Trading Off Generations: Infinitely Lived Agent Versus OLG

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Abstract: Intergenerational trade-offs are discussed in infinitely lived agent models despite the finite lifetime of individuals. We discuss these trade-offs in a continuous time OLG framework and relate the results to the infinitely lived agent setting. We identify three shortcomings of the latter: First, underlying normative assumptions about social preferences cannot be deduced unambiguously. Second, the distribution among generations living at the same time cannot be captured. Third, the optimal solution may not be implementable in overlapping generations market economies. Re-examining the recent debate on climate change, we conclude that Stern’s and Nordhaus’ infinitely lived agent models are not suitable to discuss intergenerational trade-offs, as they cannot adequately address these three aspects. In consequence, we argue to explicitly consider the generations’ life-cycles.

Keywords: climate change, infinitely lived agents, intergenerational equity, overlapping generations, time preference

JEL-Classification: D63, H23, Q54

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1. Introduction

How much should a government invest in public infrastructure? How much in basic research? Or recently heavily debated: How much CO$_2$ should be mitigated? All these decision problems exhibit two characteristics: A classical public good problem, as the private provision of these goods exhibits positive externalities, and an intergenerational equity problem, as the costs have to be borne today while the benefits spread over decades or even centuries. Economists would agree that the public good problem should be solved by internalizing the externalities, for example, via a Pigouvian tax. In long-run decision problems, however, stocks of pollutants or knowledge are affected. As these influence the well-being of later generations, it is not clear how high the respective tax should be. This depends, of course, on the planner’s objective and, in particular, on the weight attached to the utility of future generations. Most of the times the planner is supposed to maximize the utility of an infinitely lived agent (ILA) which is interpreted as a utilitarian social welfare function of an infinite succession of infinitely short-lived non-overlapping generations. Despite some agreement on this welfare function there is an ongoing dispute about the right social rate of time preference. The discussion finds its extreme positions in the normative approaches of early authors, for example, Fisher (1930), Pigou (1932), Ramsey (1928), respectively in the recent climate change debate of Cline (1992) and Stern (2007), who argue for low social time preference rates on ethical grounds, and in the positive approach by, for example, Manne et al. (1995) and Nordhaus (2007), who hold that social preferences are reflected by market outcomes.

To use an ILA’s utility as the planner’s objective may be justified by Barro (1974)’s argument that voluntary transfers between parents and children due to an altruistic bequest motive cause the representative family to behave as if it is an infinitely-lived individual. Recent empirical studies, however, indicate that the altruistic bequest motive is rather weak.$^1$ This implies that the finitely lived individuals do not behave as though they maximize a dynastic utility function and that an overlapping generations (OLG) model without altruistic bequests would better fit reality. But in this case, is it appropriate to use an ILA specification to discuss long-run decision problems such as the ones mentioned above? Can the ILA model still capture all relevant aspects of intergenerational equity, for example, via different values of time preference? And what does this imply for the normative and the positive approach to determine the “right” social rate of time preference?

In this paper, we answer these questions by examining the relationship between an OLG model in continuous time and the standard ILA economy, also known as Ramsey-Cass-Koopmans economy.

First, we examine the conditions under which an OLG and an ILA economy are observationally equivalent with respect to macroeconomic aggregates. Then we show that the ILA model can be interpreted as an unconstrained social planner that possesses full power to allocate resources in order to maximize the discounted stream of lifetime utilities of all present and future generations. That is, the ILA’s rate of time preference reflects the unconstrained social planner’s weight for the different generations’ lifetime utilities. However, this solution would require major redistribution from old to young in a decentralized OLG economy without altruistic bequests, at least when the social time preference rate is weakly lower than the individual households’ rate. We show that in this case, there is a trade-off between equality among the generations living at the same time and equality of lifetime utilities between present and future generations. The closer is the time preference of the social planner to that of the individuals, the more equally treated are generations alive at a given time at the expense of a more unequal treatment of generations living at different times. This trade-off can only be reconciled when both the individuals and the social planner exhibit a time preference rate of zero.

Moreover, the unconstrained social planner’s optimum may be difficult to implement due to the substantial redistribution requirements, which would imply a tax/subsidy regime discriminating by age in a market economy. As a consequence, we analyze the second best optimum of a constrained social planner that is not able to discriminate individual households by age but can only influence prices via taxes and subsidies. One might think of a democratically legitimized government. We show that under these circumstances, the time preference rate in the observationally equivalent ILA economy cannot be interpreted as the weight a constrained social planner attaches to the different generations’ well-being. This implies that, due to the restricted set of instruments to intratemporally redistribute consumption across generations at each point in time, the constrained social planner prefers a different intertemporal path of the economy’s aggregates than the ILA. This questions the interpretation of the ILA’s objective function as a social welfare function consisting of the weighted sum of subsequent generations’ utility.

We illustrate our results by applying them to the recent debate on climate change mitigation. We conclude that the ILA model is not able to properly account for important aspects of intergenerational equity. Our analysis further challenges the approach to deduce social time preferences from observable market outcomes, in particular the real
interest rate. First, without altruistic bequests it is not even clear whether a mandate for a social planner to deal with long-run decision problems can be positively derived. For example, if all individuals were fully selfish, they would never agree to investments the costs of which have to be borne today while the benefits accrue when they are already dead. Second, even in case there is an agreement in favor of a social planner to tackle long-run public good problems, social preferences cannot be observed by pure market outcomes when altruistic bequests are not operative. Instead, we suggest that it may be exactly state interventions (possibly by a democratically legitimized constrained social planner), such as subsidies on R&D or education, that reveal the social preferences of the society.

The paper is structured as follows. We discuss the relationship of our paper to the existing literature in Section 2. In Section 3, we develop a decentralized OLG model in continuous time. The ILA economy is introduced in Section 4. We derive conditions for observational equivalence of the decentralized OLG economy and an ILA model in Section 5. In Section 6, we examine the relationship between the latter and two social planner solutions, unconstrained and constrained. In Section 7 we apply our results to the recent debate on climate change mitigation. We conclude in Section 8.

2. Relation to the Literature

There are several papers that examine the relationship between ILA and OLG models. Aiyagari (1985) proved that under certain assumptions the OLG model with two-period lived individuals is observationally equivalent to an ILA model in discrete time. The equivalence between a Ramsey-Cass-Koopmans economy and a model with finitely lived consumers in continuous time was also established by Calvo and Obstfeld (1988). However, their main concern was with time inconsistencies in fiscal policy, arguing that these may arise if a government’s fiscal tools are too limited to allow for the decentralization of the command optimum. Our paper differs from Aiyagari (1985) in that it derives the equivalence between the Ramsey-Cass-Koopmans economy and the OLG model in continuous time and provides the explicit mapping between the two frameworks with respect to the different intertemporal elasticities of substitution and time preference rates. In contrast to Calvo and Obstfeld (1988), our focus is on aspects of intergenerational equity rather than time inconsistency of policies. In fact, all policies considered in this paper are time consistent. Two further differences are worth mentioning. First, we literally model finite life spans of individuals rather than using Blanchard’s (1985)
specification in which each individual possesses a constant probability of death at all
times. The important difference in our context is that the strictly finitely lived selfish
individuals would never consider investments the maturity of which exceed their own
lifetime, whereas the consumer of the “death-probability”-type would invest in projects
with arbitrarily long gestation periods if the rate of return is sufficiently high. Second,
switching to a discrete time two-period OLG model, Calvo and Obstfeld (1988) argue
that even a constrained planner that is not able to discriminate transfers by age can
implement the first best social optimum, given the time horizon of the social planner
is infinite rather than finite. In a similar setting in continuous time we show that the
first best optimum is not implementable, implying that Calvo and Obstfeld’s result is
strongly connected to their discrete-time-Diamond (1965)-setup.

Also in environmental economics applications, such as Howarth (1998), Howarth (2000),
Gerlach and Keyzer (2001), Gerlach and van der Zwaan (2000), and Stephan and Müller-
Fürstenberger (1997), it has been observed that ILA models can be calibrated to yield
outcomes similar to OLG models. These papers use numerical simulations of integrated
assessment models, whereas we derive the relation analytically in a continuous time
setup.

Formally, our OLG model is most closely related to d’Albis (2007) who examines the
influences of demographic structure on capital accumulation and growth. In addition to
his paper, we allow for exogenous technological change and have a clear focus on inter-
generational trade-offs. More remotely, our paper relates to the literature on overlapping
generations and debt neutrality such as Barro (1974), Blanchard (1985) and Weil (1989).

3. An Overlapping Generations Growth Model in Continuous Time

In this section, we introduce an OLG exogenous growth model in continuous time and
analyze the long-run individual and aggregate dynamics of a decentralized economy in
market equilibrium.

3.1. Households

We assume a continuum of households, each living the finite time span $T$. All house-
holds exhibit the same intertemporal preferences irrespective of their time of birth
$s \in (-\infty, \infty)$. In particular, we assume that if households are altruistic, their altruistic
preferences are not sufficiently strong for an operative bequest motive. For simplicity, we thus abstract from altruism in individual preferences. As a consequence, all households maximize welfare \( U \), which is the discounted stream of instantaneous utility derived from consumption during their own lifetime only

\[
U(s) \equiv \int_s^{s+T} c(t,s)^{1 - \frac{1}{\sigma^H}} \exp \left[ - \rho^H (t - s) \right] dt ,
\]

where \( c(t,s) \) is the consumption at calendar time \( t \) of households born at time \( s \), \( \sigma^H \) is the constant intertemporal elasticity of substitution and \( \rho^H \) is the constant rate of (pure) time preference of the households. At any time alive, each household is endowed with one unit of labor, which is supplied inelastically to the labor market at wage \( w(t) \) over the whole life span \( T \). In addition, households can accumulate assets \( b(t,s) \), which earn interest \( r(t) \), implying the following budget constraint:\footnote{Throughout the paper, partial derivatives are denoted by subscripts (e.g., \( F_k(k,l) = \partial F(k,l)/\partial k \)), derivatives with respect to calendar time \( t \) are denoted by dots and derivatives of functions depending on one variable only are denoted by primes.}

\[
\dot{b}(t,s) = r(t)b(t,s) + w(t) - c(t,s) , \quad t \in [s,s+T] .
\]

Households are born without assets and are not allowed to be indebted at time of death. Thus, the following boundary conditions apply for all generations \( s \):

\[
b(s,s) = 0 , \quad b(s + T, s) \geq 0 .
\]

The latter condition will hold with equality in the household optimum, as intertemporal welfare \( U \) of a household born at time \( s \) can always be increased by consumption at time \( s + T \), due to the non-operative bequest motive.

Maximizing equation (1) for any given \( s \) subject to conditions (2) and (3) yields the well known Euler equation

\[
\dot{c}(t,s) = \sigma^H \left[ r(t) - \rho^H \right] c(t,s) , \quad t \in [s,s+T] .
\]

The behavior of a household born at time \( s \) is characterized by the system of differential equations (2) and (4) and the boundary conditions for the asset stock (3).

At any time \( t \in (-\infty, \infty) \) the size of the population \( N(t) \) increases at the constant rate \( \nu \geq 0 \). Without loss of generality, we normalize the population at time \( t = 0 \) to unity,
which implies the following birth rate $\gamma$:

$$N(t) \equiv \exp[\nu t] \quad \Rightarrow \quad \gamma = \frac{\nu \exp[\nu T]}{\exp[\nu T] - 1}. \quad (5)$$

3.2. Firms

We assume a continuum of identical competitive firms $i \in [0,1]$. All firms produce a homogeneous consumption good under conditions of perfect competition from capital $k(t,i)$ and effective labor $A(t)l(t,i)$, where $A(t)$ characterizes the technological level of the economy and grows exogenously at a constant rate $\xi$. Without loss of generality, we normalize technological progress at $t = 0$ to unity, implying

$$A(t) \equiv \exp[\xi t]. \quad (6)$$

All firms have access to the same production technology $F(k(t,i), A(t)l(t,i))$, which exhibits constant returns to scale and positive but decreasing marginal productivity with respect to both inputs capital and effective labor. Furthermore, $F$ satisfies Inada conditions.

Constant returns to scale production and symmetry of the firms allow us to work with a representative firm, whose decision variables are interpreted as aggregate variables. With minor abuse of notation, we introduce aggregate capital per effective labor

$$k(t) \equiv \frac{\int_0^1 k(t,i) \, di}{A(t) \int_0^1 l(t,i) \, di}. \quad (7a)$$

which yields the intensive form production function $f(k(t)) = F(k(t), 1)$. For later use we also define aggregate capital per capita

$$\bar{k}(t) = \frac{\int_0^1 k(t,i) \, di}{N(t) \int_0^1 l(t,i) \, di}. \quad (7b)$$

Profit maximization of the representative firm yields the following equations for the wage

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$^3$ The equation is derived by solving $\int_0^t \gamma \exp[\nu s] \, ds = N(t)$, where $\gamma \exp[\nu s]$ denotes the cohort size of the generation born at time $s$. Applying definition (12), we obtain $\gamma = 1/Q_T(\nu)$. In addition, $\gamma \to 1/T$ for $\nu \to 0$ and $\gamma \to \nu$ for $T \to \infty$. 

