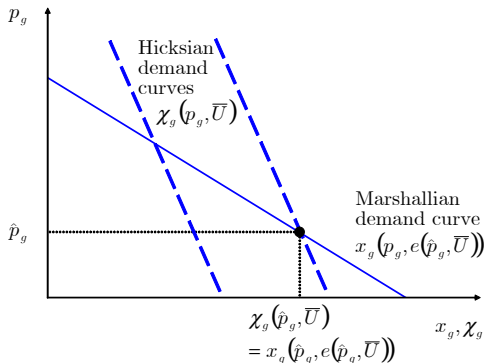
Hicksian law of demand                      for normal goods

- $g$  normal  $\Rightarrow g$  ordinary
- $g$  normal  $\Rightarrow$  effect of a price increase stronger on Marshallian demand than on Hicksian demand
- $g$  inferior  $\Rightarrow$  income effect may outweigh substitution effect  
—> Giffen good

# The Slutsky equation for the money budget

Assume  $(\hat{p}_g, \bar{U})$ .

- By duality,  $\chi_g(\hat{p}_g, \bar{U}) = x_g(\hat{p}_g, e(\hat{p}_g, \bar{U}))$ .
- $g$  normal  $\Rightarrow$  Hicksian demand curves steeper than Marshallian demand curves



# The Slutsky equation for the endowment budget

## Derivation

$$\begin{aligned} & \frac{\partial x_g^{\text{endowment}}(p, \omega)}{\partial p_k} \\ = & \frac{\partial x_g^{\text{money}}(p, p \cdot \omega)}{\partial p_k} \\ = & \frac{\partial x_g^{\text{money}}}{\partial p_k} + \frac{\partial x_g^{\text{money}}}{\partial m} \frac{\partial (p \cdot \omega)}{\partial p_k} \\ = & \frac{\partial x_g^{\text{money}}}{\partial p_k} + \frac{\partial x_g^{\text{money}}}{\partial m} \omega_k \text{ (definition of dot product)} \\ = & \left( \frac{\partial \chi_g}{\partial p_k} - \frac{\partial x_g^{\text{money}}}{\partial m} \chi_k \right) + \frac{\partial x_g^{\text{money}}}{\partial m} \omega_k \text{ (money-budget Slutsky equation)} \\ = & \frac{\partial \chi_g}{\partial p_k} + \frac{\partial x_g^{\text{money}}}{\partial m} (\omega_k - \chi_k). \end{aligned}$$

# The Slutsky equation for the endowment budget

## Implications

The Slutsky equation:

$$\frac{\partial x_g^{\text{endowment}}}{\partial p_g} = \underbrace{\frac{\partial \chi_g}{\partial p_g}}_{\leq 0} + \underbrace{\frac{\partial x_g^{\text{money}}}{\partial m}}_{> 0} \underbrace{(\omega_g - \chi_g)}_{< 0} .$$

for a normal good  $g$       for net demander

- $g$  normal and household net demander  $\Rightarrow g$  ordinary
- $g$  normal and household net supplier  $\Rightarrow g$  may be non-ordinary

# The Slutsky equation for the endowment budget

Application: consumption today versus consumption tomorrow

- The intertemporal budget equation in future value terms:

$$(1+r)x_1 + x_2 = (1+r)\omega_1 + \omega_2.$$

- The Slutsky equation:

$$\frac{\partial x_1^{\text{endowment}}}{\partial (1+r)} = \underbrace{\frac{\partial \chi_1}{\partial (1+r)}}_{\leq 0} + \underbrace{\frac{\partial x_1^{\text{money}}}{\partial m}}_{> 0} \underbrace{(\omega_1 - \chi_1)}_{> 0}.$$

for normal good  
first-period consumption

for lender



# The Slutsky equation for the endowment budget

Application: leisure versus consumption

- The budget equation:

$$w\chi_R + p\chi_C = w24 + p\omega_C.$$

- The Slutsky equation:

$$\frac{\partial \chi_R^{\text{endowment}}}{\partial w} = \underbrace{\frac{\partial \chi_R}{\partial w}}_{\leq 0} + \underbrace{\frac{\partial \chi_R^{\text{money}}}{\partial m}}_{> 0 \text{ for normal good recreation}} \underbrace{(24 - \chi_R)}_{\geq 0 \text{ by definition}}.$$

Thus, if the wage rate increases, it may well happen that the household works ...

# The Slutsky equation for the endowment budget

Application: contingent consumption

- The budget equation:

$$\frac{\gamma}{1-\gamma}x_1 + x_2 = \frac{\gamma}{1-\gamma}(A-D) + A$$

with  $\gamma K$  – payment to the insurance if  $K$  is to be paid to the insuree in case of damage  $D$ .