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and the interest rate \( r(t) \):

\[
\begin{align*}
    w(t) &= A(t) \left[ f(k(t)) - f'(k(t))k(t) \right], \quad (8a) \\
    r(t) &= f'(k(t)). \quad (8b)
\end{align*}
\]

### 3.3. Market Equilibrium and Aggregate Dynamics

Now, we investigate the aggregate dynamics of the economy. For this purpose, we introduce aggregate household variables per effective labor by integrating over all living individuals and dividing by the technological level and the labor force of the economy. With slight abuse of notation, we define

\[
x(t) = \frac{\int_{t-T}^{t} x(t,s) \gamma \exp[\nu s] \, ds}{A(t) \int_{0}^{1} l(t,i) \, di}, \quad (9a)
\]

where \( x(t) \) and \( x(t,s) \) denote aggregate per effective labor respectively individual household variables. In addition, we define aggregate household variables per capita as follows:

\[
\bar{x}(t) = \frac{\int_{t-T}^{t} x(t,s) \gamma \exp[\nu s] \, ds}{N(t)}. \quad (9b)
\]

The economy consists of three markets: the labor market, the capital market and the consumption good market. We assume the economy to be in market equilibrium at all times \( t \). This implies that labor demand equals the population size, i.e., \( \int_{0}^{1} l(t,i) \, di = N(t) \), and capital in terms of effective labor equals aggregate assets in terms of effective labor, i.e., \( k(t) = b(t) \). Then, the aggregate dynamics is characterized by

\[
\begin{align*}
    \dot{k}(t) &= f(k(t)) - (\nu + \xi)k(t) - c(t), \quad (10a) \\
    \dot{c}(t) &= \sigma \left[ r(t) - \rho H \right] - (\nu + \xi) - \frac{\Delta c(t)}{c(t)}, \quad (10b)
\end{align*}
\]

where the term

\[
\Delta c(t) = \gamma \exp[\nu(t-T)]c(t,t-T) - \gamma \exp[\nu t]c(t,t)
\]

captures the difference in aggregate consumption per effective labor between the generation born and the generation dying at time \( t \).

\footnote{Note that \( \dot{x}(t) = -(\nu + \xi)x(t) + \exp[-(\nu + \xi)t] \int_{t-T}^{t} \dot{x}(t,s) \gamma \exp[\nu s] \, ds + \gamma \left[ x(t,t) - \frac{\gamma(t)}{\exp[\nu(t-T)]} \right] \exp[-\xi t]. \)}
3.4. Steady State

Our analysis will concentrate on the long-run steady state growth path of the economy, in which both consumption per effective labor and capital per effective labor are constant over time, i.e., $c(t) = c^*$, $k(t) = k^*$. From equation (8) follows that also the interest rate $r(t) = r^* \equiv f'(k^*)$ is constant in the steady state and that the wage $w(t)$ grows at the rate of technological progress. Thus, we introduce the wage relative to the technology level which is constant in the steady state

$$w^* \equiv \frac{w(t)}{\exp[\xi t]} \bigg|_{k=k^*} = \left[ f(k^*) - f'(k^*)k^* \right].$$

For $T \in \mathbb{R}_{++}$, we define the function $Q_T : \mathbb{R} \to \mathbb{R}_+$ by

$$Q_T(r) \equiv \frac{\exp[-rT] - 1}{-r}, \quad \forall r \neq 0,$$

and $Q_T(r) = T$ for $r = 0$. Properties of the function $Q_T$ are summarized in Lemma 1 in the appendix. Just like wages we express consumption and wealth of individual households relative to the technology level. This returns a function only depending on household age $a \equiv t - s$:

$$c^*(a) \equiv \frac{c(t,s)}{\exp[\xi t]} \bigg|_{k=k^*} = w^* \frac{Q_T(r^* - \xi)}{Q_T(r^* - \sigma H(r^* - \rho H))} \exp \left[ (\sigma H(r^* - \rho H) - \xi) a \right],$$

$$b^*(a) \equiv \frac{b(t,s)}{\exp[\xi t]} \bigg|_{k=k^*} = w^* Q_a(r^* - \sigma H(r^* - \rho H)) \exp[(r^* - \xi)a] \times \left\{ \frac{Q_a(r^* - \xi)}{Q_a(r^* - \sigma H(r^* - \rho H))} - \frac{T}{a} \frac{Q_T(r^* - \xi)}{Q_T(r^* - \sigma H(r^* - \rho H))} \right\}. (13b)$$

Figure 1 illustrates these steady state paths for individual consumption and assets in terms of the technological level of the economy. The individual consumption path grows exponentially over the lifetime of each generation. Individual household assets follow an inverted U-shape, i.e., households are born with no assets, accumulate assets in their youth and consume their wealth towards their death.

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5 The calculations use the following model specifications: $f(k) = k^\alpha$, $\alpha = 0.3$, $\rho = 3\%$, $\sigma = 1$, $\xi = 1.5\%$, $\nu = 0$, $T = 75$. 8
Applying the aggregation rule (9a), we derive for the aggregate values per effective labor

\[ c^\star = \frac{w^\star Q_T(r^\star - \xi) Q_T(\nu + \xi - \sigma H(r^\star - \rho H))}{Q_T(\nu) Q_T(r^\star - \sigma H(r^\star - \rho H))}, \quad (14a) \]

\[ b^\star = \frac{w^\star}{r^\star - \nu - \xi} \left\{ \frac{Q_T(r^\star - \xi) Q_T(\nu + \xi - \sigma H(r^\star - \rho H))}{Q_T(\nu) Q_T(r^\star - \sigma H(r^\star - \rho H))} - 1 \right\}. \quad (14b) \]

The steady state is determined by the equilibrium condition \( k^\star = b^\star \). The following proposition guarantees the existence of a non-trivial steady state for a large class of production functions, in particular, CES-production functions.

**Proposition 1 (Existence of the steady state)**

There exists a \( k^\star > 0 \) that solves equation \( k^\star = \phi(k^\star) \) if

\[ \lim_{k \to 0} -kf''(k) > \begin{cases} \sigma T, & \text{if } \sigma \in (0, 1] \\ T, & \text{if } \sigma > 1 \end{cases} \quad (15) \]

The proof is given in the appendix.

In the proof of Proposition 1 we show that steady states may be equal to or larger than the golden rule capital stock, i.e., \( k^\star \geq k^{gr} \), which is implicitly defined by \( r^{gr} \equiv \nu + \xi = f' (k^{gr}) \). As our aim is to compare the decentralized OLG with an ILA economy, we are particularly interested in steady states with \( k^\star < k^{gr} \).

Let us summarize the production side of the economy, the rate of technological progress and the rate of population growth.

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\(^6\) In the ILA economy only steady states \( k^\star < k^{gr} \) may occur.
by the set $\Psi \equiv \{ f(k), \xi, \nu \}$. Given $\Psi$ and the individuals’ lifespan $T$, we define the set of individual preference parameters that lead to steady states with $k^* < k^{gr}$ by

$$\Gamma_{\Psi, T} \equiv \{ (\sigma^H, \rho^H) \in \mathbb{R}^+ \times \mathbb{R} | \exists k^* \text{ with } 0 < k^* < k^{gr} \}.$$ 

To show that $\Gamma_{\Psi, T}$ is non-empty for all $\Psi$ and $T$ satisfying condition (15) of Proposition 1, we give sufficient conditions for the existence and uniqueness of steady states with $k^* < k^{gr}$. Analogously to d’Albis (2007), we introduce the share of capital in output, $s(k)$, and the elasticity of substitution between between capital and labor, $\epsilon(k)$,

$$s(k) \equiv \frac{kf'(k)}{f(k)}, \quad \epsilon(k) \equiv -\frac{f(k) - f'(k)k}{k^2 f''(k)}.$$  

(16)

**Proposition 2 (Existence and uniqueness of dynamically efficient steady states)**

*Given condition (15) holds. Then, there exists a steady state with $k^* < k^{gr}$ if*

$$\rho^H \geq \frac{\sigma^H - 1}{\sigma^H} \xi + \nu.$$  

(17)

*There exists exactly one $k^* < k^{gr}$ if*

$$s(k) \leq \epsilon(k) \quad \land \quad \frac{d}{dk} \left( \frac{s(k)}{\epsilon(k)} \right) \geq 0,$$  

(18a)

*and, in case that $\sigma > 1$,*

$$\rho^H < \frac{\sigma^H - 1}{\sigma^H} (\nu + \xi).$$  

(18b)

The proof is given in the appendix.

Although we cannot solve the implicit equation $k^* = b^*$ analytically and, therefore, cannot calculate the steady state interest rate $r^*$, we can give a lower bound of the interest rates in a dynamically efficient OLG economy.

**Proposition 3 (Lower bound of steady state interest rate)**

*If $r^* > r^{gr}$, then*

$$f'(k^*) = r^* > \rho^H + \frac{\xi}{\sigma^H}.$$ 

The proof is given in the appendix.
The lower bound of the steady state interest rate in the decentralized OLG economy will play an important role for the comparison with the ILA economy. It will be shown that this lower bound is the corresponding interest rate in the ILA economy if the infinitely lived agent exhibits the preference parameters $\sigma^H$ and $\rho^H$. We shall trace back the difference between the two interest rates to the finite lifetimes of the households.

4. Infinitely Lived Agent Economy and Observational Equivalence

As intergenerational trade-offs are rather discussed in ILA than OLG models, we investigate how the macroeconomic observables of an OLG and ILA economy relate to each other. Therefore, we first introduce the ILA model and define under what conditions we consider the two models to exhibit coinciding outcomes.

We assume that population growth and the production side of the economy are identical in both models, i.e., $\Psi_{OLG} = \Psi_{ILA} = \Psi$. Variables of the ILA model that are not exogenously fixed to its corresponding counterparts in the OLG model are indexed by superscript $^R$. The ILA model abstracts from individual generations’ lifecycles, but only considers aggregate consumption and asset holdings. In fact, in the ILA model, optimal consumption and asset paths per capita are derived by maximizing the discounted stream of instantaneous utility of consumption per capita weighted by population size

$$U^R \equiv \int_0^{\infty} N(t) \frac{\bar{c}^R(t)^{1-\frac{1}{\sigma^R}}}{1-\frac{1}{\sigma^R}} \exp\left[-\rho^R t\right] \, dt ,$$  

subject to the following budget constraint:

$$\dot{b}^R(t) = (r^R(t) - \xi - \nu)b^R(t) + \frac{w^R(t)}{A(t)} - c^R(t) .$$  

Maximizing (19) subject to (20) yields the well known Euler equation of the ILA model

$$\frac{\dot{c}^R(t)}{c^R(t)} = \rho^R \left[ r^R(t) - \rho^R \right] - \xi .$$  

Assuming markets to be in equilibrium at all times (i.e., $\int_0^1 l(t, i) \, di = N(t)$ and $k^R(t) = b^R(t)$), the dynamics of the capital stock per effective labor in the ILA economy reads

$$\dot{k}^R(t) = f(k^R(t)) - (\nu + \xi)k^R(t) - c^R(t) .$$  

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which is formally equivalent to the corresponding equation (10a) of the OLG economy. To compare the different models we use the following definition:

**Definition 1 (Observational equivalence)**

Two economies (1) and (2) are observationally equivalent if coincidence in their current observable macroeconomic variables leads to coincidence of their future observable macroeconomic variables. Formally, if

\[ c^{(1)}(0) = c^{(2)}(0) \quad \text{and} \quad k^{(1)}(0) = k^{(2)}(0) , \]

then

\[ c^{(1)}(t) = c^{(2)}(t) \quad \text{and} \quad k^{(1)}(t) = k^{(2)}(t) \quad \text{for all } t \geq 0 . \]

The definition states that two economies are observationally equivalent if, starting from the same initial conditions, they exhibit identical paths of consumption and capital per effective labor. In all comparisons between different types of economies we will assume coinciding production side and population growth, i.e., coinciding Ψ, implying that two observationally equivalent economies have also identical real rates of interest \( r(t) \) and wages \( w(t) \). In some cases, we will be particularly interested in comparing the steady state of two economies. If we assume that both economies are in the steady state, observational equivalence is trivial because we simply assume that our macroeconomic variables do not change over time. Then we are interested in whether there exists a steady state in both economies with coinciding levels of effective capital.

5. Decentralized Overlapping Generations Versus Infinitely Lived Agent Economy

Now we investigate under what conditions a decentralized OLG economy, as outlined in Section 3, is observationally equivalent to an ILA economy, as defined in Section 4. The following proposition states the necessary and sufficient condition:

**Proposition 4 (Decentralized OLG versus ILA economy)**

(i) The decentralized OLG economy and the ILA economy are observational equivalent if and only if for all \( t \geq 0 \)

\[ \rho^R = \frac{\sigma^H}{\sigma^R} \rho^H + \left( 1 - \frac{\sigma^H}{\sigma^R} \right) r(t) + \frac{1}{\sigma^R} \left[ \frac{\Delta c(t)}{c(t)} + \nu \right] . \]
For any OLG economy in the steady state with \((\sigma^H, \rho^H) \in \Gamma_{\psi,T}\) there exists an observationally equivalent ILA economy.

The proof is given in the Appendix.

Proposition 4 says that for every given set \((\sigma^H, \rho^H) \in \Gamma_{\psi,T}\) at least the steady state of a decentralized OLG economy is observationally equivalent to the steady state of an ILA economy for an appropriate choice of \((\sigma^R, \rho^R)\). Note that \((\sigma^R, \rho^R)\) is, in general, not uniquely determined by (22), as it poses an underdetermined system of equations for the two unknowns \(\sigma^R\) and \(\rho^R\).

If we assume that the intertemporal propensity to smooth consumption between two periods is the same for the households in the OLG and the ILA economy, i.e. \(\sigma^H = \sigma^R\), we obtain the following corollary.

**Corollary 1 (Identical intertemporal elasticity of substitution)**

For \(\sigma^R = \sigma^H\) condition (22) reduces to

\[
\rho^R = \rho^H + \frac{1}{\sigma^R} \left[ \frac{\Delta c(t)}{c(t)} + \nu \right].
\]

To understand why the pure rates of time preference in the observationally equivalent ILA economy differs from the OLG economy, we analyze the two terms in brackets on the right-hand side of equation (23). The first term captures the difference in consumption between the cohort dying and the cohort just born in terms of total consumption per effective labor. The term is a consequence of the fact that every individual in the OLG model plans his own life cycle, saving while young and spending while old if the interest rate is sufficiently high. If there is no population growth, i.e. \(\nu = 0\) (\(\gamma = 1/T\)), the cohort size of the dying generation coincides with the cohort size of the generation just born. Then, the term is always positive if households accumulate assets over their lifetime, as this implies that consumption at birth is lower and consumption at death is higher than the wage. Due to this inverted U-shape saving pattern of the households, individual consumption growth is higher than aggregate consumption growth.\(^7\)

The second term reflects that instantaneous utility in the ILA model is weighted by population size. Hence, for a growing population future consumption receives an increasing

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\(^7\) Of course, this does not necessarily hold for \(\nu > 0\), as total consumption of the newborn generation may outweigh total consumption of the dying generation if the cohort size of the former is sufficiently higher than the cohort size of the latter.
weight in the objective function. A corresponding weighting does not occur in the decentralized OLG economy, where all households only maximize own lifetime utility.

Depending on whether the term in brackets is positive or negative, the pure rate of time preference of the agent in the ILA model is higher or lower than the households’ pure rate of time preference in the observational equivalent decentralized OLG economy. The following corollary elaborates on the sign of \( \Delta \left( \frac{c(t)}{c(t)} + \nu \right) \).

**Corollary 2 (Comparing time preference)**

(i) In general, the following relation holds:

\[
\left[ \frac{\Delta c(t)}{c(t)} + \nu \right] > 0 \quad \Leftrightarrow \quad \frac{\dot{c}(t,s)}{c(t,s)} > \frac{\dot{c}(t)}{c(t)} .
\]  

(ii) In the steady state \( \left[ \frac{\Delta c(t)}{c(t)} + \nu \right] > 0 \). For \( \sigma^R = \sigma^H \) this implies \( \rho^R > \rho^H \).

**Proof:** (i) Equation (24) is obtained by substituting the Euler equation of the individual household (4) into the aggregate Euler equation (10b), taking into account that \( \frac{\dot{c}(t)}{c(t)} = \frac{\dot{c}(t,s)}{c(t,s)} + \xi \) and solving for \( \left[ \frac{\Delta c(t)}{c(t)} + \nu \right] \).

(ii) For the steady state equation (10b) returns \( \left[ \frac{\Delta c(t)}{c(t)} + \nu \right] = \sigma^H \left( r(t) - \rho^H \right) - \xi \) which is positive by virtue of Proposition 3. Thus, according to equation (23), \( \rho^R > \rho^H \). □

Part (i) of the corollary states that the time preference rate in ILA model is higher than the household’s time preference rate in the observationally equivalent decentralized OLG economy whenever individual consumption grows faster than consumption per capita. Part (ii) of the corollary is particular important for estimating the pure rate of time preference. It implies that individual rates of time preference are overestimated in an ILA framework, when the structure of the underlying economy rather resembles a decentralized OLG economy.

**6. Utilitarian Overlapping Generations Versus Infinitely Lived Agent Economy**

In this section, we investigate the conditions under which an OLG economy, governed by a social planner maximizing a social welfare function, is observationally equivalent to an ILA economy.
We assume a utilitarian social welfare function in which the social planner trades off the weighted lifetime utility of different generations. The weight consists of two components. First, the lifetime utility of the generation born at time $s$ is weighted by cohort size. Second, the social planner exhibits a social rate of time preference $\rho^S \geq 0$ at which he discounts the total expected lifetime utility at birth for generations born in the future.\(^8\)

We assume that the social planner maximizes social welfare from $t = 0$ onward. Thus, the social welfare function consists of two parts. First, the weighted integral of the remaining lifetime utility of all generations alive at time $t = 0$, and second the weighted integral of all future generations

$$W \equiv \int_0^0 \left\{ \int_0^{s+T} \frac{c(t,s)^{1-\frac{1}{\sigma_H}}}{1 - \frac{1}{\sigma_H}} \exp \left[ - \rho^H (t - s) \right] dt \right\} \gamma \exp[n] \exp[-\rho^S s] ds$$

$$+ \int_0^\infty \left\{ \int_s^{s+T} \frac{c(t,s)^{1-\frac{1}{\sigma_H}}}{1 - \frac{1}{\sigma_H}} \exp \left[ - \rho^H (t - s) \right] dt \right\} \gamma \exp[n] \exp[-\rho^S s] ds . \quad (25a)$$

The term in curly braces is the (remaining) lifetime utility $U(s)$ of a household born at time $s$ as given by equation (1), the functional form of which is a given primitive for the social planner. The term $\gamma \exp[n]$ denotes the cohort size of the generation born at time $s$. Changing the order of integration and replacing $t - s$ by age $a$, we obtain

$$W = \int_0^\infty \left\{ \int_0^T \frac{c(t,t-a)^{1-\frac{1}{\sigma_H}}}{1 - \frac{1}{\sigma_H}} \gamma \exp \left[ (\rho^S - \rho^H - \nu) a \right] da \right\} \exp \left[ (\nu - \rho^S) t \right] dt . \quad (25b)$$

In the following, we consider two different scenarios. In the unconstrained utilitarian OLG economy, a social planner maximizing the social welfare function (25b) directly controls investment and household consumption. Thus, the social planner is in command of a centralized economy. In contrast, in the constrained utilitarian OLG economy the social planner relies on a market economy, in which the households own the firms and the capital stock and optimally control their savings and consumption maximizing their individual lifetime utility (1). In this second scenario, the social planner is constraint to

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\(^8\) We examine a discounted utilitarian social welfare function, as it represents the de facto standard in the economic literature (e.g. Burton 1993, Calvo and Obstfeld 1988), and is also the interpretation applied to the infinitely lived agent models in the climate change debate (Nordhaus 2007, Stern 2007). For a critical assessment see, e.g., Sen and Williams (1982). Calvo and Obstfeld (1988) show that social welfare functions which do not treat all present and future generations symmetrically, i.e. discount lifetime utility to the same point of reference (here the date of birth), may lead to time-inconsistent optimal plans.
influencing prices by a tax/subsidy regime in order to maximize social welfare (25b).

6.1. Unconstrained Utilitarian Overlapping Generations Economy

We determine the unconstrained social planner’s optimal allocation by solving

\[
\max_{\{c(t,s)\} (t,s) \in \mathbb{R}_+ \times \mathcal{S}} \int_0^\infty \left\{ \int_0^T c(t,t-a)^{1-\frac{1}{\sigma_H}} \gamma \exp\left[ (\rho^S - \rho^H - \nu) a \right] da \right\} \exp\left[ (\nu - \rho^S) t \right] dt ,
\]

subject to the budget constraint (10a). \(\mathcal{S}\) denotes the set \([-T, \infty)\) of all the birth dates of the generations under consideration.

Following the approach of Calvo and Obstfeld (1988), we interpret the optimization problem (26) as two nested optimization problems. The first problem is obtained by defining

\[
V(\bar{c}(t)) \equiv \max_{\{c(t,t-a)\} (t,a) = 0} \int_0^T c(t,t-a)^{1-\frac{1}{\sigma_H}} \gamma \exp\left[ (\rho^S - \rho^H - \nu) a \right] da ,
\]

subject to

\[
\int_0^T c(t,t-a) \gamma \exp[-\nu a] da \leq \bar{c}(t) .
\]

This maximization problem distributes aggregate consumption per capita optimally across all generations alive at any given time \(t\). From its solution we learn how the social planner optimally distributes consumption between generations alive at the same time \(t\).

**Proposition 5 (Optimal consumption distribution for given time \(t\))**

The optimal solution of the maximization problem (27) subject to condition (28) is

\[
c(t,t-a) = \bar{c}(t) \frac{Q_T(\nu)}{Q_T(\nu + \sigma_H(\rho^H - \rho^S))} \exp\left[ -\sigma_H(\rho^H - \rho^S) a \right] .
\]

As a consequence, all households receive the same amount of consumption at time \(t\) irrespective of age for \(\rho^H = \rho^S\), and receive less consumption the older (younger) they are at a given time \(t\) for \(\rho^H > \rho^S\) \((\rho^H < \rho^S)\).

The proof is given in the appendix.
Proposition 5 says that the difference between the households’ rate of time preference $\rho_H$ and the social rate of time preference $\rho_S$ determines the social planner’s optimal distribution of consumption across households of different age at some given time $t$. In particular, at any instant of time the consumption profile with respect to the individuals’ age is qualitatively opposite to that of the decentralized solution if $\rho_H > \rho_S$, as can be seen from the Euler equation (4) and as illustrated in Figure 2. That is, in the social planner’s solution households receive less consumption the older they are, whereas they would consume more the older they are in the decentralized OLG economy. The intuition for this result is as follows. The social planner weighs the lifetime utility of every individual discounted to the time of birth. Thus, the instantaneous utility at time $t$ of those who are younger (born later) is discounted for a relatively longer time at the

\[\text{Figure 2: Distribution of consumption across all generations alive at given time } t \text{ dependent on age } a \text{ for the decentralized OLG and three different utilitarian OLGs.}\]

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We do not take up a stance on the relationship between the individual and the social rate of time preference, but merely hint at the resulting consequences. This is in line with Burton (1993), who argues that “…they represent profoundly different concepts” (p. 121/122) and, thus, may differ. In fact, $\rho_H$ trades off consumption today versus consumption tomorrow within each generation, while $\rho_S$ trades off lifetime utilities across generations. If they are supposed to differ, than it is usually assumed that $\rho_H > \rho_S$ (see also Heinzl and Winkler 2007: Sec. 2).
social planner’s time preference (before birth) and for a relatively shorter time by the individual’s time preference (after birth) than is the case for the instantaneous utility at $t$ of those who are older (born earlier). For $\rho^H > \rho^S$ the social planner’s time preference is smaller and, thus, the young generation’s utility at $t$ receives higher weight.

Proposition 5 shows that the standard approach of weighted intergenerational utilitarianism poses a trade-off between intertemporal generational equity and intratemporal generational equity to the social planner whenever households exhibit a positive rate of pure time preference. Lifetime utilities of today’s and future generations receive equal weight if and only if the social rate of time preference is zero. But then, $\rho^H > \rho^S = 0$ implies that at each point in time the young enjoy higher consumption than the old. In contrast, an equal distribution of consumption among the generations alive is obtained if and only if social time preference matches individual time preference. However, a positive social rate of time preference comes at the expense of an unequal treatment of lifetime utilities of different generations. This trade-off vanishes only if the individuals’ and the social planner’s rates of time preference are both equal to zero. Of course, this trade-off can only be captured in an OLG model which explicitly considers the life cycles of different generations.

We now turn to the second maximization problem for which we obtain

$$\max_{\{\bar{c}(t)\}_{t=0}^\infty} \int_0^\infty V(\bar{c}(t)) \exp[\nu t] \exp[-\rho^S t] \, dt, \quad (30)$$

subject to the budget constraint (10a). Observe that it is formally equivalent to an ILA economy with the instantaneous utility function $V(\bar{c}(t))$, the time preference rate $\rho^S$ and the intertemporal elasticity of substitution $\sigma^H$. We obtain $V(\bar{c}(t))$ by inserting the optimal consumption profile (29) into equation (27) and carrying out the integration

$$V(\bar{c}(t)) = \left[ \frac{Q_T(\nu + \sigma^H(\rho^H - \rho^S))}{Q_T(\nu)} \right]^{\frac{1}{\sigma^H}} \frac{\bar{c}(t)^{1-\frac{1}{\sigma^H}}}{1 - \frac{1}{\sigma^H}}. \quad (31)$$

The social planner’s maximization problem (30) is invariant under affine transformations of the objective function (31), in particular, under a multiplication with the inverse of the term in square brackets. But then we see that problem (30) is identical to the optimization problem we face in the ILA economy when setting the intertemporal elasticity of substitution $\sigma^R = \sigma^H$ and the time preference rate $\rho^R = \rho^S$.

\[10\] This was already observed by Calvo and Obstfeld (1988).
Proposition 6 (Unconstrained utilitarian OLG and ILA economy)

For a social planner maximizing the social welfare (26) of overlapping generations subject to the aggregate budget constraint (10a) (unconstrained utilitarian OLG economy) the following statements hold:

(i) Outside of a steady state, the unconstrained utilitarian OLG economy is observationally equivalent to the ILA economy if and only if \( \sigma^R = \sigma^H \) and \( \rho^R = \rho^S \).

(ii) For a steady state of the unconstrained utilitarian OLG economy (ILA economy) there exists an observationally equivalent ILA economy (unconstrained utilitarian OLG economy) if and only if

\[
\rho^R = \rho^S + \xi \frac{\sigma^R - \sigma^H}{\sigma^R \sigma^H}.
\]

The proof is given in the appendix.

Proposition 6 states that, setting \( \sigma^R = \sigma^H \) and \( \rho^R = \rho^S \), maximizing the utilitarian social welfare (26) of overlapping generations yields the same aggregate consumption and capital paths of the economy as maximizing the welfare (19) in the ILA model. This result, however, does not imply that the unconstrained social planner problem can, in general, be replaced by an ILA model.

First, to derive the equivalence result, we have assumed a social planner who does not exhibit any preferences for smoothing lifetime utility across generations. The parameter \( \sigma^H \) in equation (31) stems from the individuals’ preferences to smooth consumption within the lifetime of each generation. It is therefore a given primitive to the social planner. Thus, the only normative parameter the social planner may choose is the social time preference rate \( \rho^S \). It remains an open question for future research whether a different welfare functional for the unconstrained utilitarian social planner exists that permits a normative choice of \( \sigma^S \) for the social planner and still delivers observationally equivalent to an ILA model with \( \rho^S = \rho^R \).

Second, in the ILA setting, the first best solution can easily be decentralized, for example, via taxes that ensure the optimal path of the aggregate capital stock. However, this may not be the case for the unconstrained social planner’s problem as the latter is also concerned about the intratemporal allocation of consumption across all generations alive at a certain point in time. In the next proposition we, therefore, compare the outcome in the OLG economy managed by the unconstrained social planner to that in the decentralized OLG economy.
Proposition 7 (Unconstrained utilitarian OLG and decentralized OLG)

Let \((\sigma^H, \rho^H) \in \Gamma_{\Psi,T}\). Then

(i) the unconstrained utilitarian OLG and the decentralized OLG are observationally equivalent in the steady state if and only if the social planner’s rate of pure time preference satisfies

\[
\rho^S = r^*_d - \frac{\xi}{\sigma^H} (> \rho^H),
\]

where \(r^*_d\) denotes the steady state interest rate in the decentralized economy.