- The Slutsky equation for consumption in case of damage:

$$\frac{\partial x_1^{\text{endowment}}}{\partial \frac{\gamma}{1-\gamma}} = \underbrace{\frac{\partial \chi_1}{\partial \frac{\gamma}{1-\gamma}}}_{\leq 0} + \underbrace{\frac{\partial x_1^{\text{money}}}{\partial m}}_{> 0} \underbrace{(A-D-\chi_1)}_{\leq 0} .$$

for normal good consumption in case of damage      in case of a nonnegative insurance

# Compensating and equivalent variations

## Overview

- 1 The duality approach
- 2 Shephard's lemma
- 3 The Hicksian law of demand
- 4 Slutsky equations
- 5 **Compensating and equivalent variations**

## Definition

- A variation is equivalent to an event, if both (the event or the variation) lead to the same indifference curve  $\rightarrow EV$  (event);
- A variation is compensating if it restores the individual to its old indifference curve (prior to the event)  $\rightarrow CV$  (event).

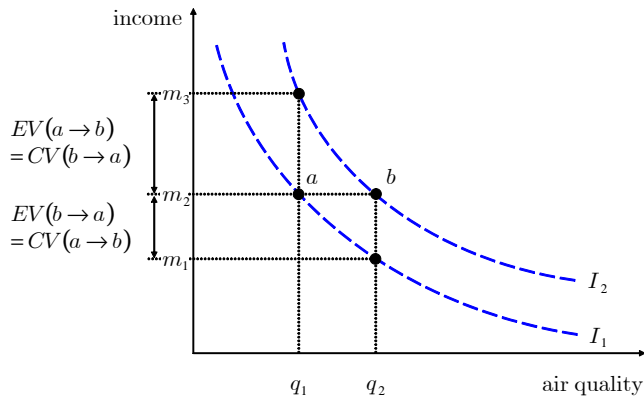
# Compensating and equivalent variations

<b>Equivalent variation</b>	<b>Compensating variation</b>
<p data-bbox="122 415 467 446"><b>in lieu of an event</b></p> <p data-bbox="122 519 670 658">monetary variation is equivalent (i.e., achieving the same utility)</p>	<p data-bbox="753 415 1130 446"><b>because of an event</b></p> <p data-bbox="753 519 1256 658">monetary variation compensates for event (i.e., holding utility constant)</p>

# Compensating and equivalent variations

The case of good air quality

Change of air quality:



# Compensating and equivalent variations

- Compensation money  $\rightarrow$  if some amount of money is given to the individual:  
 $CV$  (degr.) – the compensation money for the degradation of the air quality.
- Willingness to pay  $\rightarrow$  if money is taken from the individual.  
 $EV$  (degr.) – the willingness to pay for the prevention of the degradation.
- If the variation turns out to be negative, exchange  $-EV$  for  $EV$  or  $EV$  for  $-EV$  (similarly for  $CV$ ).

# Compensating or equivalent variation?

## Example

- Consumer's compensating variation: A consumer asks himself how much he is prepared to pay for a good.
- Consumer's equivalent variation – the compensation payment for not getting the good. You go into a shop and ask for compensation for not taking (stealing?) the good.
- Producer's compensating variation – the compensation money he gets for selling a good.
- Producer's equivalent variation: The producer asks himself how much he would be willing to pay if the good were not taken away from him.



# Price changes

- The willingness to pay for the price decrease of good  $g$ :

$$CV(p_g^h \rightarrow p_g^l) = EV(p_g^l \rightarrow p_g^h).$$

- The compensation money for the price increase of good  $g$ :

$$EV(p_g^h \rightarrow p_g^l) = CV(p_g^l \rightarrow p_g^h).$$

- $CV(p_1^h \rightarrow p_1^l) < EV(p_1^h \rightarrow p_1^l)$  (for normal goods, see below);
- $cv$  and  $ev$  – if we are not sure whether a change is good or bad.