(ii) in the steady state, the unconstrained utilitarian OLG and the decentralized OLG exhibit the same allocation of consumption across the generations alive at each point in time if and only if they are observationally equivalent.

The proof is given in the appendix.

Proposition 7 implies that if the social planner’s rate of time preference is such that he wishes to implement the same path of aggregate capital as the one that is realized in the decentralized economy, then he also agrees to the intratemporal allocation of the decentralized OLG at all times. As indicated in the proposition, this equivalence between the utilitarian social planner OLG and the decentralized OLG implies that the social planner exhibits a larger rate of time preference than the individual households.\(^{11}\) This stands in contrast to most of the literature, where it is usually assumed that \(\rho^S \leq \rho^H\).

Note that in steady state, condition (33) corresponds to (23) when \(\rho^S = \rho^R\) holds. As a consequence, we obtain the following corollary:

**Corollary 3 (Decentralized versus unconstrained utilitarian OLG economy)**

Consider an ILA economy that is observationally equivalent in the steady state to a particular decentralized OLG economy. Then a utilitarian OLG economy that is observationally equivalent in the steady state to the ILA economy with \(\sigma^H = \sigma^R\) and \(\rho^S = \rho^R\) possesses the same allocation of consumption across the generations alive in the steady state as the decentralized OLG economy (to which the ILA economy is observationally equivalent in steady state).

The corollary states that if (33) holds, the utilitarian and the decentralized OLG economy are fully equivalent and observationally equivalent to the ILA economy. However, in case that \(\rho^S\) differs such that equation (33) is not satisfied, the unconstrained social

\(^{11}\) This result follows from Proposition 3.
planner not only prefers a different path of aggregate capital than in the decentralized market outcome, but also a different intratemporal allocation of consumption. Therefore, the decentralization of the social planner solution as a market outcome needs a transfer scheme that not only depends on time, but also discriminates with respect to age for each point in time. Such a transfer scheme may hardly be implementable in democratic societies. In fact, discrimination by race, gender or age is mostly considered as inequitable. As a consequence, we examine a social planner that cannot discriminate transfers by age but only influence prices via taxes and subsidies in the following section. As we shall show, in this case the social optimum cannot be achieved in general.

6.2. Constrained Utilitarian Overlapping Generations Economy

In contrast to redistribution that explicitly depends on age, it is plausible that a government can impose taxes/subsidies on capital and labor income. Hence, we extend the market system by a tax/subsidy regime, but do not consider fully dictatorial solutions, in which the social planner can directly dictate the consumption path of individual generations.

We use $\tau_r(t)$ and $\tau_w(t)$ to denote the tax/subsidy on returns on savings and on labor income, respectively. The individual households of the overlapping generations economy base their optimal consumption and saving decisions on the effective interest rate $r^e(t, \tau_r(t))$ and the effective wage $w^e(t, \tau_w(t))$ defined by

\begin{align}
    r^e(t, \tau_r(t)) &= r(t) - \tau_r(t) , \quad (34a) \\
    w^e(t, \tau_w(t)) &= w(t)[1 - \tau_w(t)] . \quad (34b)
\end{align}

Then, the individual budget constraint reads

\begin{align}
    \dot{b}(t, s) = r^e(t, \tau_r(t)) b(t, s) + w^e(t, \tau_w(t)) - c(t, s) . \quad (34c)
\end{align}

Given this budget constraint, individual households choose consumption paths which maximize lifetime utility (1). Thus, the optimal consumption path $c^e(t, s, \{r(t'), \tau_r(t'), \tau_w(t')\}_{t'=s+T}^{s+T})$ is a function of the paths of the interest rate $r(t)$ and the taxes $\tau_r(t)$ and $\tau_w(t)$.

Note that for a given path of the interest rate and given tax/subsidy schemes $\{r(t), \tau_r(t), \tau_w(t)\}$,
the individual household’s optimal paths of consumption and assets can be characterized as in the decentralized OLG economy by (2) and (4) when using \( r^e(t, \tau_r(t)) \) and \( w^e(t, \tau_w(t)) \) instead of \( r(t) \) and \( w(t) \), respectively. Applying the aggregation rule (9a) yields aggregate consumption per effective labor \( c^e(t, \{ r(t'), \tau_r(t'), \tau_w(t') \}^{t+T}_{t' = t-T}) \).

To analyze observational equivalence between such a constrained utilitarian OLG economy and an ILA economy, we have to restrict redistribution to mechanisms which do not alter the aggregate budget constraint (10a) of the economy. This implies that the redistribution scheme has to yield a balanced government budget at all times. This requires

\[
\tau_w(t) w(t) = -\tau_r(t) b(t) .
\]  

(35)

Assuming a balanced aggregate budget at all times allows us to disregard \( \tau_w(t) \) and to drop this argument in all functions that depend on the tax/subsidy scheme. Defining the aggregate instantaneous utility of all generations alive at time \( t \) as

\[
V(t, \{ r(t'), \tau_r(t') \}^{t+T}_{t' = t-T}) = 
\int_0^T c^e(t, t-a, \{ r(t'), \tau_r(t') \}^{t-a+T}_{t' = t-a}) \gamma \exp \left[ \left( \rho_S - \rho_H - \nu \alpha \right) a \right] da ,
\]

(36)

we obtain for the social planner’s problem

\[
\max_{\{ \tau_r(t') \}^{t=\infty}} \int_0^\infty V(t, \{ r(t'), \tau_r(t') \}^{t+T}_{t' = t-T}) \exp \left[ (\nu - \rho^S) t \right] dt ,
\]

(37)

subject to the budget constraint (10a). In general, the resulting system of integro-differential equations yields no closed form solutions. We can, however, achieve an analytical solution for the steady state which is given in the following proposition.

**Proposition 8 (Constrained utilitarian OLG and ILA economy)**

Let \( (\tau_r^*, r^*) \) be the optimal interest tax and interest rate in the steady state of a social planner maximizing (37) subject to (10a). For \( \sigma^R = \sigma^H \) this constrained utilitarian OLG economy is observationally equivalent in the steady state to an ILA economy if and only if

\[
\rho^R = \rho^S + c_k^e(r^*, \tau_r^*) - \frac{\dot{V}^*_k(r^*, \tau_r^*)}{\dot{V}^*_r(r^*, \tau_r^*)} c_k^e(r^*, \tau_r^*) ,
\]

(38)
with

$$c^{\epsilon\star}(r^\star, \tau^\star_r) = (w^\star + \tau^\star_r k^\star) \frac{Q_T(r^\star - \tau^\star_r - \xi)}{Q_T(\nu)} \frac{Q_T(r^\star - \tau^\star_r - \sigma^H (r^\star - \tau^\star_r - \rho^H))}{Q_T(\nu + \xi - \sigma^H (r^\star - \tau^\star_r - \rho^H))}, \quad (39a)$$

$$\hat{V}^\star(r^\star, \tau^\star_r) = \frac{\sigma^H}{\sigma^H - 1} (w^\star + \tau^\star_r k^\star) \frac{Q_T(r^\star - \tau^\star_r - \xi)}{Q_T(\nu)} \frac{\sigma^H - 1}{\sigma^H} Q_T(1 - \sigma^H (r^\star - \tau^\star_r)) + \sigma^H - 1 \xi + \nu - \sigma^H \rho^H - \rho^S) \times \frac{Q_T(r^\star - \tau^\star_r - \sigma^H (r^\star - \tau^\star_r - \rho^H))}{Q_T(r^\star - \tau^\star_r - \rho^H)} \frac{\sigma^H - 1}{\sigma^H}. \quad (39b)$$

The proof is given in the appendix.

**Remark:** From condition (38) alone it does not necessarily follow that $\rho^R \neq \rho^S$, as the term $c^{\epsilon\star}(r^\star, \tau^\star_r) - [\hat{V}^\star_k(r^\star, \tau^\star_r)c^{\epsilon\star}_k(r^\star, \tau^\star_r)]/\hat{V}^\star_k(r^\star, \tau^\star_r)$ could be equal to zero. In fact, this would imply that the constrained social planner would be able to achieve the first best optimum of the unconstrained social planner, in which case $\rho^R = \rho^S$ holds, as shown in Proposition 6. This, however, is not possible, in general. To see this, consider the case that $\rho^H > \rho^S$. Then, the social planners would like to implement a consumption profile that is qualitatively opposite to that of the decentralized OLG economy, as shown by Proposition 5 (see Figure 2 part c). In the case of constrained social planner, the individual households’ savings and consumption paths correspond to the ones in the decentralized economy for $w(t) = w^\star(t, \tau^\star_w(t))$ and $r(t) = r^\star(t, \tau^\star_r(t))$. In the steady state the optimal interest tax in the steady state $\tau^\star_r$ is constant and, thus, the consumption and savings patterns of the individual households are qualitatively unchanged to the corresponding ones in the decentralized market economy. That is, at each point in time the young consume less than the old (see Figure 2 part a). In particular, this implies that for all $\rho^H > \rho^S$ the second best optimum of the constrained social planner falls short of the first best optimum of the unconstrained social planner.

Together with the remark, Proposition 8 states that, in general, the time preference rate in the ILA economy cannot be interpreted as the social time preference rate of the constrained social planner in an observationally equivalent overlapping generations economy, even if $\sigma^R = \sigma^H$. Or put differently, a constrained social planner whose social time preference rate is identical to the time preference rate of the ILA economy prefers an aggregate steady state consumption and capital path that is different from the one in the ILA economy. As it seems realistic that a social planner, such as a democratically legitimized government, cannot arbitrarily redistribute between the generations alive at
each point in time, the result in Proposition 8 questions the validity of the ILA model as a tool to derive policy advice.

In addition, our result stands in contrast to Calvo and Obstfeld (1988) who show that even a planner that cannot discriminate (lump sum) taxes between different generations is able to reach the first best. The reason for the difference is that Calvo and Obstfeld (1988) use an OLG model of the Diamond (1965)-type with only two generations alive at each point in time. As only one generation – the young – decide on savings, one instrument is sufficient to effectively target transfers to one of the generations. In our model there exists a continuum of generations at each point in time prohibiting to effectively direct transfers to different generations when only market prices can be influenced.

7. Stern Versus Nordhaus – A Critical Review of Choosing the Social Rate of Time Preference

A prime example for questions of intergenerational equity is the mitigation of anthropogenic climate change, as most of its costs have to be borne today while the benefits spread over decades or even centuries. The question of optimal greenhouse gas abatement has been analyzed in so called integrated assessment models combining an ILA economy with a climate model. Interpreting the ILA’s utility function (19) as a utilitarian social welfare function, intergenerational equity concerns are closely related to the choice of intertemporal elasticity of substitution $\sigma^R$ and the pure rate of time preference $\rho^R$.

This is illustrated well by Nordhaus (2007), who compares two runs of his open source integrated assessment model DICE-2007. The first run uses his preferred specifications $\sigma^R = 0.5$ and $\rho^R = 1.5\%$. The second run applies $\sigma^R = 1$ and $\rho^R = 0.1\%$, which are the parameter values chosen by Stern (2007). These different parametrizations cause a difference in the optimal reduction rate of emissions in the period 2010–2019 of 14% versus 53% and a difference in the optimal carbon tax of 35$ versus 360$ per ton C.

In the following, we apply our results derived in Sections 5 and 6 to critically review recent approaches to the evaluation of climate change mitigation scenarios. We focus on Stern (2007), which we consider as a paradigm for the normative approach, and its critique by Nordhaus (2007), representative for the observational or positive approach. We find that neither Nordhaus’ (2007) observational nor Stern’s (2007) normative approach spell out the normative assumptions hidden in their respective descriptions of the intergenerational allocation problem. We identify the implicit assumptions and the
shortcomings in the current debate and lay out a more comprehensive foundation for approaching the valuation problem in the integrated assessment of climate change.

7.1. The observational or ‘Nordhaus’ approach

The majority of economists in the climate change debate takes an observation-based approach to social discounting. This view is exemplarily laid out in Nordhaus’ (2007) critical review of the Stern (2007) review of climate change. Individual preferences towards climate change mitigation cannot be observed directly in market transactions because of the public good characteristic of greenhouse gas abatement. However, we observe everyday investment decisions on capital markets that carry information on intertemporal preferences. In particular, we observe the steady state growth rate of the economy. Under the assumption of an ILA economy, the observed market interest rate $r^*$ translates into pairs of intertemporal elasticity of substitution $\sigma^R$ and pure time preference $\rho^R$. The observational approach transfers the observed preference information into the welfare function of a social planner. Reinterpreting the ILA economy as a utilitarian social planner model, the latter confronts the climate problem in an integrated assessment model.

Our paper provides the tools to critically reexamine the observational approach accounting for the finite life span of individuals living in an overlapping generations economy. In the first part of the discussion, we build on our insights from section 5 on observational equivalence (and numerical difference) between the OLG and the ILA economies only. Here, we focus on the case where the intertemporal elasticity of substitution is chosen to coincide between the representative agent and the OLG households. In the second part of the discussion, we analyze the observational approach from the perspective of our combined social planner OLG model derived in section 6 and comment on equity concerns and decentralizability.

In section 5 we have shown that, indeed, there exists an observationally equivalent ILA economy for the continuous time OLG economy. However, we have also shown that the pure rate of time preference of the imaginary infinitely lived agent does not reflect the actual time preference of the (homogeneous) individuals in the economy. The ILA model overestimates time preference for two reasons. First, the ILA model assumes that every individual plans for an infinite future when taking their market decisions. However, our overlapping generations only make plans for their own live span when revealing their preferences on the market. The mismatch between the individual optimization problems

25
underlying their market decisions and the assumptions of an infinitely lived agent evokes
the impression that individuals have a higher rate of pure time preference. Second, the
ILA model assumes that the representative consumer accounts for population growth
by giving more weight to the welfare of the larger future population. If our overlapping
generations dismiss this farsighted altruistic reasoning, the ILA approach once more
overestimates individual time preference.

The second step of the observational approach passes the preferences of the imaginary
infinitely lived agent on to a social planner. This step calls for an explicit justification,
given that we have shown these preferences to be different from those of any individual
in the economy. Thus, a serious problem of the ‘observational approach’ is that the
normative content of the preferences assigned to the social planner is concealed. They
are a combined estimate of agent’s preferences and macroeconomic characteristics.

A comprehensive discussion of the normative aspects involved in the observational ILA
approach must address the following issue. The social planner is endowed with an infinite
planning horizon and a welfare function that gives additional weight to a larger (future)
population. In exchange, his impatience is increased to match the equilibrium behavior of
decision makers that only plan for their own future. While the infinite time horizon and
the incorporation of population size into the welfare weight clearly show their normative
content, the normative content of ‘calibrating up’ time preference to match observed
behavior is less obvious. If we would like to capture only observed current preferences
a clear cut approach would terminate the time horizon of the social planner $T$ periods
into the future, use individual rates of time preference, and introduce a weight that
reflects the current individuals still alive at a given point of time in the future. If we
acknowledge that climate change is a problem where individuals agree to adopt time
horizons that exceed those relevant for their own life cycle plans, we can adopt an
infinite planning horizon. Then, however, it is not clear why we should increase individual
impatience in order to match short sighted equilibria. The same question arises in the
context of increasing individual impatience in order to match the fact that observed
individuals do not take account of future population growth. If we consider it adequate
to endow the social planner with a welfare function that gives more weight to larger
(future) populations, then why would we want to reduce the time preference to crowd
out this effect? We do not provide an answer to these normative questions, but point
out the normative content of the observational approach and its possible normative
inconsistencies. Moreover, our proposition 4 and corollary 1 provide a starting point for
extracting individual household’s time preference from the macroscopic ILA description
A numerical illustration shows how the inferred ILA preferences differ from actual household preferences. For our non-preference parameters underlying the economies we choose $\alpha = 0.3$, $\xi = 0.2$, $\nu = 0\%$, $T = 50$ and $r = 5.5\%$ close to Nordhaus (2007). For $\sigma^H = \sigma^R = 0.5$, as assumed in Nordhaus (2008) latest version of DICE, the ILA model implies a rate of pure time preference of the representative household (and social planner) of $\rho^R = 1.5\%$, while the individuals of the OLG economy have a time preference of $\rho^H = -5.5\%$. The surprising finding of a negative rate of pure time preference poses the question how reasonable the above specifications are. Analyzing the sensitivity of the negativity result with respect to the assumed parameter specifications, we find that the most promising parameter to change is the intertemporal elasticity of substitution $\sigma^H = 0.5$. However, the necessary change can as well be rooted back to a very different phenomenon. Vissing-Jørgensen and Attanasio (2003) estimate intertemporal substitutability in an approach building on Epstein and Zin (1991) and Campbell (1996) disentangling risk aversion from intertemporal substitutability. Their best guess is $\sigma = 1.5$, which is also part of a parameter constellation explaining best the equity premium puzzle. Employing their estimate for the households and the Ramsey consumer we find $\rho^H = 1.9\%$ as opposed to $\rho^R = 4.2\%$. Then, a social planner in the ILA model who would be equipped with the corresponding household preferences but an infinite time horizon would go along with a social discount rate of $\rho^H + \xi = 1.9\% + \frac{2\%}{1.5} = 3.2\%$ rather than $r = 5.5\%$.

Another approach to reveal individual household preferences for projects exceeding their life spans would be to explicitly analyze situations where intergenerational altruism can be observed even if the bequest motive is non-operative in every day consumption decisions (as is the assumption underlying this paper). It may well be that individuals are not sufficiently altruistic to make private bequests in capital but would be willing to pay taxes to finance a public good that benefits future generations such as the mitigation of climate change. For example, by extending the Diamond (1965) setting, as considered by Weil (1987), it can be shown that there exist values of the altruism parameter for

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13 The argument is similar to the “isolation paradox” in, e.g., Marglin (1963). In our model, such an interpretation requires that the lifetime utility of individual households possess an additional term reflecting altruism. For reasons of simplicity we have neglected this part in the analysis and simply assumed that the bequest motive is non-operative. As long as the latter assumption holds, the functional form of altruism does not matter for the optimal consumption and investment paths in our analysis. However, the functional form determines the level of taxes people are willing to contribute to the provision of the public good.
which the above situation occurs. In this way, a (constrained) social planner can find a positive foundation, for example, as a democratically legitimized government. However, such an interpretation also implies that market outcomes in the absence of government interventions would not provide information about the individual preferences towards the provision of public goods that mostly benefit future generations. Instead, it is exactly the state interventions that reveal altruistic intertemporal preferences. For example, in many economies we observe subsidies on different kinds of long-run investments such as education, retirement savings and research & development. These observed subsidies could, at least in principle, be interpreted as the households’ revealed preference on intergenerational equity.

In general, there exists an infinite set of combinations of \((\sigma^R, \rho^R)\) to match the observed market interest rate \(r^*\). Now, the crucial question is in how far is it justified to interpret such an observationally equivalent ILA economy as a utilitarian social planner economy? Interpreting the ILA economy as a social planner setting the following important aspects have to be recognized. First, the social welfare function \((25b)\) we considered does not include any preferences for smoothing lifetime utility of different generations over time. Of course, such functional forms are conceivable but it is not clear whether and how such a utilitarian OLG economy translates into an observationally equivalent ILA economy.

Second, the interpretation of the ILA economy as a utilitarian social planner OLG economy not only involves the intertemporal allocation of aggregates – which is given when the two are observationally equivalent – but also the intratemporal allocation of consumption across all generations alive at each point in time. Nordhaus (2007) ignores this second issue when arguing that the details of the ILA specification as well as individual households’ preferences can be neglected as long as the interest rates in the observed decentralized OLG and in the ILA economy coincide. As we have shown in Proposition 7, the intratemporal allocation of consumption of the utilitarian social planner only corresponds to that in the observationally equivalent decentralized OLG if the social time preference rate and the households’ intertemporal elasticity of substitution stand in a particular relation, as specified by condition \((33)\). Only in this case, all three economies, the decentralized OLG, the ILA and the utilitarian social planner OLG economy are

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14 A formal argument can be obtained upon request. Also Jouvet et al. (2000) show that in an OLG economy with a pollution externality, the old generation may abate pollution to the benefit of future generations but refrains from private bequests. This argument applies even more if man made capital and natural capital are not perfect substitutes but are at least partly complementary. For example, Cardia and Michel (2004) show that in an OLG model where transfers of time and bequests are available, there exist parameter values for which altruistic intergenerational time transfers take place but no bequests.
observationally equivalent in steady state and the decentralized and the social planner OLG economy exhibit the same steady state consumption profile. Interestingly – and in contrast to most of the normative literature – this specification implies that the social time preference rate is larger than the time preference rate of the individual households \( \rho^S > \rho^H \). That is, the social planner is less patient than the individual households.

Certainly, one could argue that there are other specifications of the unconstrained social planner in the utilitarian OLG economy that would achieve observational equivalence in the steady state, particularly if one allows the social planner to additionally choose \( \sigma \) (which, as pointed out above, should correspond to \( \sigma^H \)). However, in general this implies that condition (33) is violated and the intratemporal allocation of consumption in the unconstrained social planner solution differs from that in the decentralized OLG economy. Hence, depending on the magnitude of this difference, the implementation of such a social planner solution may involve substantial redistribution conditional on age. We have argued that such fiscal policies may not always be feasible in democratic societies. In this way, the particular interpretation of the ILA model as a social planner and hence individual and social preference parameters are important, as they can have a major impact on the political viability of an optimal policy derived in the ILA model.

The observationally equivalent ILA economy neglects aspects of intratemporal allocation across generations and is only concerned with the intertemporal allocation of aggregates. However, we have shown in Section 6.2 that the intratemporal allocation is an important issue, as, in the likely case that a government cannot fully discriminate the households by age, the second best solution with respect to the intertemporal allocation of aggregates is different from the first best derived in the ILA economy (Proposition 8).

We illustrate the implementation problem. Suppose that the social planner possesses a time preference rate of \( \rho^S = 1.5\% \) which corresponds to the specification of Nordhaus’ (2007) ILA economy. Further let \( \sigma^H = \sigma^R \). Consequently the ILA economy and the unconstrained utilitarian social planner OLG economy are observationally equivalent in steady state exhibiting an interest rate of \( r^* = 5.5\% \). The optimal capital path corresponding to this interest rate could be implemented if full discrimination with respect to age is possible. However, a constrained social planner who can only influence market prices and exhibits a pure rate of time preference of \( \rho^S = 1.5\% \) prefers a substantially different path of capital. Depending on the combination of \( (\sigma^H, \rho^H) \) ranging between \((0.79, 1\%)\) and \((1.42, 3\%)\), all yielding \( r^* = 5.5\% \) in the decentralized OLG economy, the constrained social planner OLG economy translates into an observationally equivalent ILA economy with \( (\rho^R, r^*) \) ranging from \((1.06\%, 3.59\%)\) to \((0.93\%, 2.35\%)\).
7.2. The normative or ‘Stern’ approach

The Stern (2007) review of climate change takes a purely normative approach to social discounting. Arguing that a positive rate of pure time preference is non-ethical, the Stern review effectively uses a zero rate of pure time preference. It adopts a parameter value $\rho^R = 0.1\%$ in order to capture a small but positive probability that society becomes extinct.\textsuperscript{15} By setting the pure time preference to zero the authors of the Stern review build on a series of statements and arguments supporting their choice from a normative perspective (Ramsey 1928, Pigou 1932, Harrod 1948, Koopmans 1965, Solow 1974, Broome and Schmalensee 1992). Less well founded from a normative perspective is the choice $\sigma^R = 1$ for the intertemporal elasticity of substitution. The inverse of the substitution elasticity is a measure for the preference to smooth consumption over time.

The Stern review interprets the ILA economy as a succession of infinitely short-lived non-overlapping generations whose wellbeing is measured by their consumption level. In this interpretation consumption smoothing translates into a measure for intergenerational equity. As pointed out by Dasgupta (2007), the choice $\sigma^R = 1$ corresponds to a rather low preference for intergenerational equity when judged side by side with a zero rate of pure time preference.

In a purely normative approach to social discounting it seems more natural to jump straight to an ILA model. By normatively justified assumptions the social planner exhibits an infinite planning horizon and particular values of pure time preference and intertemporal substitutability. However, it is obvious that a distinction or interaction between intergenerational weighting and individual time preference cannot be captured in such a model. Nevertheless, proposition 6 shows that a social planner fully controlling an OLG economy is observationally equivalent to his colleague controlling an ILA economy if the parameters $\sigma^R$ and $\rho^R$ are appropriately chosen. In particular, the intertemporal path of aggregate consumption does not depend on the individual time preference $\rho^H$, but only on the time preference of the social planner $\rho^S$, which coincides to the pure rate of time preference $\rho^R$ of the observationally equivalent ILA economy. This finding provides some support for Stern’s (2007) normative approach to intergenerational equity in the ILA model. However, the shortcut of setting up an ILA economy exhibits a number of caveats showing that questions of intergenerational equity are more complex than the ILA model reveals.

\textsuperscript{15} Strictly speaking this is not pure time preference, but Yaari (1965) shows the equivalence of discounting because of a constant probability of death/extinction and a corresponding rate of pure time preference.
First, according to Proposition 6, the interpretation of the time preference rate of the ILA economy as the time preference rate of a social planner in an observationally equivalent social planner OLG economy \( \rho^R = \rho^S \) requires that the intertemporal elasticity of substitution of the ILA has to be equal to that of the individual households in the OLG economy, i.e., \( \sigma^R = \sigma^H \). This, however, implies that the intertemporal elasticity of substitution is a primitive to the social planner and cannot be chosen to match particular normative considerations.\(^{16}\)

Second, our model allows us to analyze the resulting intratemporal distribution of consumption across all generations alive at some particular instant of time. In the most likely scenario, the normatively chosen social rate of time preference \( \rho^S \) is smaller than the individual rate of time preference \( \rho^H \). Then, according to Proposition 5, among all generations alive the oldest generation receives least consumption while the newborns get most (see Figure 2 part c). Thus, the standard approach of discounted utilitarianism poses a trade-off between intertemporal generational equity and intratemporal generational equity to the social planner whenever households exhibit a positive rate of pure time preference. This trade-off between the two different meanings of generational equity, one over time and one over age, is of course directly related to the welfare function adopted in equation (26). It is Calvo and Obstfeld’s (1988) straightforward continuous time extension of Samuelson’s 1967 welfare criterion. Calvo and Obstfeld (1988) show that many other functional forms run into time inconsistency problems.

Third, apart from the question whether consumption discrimination by age is justified on ethical grounds, it is questionable whether it could be implemented in a market economy. We have analyzed the second-best optimal solution in which the social planner’s choice of policy instruments is limited to non-age-discriminating taxes and subsidies. Proposition 8 establishes that the second best optimum is observationally equivalent (at least in the steady state) to an ILA economy. However, the pure rate of time preference \( \rho^S \) of the social planner in the OLG economy no longer coincides with the time preference \( \rho^R \) in the observationally equivalent ILA economy. As already discussed with respect to the observational approach, the desired intertemporal allocation of the aggregates of a social planner in the more realistic (non-dictatorial) OLG model is not reached by setting the rate of time preference \( \rho^R \) in the ILA economy to the normatively desired pure rate of time preference \( \rho^S \).

Again, we provide some illustrative numbers. Stern (2007) chooses \( \sigma^R = 1, \rho^R = 0.1\% \).

\(^{16}\) It is important to emphasize that we consider a specific utilitarian welfare function (25b) without intergenerational smoothing of lifetime utility.
$\xi = 1.3\%$ and $\nu = 0\%$ implying a real rate of interest $r^* = 1.4\%$. For the ILA economy to be observationally equivalent to the utilitarian OLG economy with the interpretation that $\rho^R = \rho^S$, then $\sigma^R = \sigma^H$ has to hold. As already pointed out by Nordhaus (2007), the choice of $\sigma^R$ has a crucial impact on the resulting steady state interest rate $r^*$. For $\sigma^H$ ranging from 0.25–2 real interest rates span the range from 5.3–0.75%. In particular, this implies that the Stern review drastically overestimates optimal carbon taxes if the households’ intertemporal elasticity of substitution is substantially smaller than one ($\sigma^H \ll 1$), as in this case the interest rate is much higher than $r^* = 1.4\%$.