## Lemma

Consider the event of a price change from  $p^{old}$  to  $p^{new}$ . Then:

$$U^{old} : = V(p^{old}, m) = V(p^{new}, m + cv), \quad CV = |cv| \quad \text{and}$$

$$U^{new} : = V(p^{new}, m) = V(p^{old}, m + ev), \quad EV = |ev|.$$

# Price changes

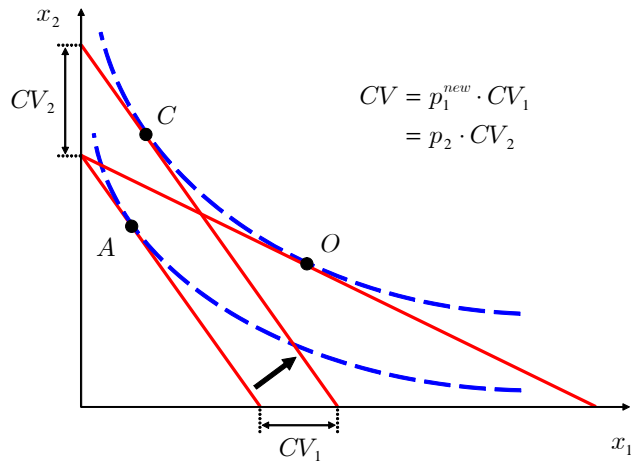
## Exercise

### Problem

*Tell the sign of  $cv$  and  $ev$  for a price increase of all goods.*

# Price changes

Price increase of good 1:



## Example

Cobb-Douglas utility function:  $u(x_1, x_2) = x_1^a x_2^{1-a}$  with  $(0 < a < 1)$ .  
By a price decrease from  $p_1^h$  to  $p_1^l < p_1^h$  (for example, Bahncard 50)

$$\underbrace{\left( a \frac{m}{p_1^h} \right)^a \left( (1-a) \frac{m}{p_2} \right)^{1-a}}_{\text{utility at the old, high price}}$$
$$= \underbrace{\left( a \frac{m + cv(p_1^h \rightarrow p_1^l)}{p_1^l} \right)^a \left( (1-a) \frac{m + cv(p_1^h \rightarrow p_1^l)}{p_2} \right)^{1-a}}_{\text{utility at the new, lower price and compensating variation}}.$$

$$cv(p_1^h \rightarrow p_1^l) = - \left( 1 - \left( \frac{p_1^l}{p_1^h} \right)^a \right) m < 0.$$

### Problem

*Determine the equivalent variation for a price decrease in case of Cobb-Douglas utility preferences.*

### Problem

*Determine the compensating variation and the equivalent variation for the price decrease from  $p_1^h$  to  $p_1^l < p_1^h$  and the quasi-linear utility function given by*

$$u(x_1, x_2) = \ln x_1 + x_2 \quad (x_1 > 0)!$$

*Assume  $\frac{m}{p_2} > 1$ ! Hint: the household optimum is  $x(m, p) = \left(\frac{p_2}{p_1}, \frac{m}{p_2} - 1\right)$ .*

# Applying duality

## Implicit definition of compensating variation

Implicit definition:  $U^{old} := V(p^{old}, m) = V(p^{new}, m + cv)$

Duality equation  $e(p, V(p, m)) = m$  leads to

$$e(p^{old}, V(p^{old}, m)) = m \quad (1)$$

$$e(p^{new}, V(p^{new}, m + cv)) = m + cv \quad (2)$$

$\Rightarrow$

$$cv = e(p^{new}, V(p^{new}, m + cv)) - m \quad (2)$$

$$= e(p^{new}, U^{old}) - e(p^{old}, U^{old}) \quad (1) \text{ and implicit definition}$$

The household is given, or is relieved of, the money necessary to uphold the old utility level.

# Applying duality

## Implicit definition of equivalent variation

Implicit definition:  $U^{new} := V(p^{new}, m) = V(p^{old}, m + ev)$

Duality equation  $e(p, V(p, m)) = m$  leads to

$$e(p^{new}, V(p^{new}, m)) = m \quad (1)$$

$$e(p^{old}, V(p^{old}, m + ev)) = m + ev \quad (2)$$

$\Rightarrow$

$$ev = e(p^{old}, V(p^{old}, m + ev)) - m \quad (2)$$

$$= e(p^{old}, U^{new}) - e(p^{new}, U^{new}) \quad (1) \text{ and implicit definition}$$

Assume  $p^{new} < p^{old}$ . The equivalent variation is the amount of money necessary to increase the household's income from  $m = e(p^{new}, U^{new})$  to  $e(p^{old}, U^{new})$ .