Next, we illuminate the relationship between the ILA economy and the constrained social planner OLG economy. According to Proposition 8, the specification of Stern’s ILA economy is observationally equivalent to a constrained utilitarian social planner OLG economy exhibiting a time preference rate $\rho^S$ between 0.12% and 0.25% and subsidizing savings by 0.09–0.42%, depending on the individual time preference rate $\rho^H$. This means that if the optimal path derived in the ILA economy is implemented by a constrained social planner, this planner is much less patient than normatively desired.

On the other hand, specifying the constrained social planner with the normatively desired pure rate of time preference of $\rho^S = 0.1\%$ translates, depending on the combination of $\sigma^H$ and $\rho^H$ in the decentralized OLG economy, into an observationally equivalent ILA economy with $\rho^R$ ranging from 0.08–0.06% and a real interest rate $r^*$ between 1.38% and 1.36%. Hence, the constrained social planner would then prefer an even lower interest rate respectively even higher savings than the corresponding unconstrained social planner.

8. Conclusions

Although the lifetime of individuals is finite, intergenerational trade-offs are most often discussed within ILA frameworks, which are interpreted as a utilitarian social welfare function. In this paper, we analyzed to what extend this interpretation is justified, in particular, if we assume a non-operative bequest motive.

Therefore, we examined under which conditions an ILA economy is observationally equivalent to (i) a decentralized OLG economy and (ii) an OLG economy in which a social planner maximizes a utilitarian welfare function. We found that the utilitarian social planner faces a trade-off between intergenerational and intragenerational equity that cannot be captured in the ILA model. Furthermore, we have argued that even if the
optimal intertemporal allocation of the economy’s aggregates in the ILA economy corresponds to that of a utilitarian social planner, it may not be implementable in an OLG market economy. We applied our results to the recent dispute between Stern (2007) and Nordhaus (2007) in the discussion on the mitigation of climate change.

Our main insights are that, in contrast to Nordhaus (2007), if the ILA economy is supposed to be interpreted as a social planner OLG economy, the relationship between individual and social preference parameters do matter, as they determine the intragenerational distribution of consumption and, as a consequence, the political viability of the ILA’s optimal policy. Moreover, we argue that without an operative bequest motive intergenerational preferences cannot be deduced by observing market outcomes, in particular the steady state real interest rate. In addition, Stern’s (2007) normative choice of using an ILA model with a near zero rate of time preference, which is driven by the desire to treat all generations alike, neglects that this comes at the cost of a more unequal treatment of all generations alive at any point in time, at least if individuals possess a positive rate of pure time preference. For these reasons we conclude that ILA models are inappropriate for the analysis of intergenerational equity in long-run decision problems which should rather be examined in OLG frameworks.

Our analysis employs two central assumptions. First, we assume individual households to be selfish. Although several empirical studies support this view, extending the model to include different degrees of altruism is an interesting venue for future research. Second, we assume a specific utilitarian social welfare function. Although commonplace in the literature, other functional specifications are conceivable. It would be interesting to know how other specifications relate to the one we have chosen and to the standard ILA model.
A. Appendix

A.1. Proof of Proposition 1

We prove the existence of a non-trivial steady state, i.e. \( k^* \neq 0 \). For this purpose, we define the function \( J : \mathbb{R} \rightarrow \mathbb{R} \) by

\[
J(r) \equiv \frac{Q_T(r - \xi) Q_T(\nu + \xi - \sigma H(r - \rho H))}{Q_T(\nu) Q_T(r - \sigma H(r - \rho H))}, \quad \forall r \in \mathbb{R}.
\] (A.1)

Lemma 2 in the appendix summarizes some properties of \( J(r) \). Then, the steady state is given by the solution of the equation

\[
k = \phi(k) \equiv f'(k) - \left( \nu + \xi \right) \frac{f(k) - f'(k)}{f(k) - f'(k)k}.
\] (A.2)

Observe that \( \phi(k) \) is also well defined at the golden rule capital stock \( k^{gr} \), which is implicitly defined by \( r^{gr} \equiv \nu + \xi = f'(k^{gr}) \), as we obtain

\[
\lim_{k \rightarrow k^{gr}} \phi(k) = \left[ (f(k^{gr}) - f'(k^{gr})k^{gr}) \right] f'(k^{gr}) \in (0, \infty).
\] (A.3)

For all \( k \neq k^{gr} \), the equation \( k = \phi(k) \) is equivalent to

\[
\frac{f(k) - (\nu + \xi)k}{f(k) - f'(k)k} = J(f'(k)).
\] (A.4)

We shall discuss the solutions to this equation in terms of the interest rate \( r \) instead of the capital stock \( k \). Therefore, we define

\[
F(r) \equiv \frac{f(k(r)) - (\nu + \xi)k(r)}{f(k(r)) - f'(k(r))k(r)},
\] (A.5)

where \( k(r) = f^{-1}(r) \), which is well defined due to the strict monotonicity of \( f'(k) \). Also observe that \( k'(r) = 1/f''(k(r)) \). We seek the steady state interest rate \( r^* \), which is solution of the equation \( F(r^*) = J(r^*) \). The corresponding steady state capital stock is given by \( k^* = f^{r^*}(r^*) \).

First, observe that for \( r = r^{gr} \equiv \nu + \xi \)

\[
F(r^{gr}) = 1 = J(r^{gr}).
\] (A.6)

However, the corresponding golden rule capital stock \( k^{gr} \equiv f^{r^{gr}}(r^{gr}) \) is a steady state if and
only if
\[
  k^{gr} = \lim_{k \to k^{gr}} \phi(k) = \left[ f(k^{gr}) - f'(k^{gr})k^{gr} \right] J'(f'(k^{gr})) \\
  \iff F'(r^{gr}) = \frac{k^{gr}}{f(k^{gr}) - r^{gr}k^{gr}} = J'(r^{gr}) .
\]  
(A.7)

Thus, we can distinguish three different cases depending on whether \( F'(r^{gr}) \) is (i) equal, (ii) smaller or (iii) greater than \( J'(r^{gr}) \).

(i) \( F'(r^{gr}) = J'(r^{gr}) \)

As just shown, in this case there exists a steady state \( k^* = k^{gr} \).

(ii) \( F'(r^{gr}) < J'(r^{gr}) \)

In this case, there exists a steady state \( k^* > k^{gr} \) (inefficient steady state), because \( J(0) > 0 \) and

\[
  \lim_{r \to 0} F(r) = \lim_{r \to 0} \left( \frac{f(k(r))/k(r) - (\nu + \xi)}{f(k(r))/k(r) - r} \right) = -\infty .
\]  
(A.8)

(iii) \( F'(r^{gr}) > J'(r^{gr}) \)

A sufficient condition for the existence of a steady state \( k^* < k^{gr} \) (efficient steady state) is that

\[
  \lim_{r \to \infty} \frac{J'(r)}{J(r)} > \lim_{r \to \infty} \frac{F'(r)}{F(r)} .
\]  
(A.9)

We know from part (iii) and (v) of Lemma 2 that

\[
  \lim_{r \to \infty} \frac{J'(r)}{J(r)} = \begin{cases} 
    \sigma T , & \text{if } \sigma \in (0,1] \\
    T , & \text{if } \sigma > 1
  \end{cases} .
\]  
(A.10)

In addition, we obtain for

\[
  \lim_{r \to \infty} \frac{F'(r)}{F(r)} = \lim_{r \to \infty} \left[ \frac{f'(k(r)) - (\nu + \xi)}{k(r)f''(k(r)) [f(k(r)) - (\nu + \xi)]} + \frac{k(r)}{f(k(r)) - rk(r)} \right] .
\]  
(A.11)

Thus, a sufficient condition for (A.9) to hold is

\[
  \lim_{r \to \infty} \frac{k(r)}{f(k(r)) - rk(r)} = \lim_{k \to 0} \frac{k}{f(k) - f'(k)k} \\
  = \lim_{k \to 0} \frac{1}{f''(k)k} < \begin{cases} 
    \sigma T , & \text{if } \sigma \in (0,1] \\
    T , & \text{if } \sigma > 1
  \end{cases} ,
\]  
(A.12)

which implies condition (15).
A.2. Proof of Proposition 2

Given that condition (15) holds, we know from the proof of Proposition 1 that there exists a steady state with $k^* < k^{gr}$ if $F'(r^{gr}) > J'(r^{gr})$. As $F'(r^{gr}) > 0$, a sufficient condition for $F'(r^{gr}) > J'(r^{gr})$ to hold is that

$$J'(r^{gr}) = q(\nu - q((\nu + \xi)(1 - \sigma^H) + \sigma^H \rho^H)) \leq 0.$$  \hfill (A.13)

As $q'(r) > 0$ due to part (v) of Lemma 1, condition (A.13) holds if and only if (17) is satisfied.

We now derive sufficient conditions such that there exists only one steady state $k^* < k^{gr}$. Suppose that condition (15) holds, which guarantees existence of a dynamically efficient steady state. There exists only one steady state interest rate $r^*$ with $r^* > r^{gr}$ if and only if

$$F'(r)|_{r=r^*} < J'(r)|_{r=r^*}, \quad \forall r^* > r^{gr}$$

$$\Leftrightarrow \frac{F'(r)}{F(r)}|_{r=r^*} < \frac{J'(r)}{J(r)}|_{r=r^*}, \quad \forall r^* > r^{gr}. \hfill (A.14)$$

The second line holds, as $F(r) = J(r)$ for all $r = r^*$. A sufficient condition for (A.14) to hold is that

$$\frac{d}{dr} \left( \frac{F'(r)}{F(r)} \right)|_{r=r^*} < 0 \quad \land \quad \frac{d}{dr} \left( \frac{J'(r)}{J(r)} \right)|_{r=r^*} > 0, \quad \forall r^* > r^{gr}. \hfill (A.15)$$

From part (ii) and (iv) of Lemma 2 we know that the second condition holds for all $r > r^{gr}$ if, in case that $\sigma > 1$, also condition (18b) holds.

From the second to the third line we employed $F(r) = J(r)$ for all $r = r^*$. We show in the following that $g_1'(r) \leq 0$ and $g_2'(r) \geq 0$ are sufficient for $\frac{d}{dr} \left( \frac{F'(r)}{F(r)} \right)|_{r=r^*} < 0$.

First, observe from equation (A.2) that $J(r^*) > 1$ for all $r^* > r^{gr}$. As $J(r)$ is U-shaped on $r \in (r^{gr}, \infty)$ because of part (ii) and (iv) of Lemma 2 and $J(r^{gr}) = 1$, this implies that $J'(r^*) > 0$ for all $r^* > r^{gr}$.
Second, we show that \( \frac{F'(r)}{F(r)} \bigg|_{r=r^*} > 0 \) for all \( r^* > r^{\sigma r} \) if \( g_2'(r) \geq 0 \). Observe that

\[
\lim_{r \to \infty} \frac{F'(r)}{F(r)} \bigg|_{r=r^*} = \lim_{r \to \infty} \left[ \frac{1}{k(r) f''(k(r))} \left( 1 - \frac{1}{J(r)} \right) + \frac{k(r)}{f(k(r)) - r k(r)} \right] = \frac{1}{k(r) f''(k(r))} + \frac{k(r)}{f(k(r)) - r k(r)} \]

(A.17a)

\[
= \lim_{r \to \infty} \left[ \frac{1}{k(r) f''(k(r))} - \frac{k(r)}{f(k(r)) - r k(r)} \right] = 0 .
\]

(A.17b)

In addition, we know that \( g_1(r) > 0 \) for all \( r > 0 \) and

\[ \lim_{r \to \infty} g_1(r) = \lim_{r \to \infty} \frac{1}{k(r) f''(k(r))} > 0 . \]

(A.18)

The latter implies together with equation (A.17)

\[ \lim_{r \to \infty} g_2(r) \left( 1 - \frac{1}{J(r)} \right) = 1 . \]

(A.19)

As \( g_2(r) \left( 1 - \frac{1}{J(r)} \right) \) equals zero at \( r = r^{\sigma r} \) and is monotonically increasing in \( r \) for \( g_2'(r) \geq 0 \), this implies that \( \frac{F'(r)}{F(r)} \bigg|_{r=r^*} > 0 \) for all \( r^* > r^{\sigma r} \). Then, we obtain for \( g_1'(r) \leq 0 \) and \( g_2'(r) \geq 0 \)

\[
\frac{d}{dr} \left( \frac{F'(r)}{F(r)} \bigg|_{r=r^*} \right) = g_1'(r) \left[ 1 - \left( 1 - \frac{1}{J(r)} \right) g_2(r) \right] - g_1(r)g_2'(r) \left( 1 - \frac{1}{J(r)} \right) < 0 .
\]

(A.20)

The conditions \( s(k) \geq e(k) \) and \( \frac{d}{dk} \left( \frac{\varepsilon(k)}{\tau(k)} \right) \) are sufficient for \( g_1'(r) \leq 0 \) and \( g_2'(r) \geq 0 \).

\( \square \)

### A.3. Proof of Proposition 3

We show that \( \sigma(r^* - \rho^H) - \xi > 0 \) is a necessary condition for aggregate assets \( b^* \) to be strictly positive in a dynamically efficient steady state, i.e., \( (\sigma^H, \rho^H) \in \Gamma^\psi_T \). As \( b^* = k^* \) holds, this implies that for \( k^* > 0 \) the steady state real interest rate must exceed \( \rho^H + \frac{\xi}{\psi} \).

The household’s wealth, as given by equation (13b), can be re-written to yield

\[
b^*(a) = \frac{w^*}{r^* - \xi} \left[ \theta \exp \left( [\sigma(r^* - \rho^H) - \xi] a \right) + (1 - \theta) \exp \left( (r^* - \xi) a \right) - 1 \right] .
\]

(A.21)
with
\[
\theta = \frac{1 - \exp[-(r^* - \xi)T]}{1 - \exp[-(r^* - \sigma_H (r^* - \rho))T]}.
\] (A.22)

Assuming a dynamically efficient steady states implies that \( r^* - \xi > 0 \) and we obtain from (A.22)
\[
\theta \begin{cases} 
< 1, & \text{if } \sigma(r^* - \rho_H) - \xi < 0 \\
= 1, & \text{if } \sigma(r^* - \rho_H) - \xi = 0 \\
> 1, & \text{if } \sigma(r^* - \rho_H) - \xi > 0
\end{cases}.
\] (A.23)

Thus, we can directly infer from (A.21) that \( b^*(a) = 0 \) for all \( a \in [0, T] \) for \( \sigma(r^* - \rho_H) - \xi = 0 \). As all households hold no assets, the aggregate capital stock equals zero. To show that \( \sigma(r^* - \rho_H) - \xi < 0 \) precludes strictly positive capital stocks, we analyze the second derivative of \( b^*(a) \)
\[
\frac{d^2 b^*(a)}{da^2} = \frac{w^*}{r^* - \xi} \left\{ \theta \left( \sigma(r^* - \rho_H) - \xi \right)^2 \exp \left[ (\sigma(r^* - \rho_H) - \xi) a \right] \\
+ (1 - \theta) (r^* - \xi)^2 \exp[(r^* - \xi) a] \right\}.
\] (A.24)

For \( \sigma(r^* - \rho_H) - \xi < 0, \theta < 1 \) holds, which implies that \( \frac{d^2 b^*(a)}{da^2} > 0 \). Hence, the household’s wealth profile is strictly convex. Together with the boundary conditions \( b^*(0) = 0 = b^*(T) \) this implies that all households possess non-positive wealth at all times. This, in turn, precludes \( k^* > 0 \).

Further, it is obvious from (A.21) and (A.24) that \( \sigma(r^* - \rho_H) - \xi > 0 \) does not contradict strictly positive wealth of the individual households and, therefore, is a necessary condition for \( k^* > 0 \).

**A.4. Proof of Proposition 4**

(i) Both economies exhibit the same \( \Psi \) by assumption and, thus, the market equilibria on the capital and the labor market imply that the equations of motion for the aggregate capital per effective labor (21b) and (10a) coincide. If condition (22) holds, then also the Euler equations (10b) and (21a) coincide.

“\( \Rightarrow \)”: Suppose that condition (22) holds, then the system dynamics of both economies is governed by the same system of two ordinary first order differential equations, the solution of which is uniquely determined by some initial conditions on \( c \) and \( k \). Thus, if the two economies coincide in the levels of consumption and capital at one point in time they also do so for all future times. As a consequence, the two economies are observationally equivalent.

“\( \Leftarrow \)”: Suppose the two economies are observationally equivalent, i.e., they coincide in the levels of consumption and capital at all future times if they coincide in these levels today. For this to hold the system dynamics of the two economies has to be governed by the same system of differential equations. Equating the Euler equations (10b) and (21a) yields condition (22).
(ii) The left-hand side of condition (22) is constant. Thus, for the condition to hold for all $t \geq 0$ the right-hand side has to be constant as well. In a steady state this holds, as $r(t) = r^*$ and $\Delta c(t) / c(t) = \nu + \xi - \sigma^H (r^* - \rho^H)$ are constant. Then, for any given set $\{\sigma^H, \rho^H\} \in \Gamma_{\Psi, T}$ there exists a steady state with $r^* = \rho^R$. This is also a steady state in the ILA economy if the set $\{\sigma^R, \rho^R\} \in \mathbb{R}^+ \times \mathbb{R}^+$ is such that condition (22) holds. Such a set always exists, as it has to satisfy
\[
r^* = \rho^R + \frac{\xi}{\sigma^R}.
\] \hfill (A.25)

\section*{A.5. Proof of Proposition 5}

The optimization problem (27) subject to condition (28) is equivalent to a resource extraction model (or an isoperimetrical control problem). We denote consumption at time $t$ of an individual of age $a$ by $C(a) = c(t, t - a)$ and define the stock of consumption left to distribute among those older than age $a$ by
\[
y(a) = \bar{c}(t) - \int_0^a C(a') \gamma \exp[-\nu a'] da'.
\] \hfill (A.26)

Then the problem of optimally distributing between the age groups is equivalent to optimally ‘extracting’ the consumption stock over age (instead of time). The equation of motion of the stock is $dy/da = -C(a)\gamma \exp[-\nu a]$, the terminal condition is $y(T) \geq 0$, and the present value Hamiltonian reads
\[
H = \frac{C(a)^{1 - \frac{\sigma^H}{\rho^R}}}{1 - \frac{\sigma^H}{\rho^R}} \gamma \exp [(\rho^S - \rho^H - \nu)a] - \lambda(a)C(a)\gamma \exp[-\nu a],
\] \hfill (A.27)

where $\lambda(a)$ denotes the co-state variable of the stock $y$. The first order conditions yield
\[
\lambda(a) = C(a) \gamma \exp [(\rho^S - \rho^H - \nu)a], \quad \hfill (A.28a)
\]
\[
\dot{\lambda}(a) = 0, \quad \hfill (A.28b)
\]

which imply that
\[
C(a) = C(0) \exp [\sigma^H (\rho^S - \rho^H)a]. \quad \hfill (A.29)
\]

As $\lambda(T)$ is obviously not zero, transversality implies that $y(T) = T$. From equation (A.26), acknowledging $Q_T(\nu) = \frac{1}{\gamma}$, we therefore obtain
\[
C(0) = \bar{c}(t) \frac{Q_T(\nu)}{Q_T(\nu + \sigma^H (\rho^H - \rho^S))}. \quad \hfill (A.30)
\]
which together with equation (A.29) returns equation (29).

\[ \square \]

A.6. Proof of Proposition 6

(i) The equivalence of the unconstrained social planner problem and of the optimization problem in the ILA economy pointed out in relation to equations (30) and (31) implies the Euler equation of the unconstrained social planner economy

\[
\frac{\dot{c}(t)}{c(t)} = \sigma^H \left[ r(t) - \rho^S \right] - \xi. \tag{A.31}
\]

For both economies the Euler equation implies that a time varying consumption rate also implies a time varying interest rate (and obviously so does a time varying capital stock).

For observational equivalence to hold, consumption and interest rate of the unconstrained utilitarian OLG economy have to coincide with that of the ILA economy, implying the following equality of the Euler equations

\[
\sigma^H \left[ r(t) - \rho^S \right] - \xi = \sigma^R \left[ r(t) - \rho^R \right] - \xi \tag{A.32}
\]

\[
\Leftrightarrow \sigma^R \rho^R - \sigma^H \rho^S = (\sigma^R - \sigma^H) r(t). \tag{A.32}
\]

For a time varying interest rate this equation can only be satisfied if \( \sigma^R = \sigma^H \) and \( \rho^R = \rho^S \).

If \( \sigma^R = \sigma^H \) and \( \rho^R = \rho^S \) hold, the equivalence of the two problems was explained in relation to equations (30) and (31).

(ii) Existence of an observationally equivalent ILA economy implies that, first, the ILA economy has to be in a steady state as well and, second, that the steady state Euler equations have to coincide implying

\[
\frac{\dot{r}}{\sigma^R} = \frac{\dot{\rho^S}}{\sigma^H} = \rho^S - \frac{\xi}{\sigma^H} \Rightarrow \rho^R - \rho^S = \xi \frac{\sigma^R - \sigma^H}{\sigma^R \sigma^H}. \tag{A.33}
\]

The exact same reasoning applies when starting from the ILA economy steady state and assuming an observationally equivalent unconstrained utilitarian OLG economy.

If equation (32) is satisfied and the unconstrained utilitarian OLG economy is in a steady state we have by equation (A.31) that

\[
r^S = \rho^S + \frac{\xi}{\sigma^H}. \tag{A.33}
\]
Using equation (32) to substitute $\rho^S$ on the right hand side yields

$$r^S = \rho^R - \xi \frac{\sigma^R - \sigma^H}{\sigma^R \sigma^H} + \xi \frac{\sigma^H}{\sigma^R} = \rho^R + \frac{\xi}{\sigma^R} = r^R .$$

(A.34)

Thus, also the ILA economy is in a steady state (see section 4) with the coinciding interest rate. As the interest rates coincide, so does the capital stock and so do the consumption paths. Starting with the ILA steady state with interest rate $r^R$ yields a coinciding unconstrained utilitarian OLG steady state by the same procedure. $\square$

A.7. Proof of Proposition 7

(i) According to the proof of Proposition 6, the Euler equation of the unconstrained social planner solution is

$$\frac{\dot{c}(t)}{c(t)} = \sigma^H \left[ r(t) - \rho^S \right] - \xi .$$

In steady state we obtain for the interest rate:

$$r^*_S = \rho^S + \frac{\xi}{\sigma^H} .$$

Observational equivalence holds if and only if $r^*_S = r^*_d (= r^*_r)$, and consequently if and only if

$$\rho^S = r^*_d - \frac{\xi}{\sigma^H} .$$

(ii) Using (29), we can write the intratemporal allocation of consumption across the generations alive in steady state in the unconstrained utilitarian OLG as

$$c^*_S(a) = \frac{c(t, t-a)}{\exp[\xi t]} = c^* \frac{Q_T(\nu)}{Q_T(\nu + \sigma^H(\rho^H - \rho^S))} \exp[-\sigma^H(\rho^H - \rho^S)a] .$$

(A.35)

The intratemporal allocation of consumption in the decentralized OLG economy is given by (13a) and can be written as

$$c^*_d(a) = c^* \frac{Q_T(\nu)}{Q_T(\nu + \xi - \sigma^H(r^*_d - \rho^H))} \exp[(\sigma^H(r^*_d - \rho^H) - \xi)a] .$$

(A.36)

$\Rightarrow$: Suppose that the allocation of consumption across all generations alive at each point is...
For this to be the case, we must necessarily have for all \( a \in [0, T] \)

\[
\exp[-\sigma^H(\rho^H - \rho^S)a] = \exp[(\sigma^H(r^*_d - \rho^H) - \xi)a] \text{ and }
\sigma^H(\rho^H - \rho^S) = \xi - \sigma^H(r^*_d - \rho^H).
\]

Minor mathematical transformations show that the only value for \( \rho^S \) that solves the two equations is the one given in condition (33). This is the condition for the unconstrained utilitarian OLG and the decentralized OLG to be observationally equivalent in steady state. \( \Leftarrow \): Now suppose that the unconstrained utilitarian OLG and the decentralized OLG are observationally equivalent in steady state, i.e. that (33) is satisfied.

Inserting \( \rho^S \) as given by (33) into (A.35) yields

\[
c^*_S(a) = c^* \frac{Q_\tau(\nu)}{Q_\tau(\nu + \xi - \sigma^H(r^*_d - \rho^H))} \exp[(\sigma^H(r^*_d - \rho^H) - \xi)a].
\]

The comparison with (A.36) immediately reveals that both are identical. Hence, observational equivalence in steady state is also sufficient for identical allocations across the generations alive in both economies. \( \Box \)

**A.8. Proof of Proposition 8**

First, we seek the optimal steady state values \((\tau^*_e, r^*_e)\) of a social planner maximizing (37) subject to (10a). The (present value) Hamiltonian reads

\[
\mathcal{H} = V(t, \{\tau_\tau(t')\}_{t'=t-T}^{t+T}) \exp [(\nu - \rho^S)t] \\
+ \lambda(t) \left[f(k(t)) - (\nu + \xi)k(t) - c^e(t, \{\tau_\tau(t')\}_{t'=t-T}^{t+T})\right],
\]

(A.37)
yields the following first-order conditions:

\[
\lambda(t) = \exp [(\nu - \rho^S)t] \frac{V_{\tau_\tau}(t, \{\tau_\tau(t')\}_{t'=t-T}^{t+T})}{c^e_\tau(t, \{\tau_\tau(t')\}_{t'=t-T}^{t+T})},
\]

(A.38a)

\[
\dot{\lambda}(t) = \lambda(t) \left[c^e_\tau(t, \{\tau_\tau(t')\}_{t'=t-T}^{t+T}) + \nu + \xi - r(t)\right] \\
- V_k(t, \{\tau_\tau(t')\}_{t'=t-T}^{t+T}) \exp [(\nu - \rho^S)t].
\]

(A.38b)

We now solve for the steady state Euler equation. Therefore, we calculate \( c^e(t, \{\tau_\tau(t')\}_{t'=t-T}^{t+T}) \) and \( V(t, \{\tau_\tau(t')\}_{t'=t-T}^{t+T}) \) in the steady state. We obtain aggregate consumption per effective labor in the steady state by replacing \( r^* \) and \( w^* \) by \( r^{e*} = r^* - \tau^*r \) and \( w^{e*} = w^* + \tau^*k^* \) in equation
Then, we obtain for instantaneous utility in the steady state

\[
\epsilon^*(\tau^*_r, r^*) = (w^* + \tau^*_r k^*) \frac{Q_T(r^* - \tau^*_r - \xi) Q_T(r^* - \tau^*_r - \sigma^H (r^* - \tau^*_r - \rho^H))}{Q_T(\nu + \xi - \sigma^H (r^* - \tau^*_r - \rho^H))} .
\]  
(A.39)

To calculate instantaneous utility in steady state, we need individual consumption \(\epsilon^*(t, t - a, \{\tau_r(t')\}_{t'=s}, \{\tau_w(t')\}_{t'=s})\) in the steady state, which we obtain from equation (13a) by replacing \(r^*\) and \(w^*\) by \(r^{es} = r^* - \tau^*_r\) and \(w^{es} = w^* + \tau^*_r k^*\)

\[
\epsilon^*(t, a, \tau^*_r, r^*) = (w^* + \tau^*_r k^*) \frac{Q_T(r^* - \tau^*_r - \xi) Q_T(r^* - \tau^*_r - \sigma^H (r^* - \tau^*_r - \rho^H))}{Q_T(\nu + \xi - \sigma^H (r^* - \tau^*_r - \rho^H))} 
\times \exp[\xi t] \exp \left[ (\sigma^H (r^* - \rho^H) - \xi) a \right] .
\]  
(A.40)