# Variations for a price change and Hicksian demand

Applying the fundamental theorem of calculus

$$cv(p_g^h \rightarrow p_g^l) = - \int_{p_g^l}^{p_g^h} \chi_g(p_g, V(p_g^h, m)) dp_g$$

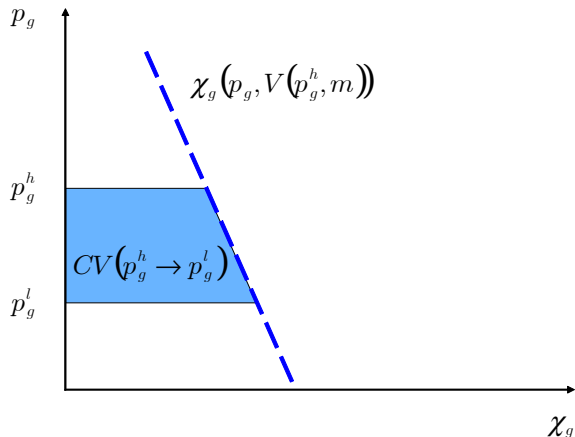
by (if you want)

$$\begin{aligned} cv(p_g^h \rightarrow p_g^l) &= e(p_g^l, V(p_g^h, m)) - e(p_g^h, V(p_g^h, m)) \\ &= - \left[ e(p_g^h, V(p_g^h, m)) - e(p_g^l, V(p_g^h, m)) \right] \\ &= - e(p_g, V(p_g^h, m)) \Big|_{p_g^l}^{p_g^h} \\ &= - \int_{p_g^l}^{p_g^h} \frac{\partial e(p, V(p_g^h, m))}{\partial p_g} dp_g \quad (\text{Fundamental Theorem}) \\ &= - \int_{p_g^l}^{p_g^h} \chi_g(p_g, V(p_g^h, m)) dp_g \quad (\text{Shephard's lemma}). \end{aligned}$$



# Variations for a price change and Hicksian demand

Applying the fundamental theorem of calculus



# Variations for a price change and Hicksian demand

## Comparisons

### Theorem

Assume any good  $g$  and any price decrease from  $p_g^h$  to  $p_g^l < p_g^h$ .

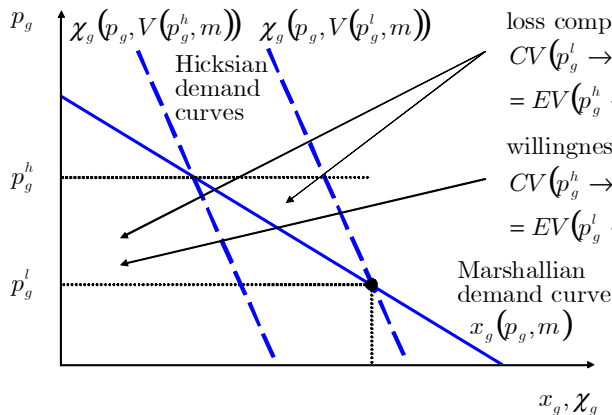
$$cv \left( p_g^h \rightarrow p_g^l \right) = - \int_{p_g^l}^{p_g^h} \chi_g \left( p_g, V \left( p_g^h, m \right) \right) dp_g.$$

If  $g$  is a normal good:

$$\underbrace{CV \left( p_g^h \rightarrow p_g^l \right)}_{\substack{\text{(Hicksian)} \\ \text{willingness to pay}}} \leq \underbrace{\int_{p_g^l}^{p_g^h} x_g \left( p_g \right) dp_g}_{\substack{\text{Marshallian} \\ \text{willingness to pay}}} \leq \underbrace{CV \left( p_g^l \rightarrow p_g^h \right)}_{\substack{\text{(Hicksian)} \\ \text{loss compensation}}}.$$

# Variations for a price change and Hicksian demand

Comparisons for normal goods



# Variations for a price change and Hicksian demand

## Consumers' rent

### Definition

The Hicksian consumer's rent at price  $\hat{p}_g < p_g^{proh}$  is given by

$$\begin{aligned} CR^{Hicks}(\hat{p}_g) &: = CV(p_g^{proh} \rightarrow \hat{p}_g) \\ &= \int_{\hat{p}_g}^{p_g^{proh}} \chi_g(p_g, V(p_g^{proh}, m)) dp_g. \end{aligned}$$

# Further exercises I

## Problem 1

Determine the expenditure functions and the Hicksian demand function for  $U(x_1, x_2) = \min(x_1, x_2)$  and  $U(x_1, x_2) = 2x_1 + x_2$ . Can you confirm the duality equations

$$\begin{aligned}\chi(p, V(p, m)) &= x(p, m) \text{ and} \\ x(p, e(p, \bar{U})) &= \chi(p, \bar{U})?\end{aligned}$$

## Further exercises II

### Problem 2

Derive the Hicksian demand functions and the expenditure functions of the two utility functions:

(a)  $U(x_1, x_2) = x_1 \cdot x_2$ ,

(b)  $U(x_1, x_2) = \min(a \cdot x_1, b \cdot x_2)$  with  $a, b > 0$ .

### Problem 3

Verify Roy's identity for the utility function  $U(x_1, x_2) = x_1 \cdot x_2$ !

### Problem 4

Draw a figure that shows the equivalent variation following a price increase.