Then, we obtain for instantaneous utility in the steady state

\[
V^*(t, \tau^*_r, r^*) = \int_0^T \frac{\epsilon^*(t, a, \tau^*_r, r^*)}{1 - \frac{\gamma}{\sigma^H}} da \exp \left[ (\rho^S - \rho^H - \nu) a \right] \exp \left[ \frac{\sigma^H - 1}{\sigma^H} \xi t \right] .
\]  
(A.41a)

\[
= \frac{\sigma^H}{\sigma^H - 1} \left( w^* + \tau^*_r k^* \right)^{\frac{\sigma^H}{\sigma^H - 1}} \frac{Q_T(r^* - \tau^*_r - \xi) Q_T(r^* - \tau^*_r - \sigma^H (r^* - \tau^*_r - \rho^H))^{\frac{\sigma^H}{\sigma^H - 1}}}{Q_T(\nu)} \times \exp \left[ (\sigma^H (r^* - \rho^H) - \xi) a \right] \exp \left[ (\rho^S - \rho^H - \nu) a \right],
\]  
(A.41b)

\[
= \frac{\sigma^H}{\sigma^H - 1} \exp \left[ \frac{\sigma^H - 1}{\sigma^H} \xi t \right] \left( w^* + \tau^*_r k^* \right)^{\frac{\sigma^H}{\sigma^H - 1}} Q_T(r^* - \tau^*_r - \xi) Q_T(\nu) \times \exp \left[ \frac{1 - \sigma^H}{\sigma^H} (r^* - \tau^*_r) + \frac{\sigma^H - 1}{\sigma^H} \xi + \nu - \sigma^H \rho^H - \rho^S \right] \frac{Q_T(r^* - \tau^*_r - \sigma^H (r^* - \tau^*_r - \rho^H))^{\frac{\sigma^H}{\sigma^H - 1}}}{Q_T(\nu)} 
\]  
(A.41c)

We also introduce the following definition

\[
\hat{V}^*(\tau^*_r, r^*) \equiv V^*(t, \tau^*_r, r^*) \exp \left[ -\frac{\sigma^H - 1}{\sigma^H} \xi t \right] .
\]  
(A.42)

Inserting \(\epsilon^*(\tau^*_r, r^*)\) and \(V^*(t, \tau^*_r, r^*)\) into equation (A.38a) and differentiating with respect to time yields:

\[
\dot{\lambda}(t) = \left( \frac{\sigma^H - 1}{\sigma^H} \xi + \nu - \rho^S \right) \lambda(t) ,
\]  
(A.43)

Inserting equations (A.43) and (A.38a) into equation (A.38b) yields the Euler equation

\[
\rho^S = \frac{\dot{V}_k(\tau^*_r, r^*)}{\hat{V}_\tau(\tau^*_r, r^*)} \epsilon^*(\tau^*_r, r^*) - c^*_T(\tau^*_r, r^*) + r^* - \frac{\xi}{\sigma^H} .
\]  
(A.44)

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This equation, together with the aggregate budget constraint (10a) in steady state, determines the optimal investment tax \( \tau^* \) and the optimal interest rate \( r^* \) (or the capital stock \( k^* \)) of the social planner in the steady state.

Now, suppose that \((\tau^*, r^*)\) are the optimal steady state values of a social planner maximizing (37) subject to (10a). Then, an ILA economy is observationally equivalent in steady state to this constrained utilitarian OLG economy, if and only if it exhibits the same steady state interest rate \( r^* \). From the Euler equation (21a) of the ILA economy we know:

\[
\rho^R = r^* + \frac{\xi}{\sigma H}.
\] (A.45)

Replacing \( r^* - \frac{\xi}{\sigma H} \) by \( \rho^R \) yields equation (38).

\[\square\]

**A.9. Characteristics of the functions characterizing the steady state capital stock**

**Lemma 1**
The function \( Q_T(r) \) defined in (12) satisfies:

(i) \( Q_T(r) > 0 \) for all \( r \in \mathbb{R} \),

(ii) \( Q'_T(r) < 0 \) for all \( r \in \mathbb{R} \).

The function

\[
q(r) = \frac{Q'_T(r)}{Q_T(r)} = \frac{T}{\exp(rt) - 1} - \frac{1}{r},
\] (A.46)

satisfies

(iii) \( q(r) < 0 \) for all \( r \in \mathbb{R} \),

(iv) \( \lim_{r \to -\infty} q(r) = 0 \) and \( \lim_{r \to -\infty} q(r) = -T \),

(v) \( q'(r) = q'(-r) > 0 \) for all \( r \in \mathbb{R} \),

(vi) \( q'(r) > z^2q'(zr) \) for all \( r \in \mathbb{R}, z \in (0,1) \),

(vii) \( y^2q'(yr) > z^2q'(zr) \) for all \( r \in \mathbb{R}, y > z \geq 1 \),

(viii) \( q''(r) < 0 \) for all \( r \in \mathbb{R}_{++} \).

**Proof:** (i) Obviously, \( Q_T(r) > 0 \) for all \( r \neq 0 \). In addition, \( \lim_{r \to 0} Q_T(r) = T > 0 \).

(ii) We obtain

\[
Q'_T(r) = -\frac{1 - \exp[-rT](1 + rT)}{r^2}.
\]
For all \( r \neq 0 \):

\[
Q_\alpha(r) < 0 \iff \exp[-rT](1 + rT) < 1 \iff 1 + rT < \exp[rT].
\]

The last inequality holds as \( x + 1 < \exp[x] \) for all \( x \in \mathbb{R} \). In addition, \( \lim_{r \to 0} Q_\alpha(r) = -\frac{T^2}{2} < 0 \).

(iii) Follows directly from items (i) and (ii).

(iv) Follows directly from the definition (A.46).

(v) We obtain:

\[
q'(r) = -\frac{1}{r^2} - \frac{T^2 \exp[-rT]}{(1 - \exp[-rT])^2} = \frac{1}{r^2} - \frac{T^2}{2(\cosh[rT] - 1)}.
\]

For all \( r \neq 0 \):

\[
q'(r) > 0 \iff 2(\cosh[rT] - 1) > r^2T^2 \iff \cosh[rT] > 1 + \frac{r^2T^2}{2}.
\]

The last inequality holds as \( \cosh[x] > 1 + \frac{x^2}{2} \) for all \( x \in \mathbb{R} \). In addition, \( \lim_{r \to 0} q'(r) = \frac{T^2}{12} > 0 \).

(vi) The statement holds if and only if:

\[
q'(r) - z^2q'(zr) = -\frac{z^2T^2}{2(\cosh[zrT] - 1)} - \frac{T^2}{2(\cosh[rT] - 1)} > 0
\]

\[
\iff z^2(\cosh[rT] - 1) > \cosh[zrT] - 1.
\]

To see that the last inequality holds, we employ the infinite series expansion of \( \cosh[x] \):

\[
z^2(\cosh[x] - 1) - (\cosh[zr] - 1) = z^2 \left( \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} - 1 \right) - \left( \sum_{n=0}^{\infty} \frac{(zx)^{2n}}{(2n)!} - 1 \right)
\]

\[
= z^2 \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!} - \sum_{n=1}^{\infty} \frac{(zx)^{2n}}{(2n)!} = \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!} \left( z^2 - z^{2n} \right) > 0.
\]

The inequality holds, as the first summand is zero and all other terms are strictly positive for all \( z \in (0, 1) \).

(vii) The statement holds if and only if:

\[
y^2q'(yr) - z^2q'(zr) = -\frac{z^2T^2}{2(\cosh[zrT] - 1)} - \frac{y^2T^2}{2(\cosh[yrT] - 1)} > 0
\]

\[
\iff z^2(\cosh[yrT] - 1) > y^2 \cosh[zrT] - 1.
\]

Employing the infinite series expansion of \( \cosh[x] \), we obtain

\[
z^2(\cosh[yr] - 1) - y^2(\cosh[zr] - 1) = z^2 \left( \sum_{n=0}^{\infty} \frac{(yx)^{2n}}{(2n)!} - 1 \right) - y^2 \left( \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} - 1 \right)
\]

\[
= z^2 \sum_{n=1}^{\infty} \frac{(yx)^{2n}}{(2n)!} - y^2 \sum_{n=1}^{\infty} \frac{(yx)^{2n}}{(2n)!} = \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!} z^2y^2 \left( y^{2(n-1)} - z^{2(n-1)} \right) > 0.
\]
The inequality holds, as the first summand is zero and all other terms are strictly positive for all \( y > z \geq 1 \).

(viii) We obtain:

\[
q''(r) = -\frac{2}{r^3} + \frac{2T^3 \sinh[rT]}{(2 \cosh[rT] - 2)^2} = -2T^3 \left( \frac{1}{(rT)^3} + \frac{\sinh[rT]}{(2 \cosh[rT] - 2)^2} \right)
\]

Then, the statement holds if and only if \((\cosh[x] - 2)^2 > x^3 \sinh[x]\). To see this, we employ the infinite series expansion of \(\cosh[x]\) and \(\sinh[x]\)

\[
\frac{2 \sum_{n=0}^{\infty} x^{2n}}{(2n)!} - \frac{2}{2} \frac{\sum_{n=0}^{\infty} x^{2n+1}}{(2n+1)!} = \left( 2 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \right)^2 - \frac{x^{2n+4}}{(2n+1)!}
\]

Both series exhibit all even powers of \(x\) starting with \(x^4\):

\[
x^4 \left( \frac{4}{2!2!} - 1 \right) + x^6 \left( \frac{2 \cdot 4}{2!4!} - \frac{1}{3!} \right) + x^8 \left( \frac{2 \cdot 4}{2!6!} + \frac{4}{4!4!} - \frac{1}{5!} \right) + \cdots \geq 0.
\]

The inequality holds as the first term is zero and all other terms are strictly positive for all \(x \in \mathbb{R}_{++}\). □

**Lemma 2**

For all \(\xi, \nu \in \mathbb{R}_{++}\) the function \(J\) defined in \((A.1)\) satisfies

(i) \(J(r) > 0\).

For all \(\xi, \nu \in \mathbb{R}_{++}\) and \(\sigma^H \in (0, 1]\) the function \(J\) satisfies

(ii) \(\frac{d}{dr} \left( \frac{J'(r)}{J(r)} \right) > 0\) for all \(r \geq \xi\),

(iii) \(\lim_{r \to \infty} \frac{J'(r)}{J(r)} = \sigma^H T\).

For all \(\xi, \nu \in \mathbb{R}_{++}\) and \(\sigma^H > 1\) the function \(J\) satisfies

(iv) \(\frac{d}{dr} \left( \frac{J'(r)}{J(r)} \right) > 0\) for all \(r \geq \nu + \xi\) and \(\rho^H < \frac{\sigma^H}{\sigma^H - 1}(\nu + \xi)\),

(v) \(\lim_{r \to \infty} \frac{J'(r)}{J(r)} = T\).

**Proof:** (i) Follows immediately from \(Q_T(r) > 0\) for all \(r \in \mathbb{R}\) as shown in Lemma 1.

(ii) Using the definition \((A.46)\), we obtain

\[
\frac{J'(r)}{J(r)} = q(r - \xi) - \sigma^H q (\nu + \xi - \sigma^H (r - \rho^H)) - (1 - \sigma^H) q (r - \sigma^H (r - \rho^H)), \quad (A.47a)
\]
and

\[ M(r) \equiv \frac{d}{dr} \left( \frac{J'(r)}{J(r)} \right) = \frac{J''(r)}{J(r)} - \left( \frac{J'(r)}{J(r)} \right)^2 \]  
(A.47b)

\[ = q'(r - \xi) + (\sigma^H)^2 q' (\nu + \xi - \sigma^H (r - \rho^H)) - (1 - \sigma^H)^2 q' (r - \sigma^H (r - \rho^H)). \]

For \( \sigma^H \in (0, 1] \) set \( x = r - \xi \) and restrict attention to all \( x \geq 0 \)
\[ M(x) = q'(x) + (\sigma^H)^2 q' (\nu + (1 - \sigma^H) \xi - \sigma^H (x - \rho^H)) \]
\[ - (1 - \sigma^H)^2 q' ((1 - \sigma^H) x + (1 - \sigma^H) \xi + \sigma^H \rho^H) \]
\[ > q'(x) - (1 - \sigma^H)^2 q' ((1 - \sigma^H) x + (1 - \sigma^H) \xi + \sigma^H \rho^H) \]
\[ \geq q'(x) - (1 - \sigma^H)^2 q' ((1 - \sigma^H) x) \geq 0. \]

The first inequality holds due to part (v), the second inequality due to part (viii) and the last inequality due to part (vi) of Lemma 1.

(iii) Follows directly from equation (A.47a) and part (iv) of Lemma 1.

(iv) For \( \sigma^H > 1 \) and \( \rho^H < \frac{\sigma^H - 1}{\sigma^H} (\nu + \xi) \) consider only \( r \geq \nu + \xi \)
\[ M(r) = q'(r - \xi) + (\sigma^H)^2 q' (\sigma^H r - \sigma^H \rho^H - (\nu + \xi)) - (\sigma^H - 1)^2 q' ((\sigma^H - 1) r + \sigma^H r) \]
\[ > (\sigma^H)^2 q' (\sigma^H r - \sigma^H \rho^H - (\nu + \xi)) - (\sigma^H - 1)^2 q' ((\sigma^H - 1) r + \sigma^H r) \]
\[ > (\sigma^H)^2 q' (\sigma^H r) - (\sigma^H - 1)^2 q' ((\sigma^H - 1) r) \geq 0 \]

The first inequality holds due to part (v), the second inequality due to part (viii) and the last inequality due to part (vii) of Lemma 1.

(v) Follows directly from equation (A.47a) and part (iv) of Lemma 1. □
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