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Why *prāṇa* is the most excellent among the vital functions,  
or: the Shapley value in the  
*Upaniṣads*

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Abstract:

The contest among vital functions is a well-known topic in the *Āraṇyakas* and the *Upaniṣads*. We are interested in two different versions that can be linked to the Shapley value from modern-day cooperative game theory.

Keywords:

Shapley value; cooperative game theory; balancing operation; vital functions; superiority

## I. Introduction

Comparisons of the natural body with a political one have been common in many cultures, Egyptian, European, Greek, Indian, and Roman.<sup>2</sup> For India, the Ṛgvedic Hymn of the Man (*puruṣasūkta*) identifies four parts of the body with the four castes.<sup>3</sup> The Indian organic theory of the state is put succinctly in the Śukranīti (about as old as Kauṭilya's Arthaśāstra, give or take a few centuries)<sup>4</sup>:

The kingdom is an organism of seven limbs, *viz.*, the Sovereign, the Minister, the Friend, the Treasure, the State, the Fort and the Army. Of these seven constituent elements of the kingdom, the king or Sovereign is the head, the Minister the eye ...<sup>5</sup>

We focus on a specific aspect of organic theories, rank order disputes. From the Western point of view, one of Aesop's fables is most relevant, the quarrel between the belly and the feet about their relative importance:

The belly and the feet were arguing about their importance, and when the feet kept saying that they were so much stronger that they even carried the stomach around, the stomach replied, "But, my good friends, if I didn't take in food, you wouldn't be able to carry anything."<sup>6</sup>

This fable's dating is very difficult, sometime between 1000 BCE and 100 CE.<sup>7</sup> In the Indian context, we find the contest of the "vital functions"<sup>8</sup> for superiority in the Bṛhadāraṇyaka Upaniṣad and the Chāndogya Upaniṣad (both of them from 7th to 6th centuries BCE<sup>9</sup>), in the Aitareya Āraṇyaka (6th to 5th centuries BCE<sup>10</sup>) and others. The vital functions comprise breath, sight, hearing, etc. Their contest is present in different versions in the *Āraṇyakas* and the *Upaniṣads*:

- In BĀU 1.5.21, death succeeds in capturing the vital functions with the exception of breath. This fact shows breath's superiority.
- In BĀU 1.3.1-7, the vital functions (speech, breath, sight, hearing, mind, breath in the mouth) have to sing the High Chant. The demons "riddle

<sup>2</sup> A brief introduction is presented by Ilsley Hicks (1963). Harvey (2007) is a book-length introduction that deals with the body politic from ancient times into the present. Nederman (2004) covers the use of organic metaphors in the Late Middle Ages. Shogimen (2008) shows how the healing of human bodies is transferred to dealing with political problems, juxtaposing Late Medieval Europa and Japan.

<sup>3</sup> See, for example, Jamison/Brereton (2014, pp. 57-58).

<sup>4</sup> Singh, G.P. (1993, p. 10) discusses the dating problems surrounding the "original *Śukra-Nītisāra*" and its re-visions.

<sup>5</sup> Sarkar (1975, p. 11)

<sup>6</sup> Daly (1961, p. 148)

<sup>7</sup> See Harvey (2007, pp. 4-5). The Egyptian fragmented version (see Erman 1927, pp. 173-174) may be older, from the first half of the first century BCE (see Ilsley Hicks 1963, p. 29).

<sup>8</sup> This term has been introduced by Olivelle (1998). Indologists have, of course, noted the "Rangstreitfabel" (Ruben 1947) and the importance of breath (Frauwallner 1997, pp. 41-45). Bodewitz (1992) discusses the superiority of *Vāyu* and *Prāṇa* in connection with kingship.

<sup>9</sup> This chronology is due to Olivelle (1998, p. 12) who cautiously adds "give or take a century or so".

<sup>10</sup> See Olivelle (1998, pp. 8, 12-13).

with evil”<sup>11</sup> the functions from speech to mind, but they fail to do the same with breath in the mouth.

- In ChU 4.3.3, breath is characterized as the “gatherer” into whom the other vital functions pass when a man sleeps.<sup>12</sup>

The first of these examples is explicitly framed as a competition between the vital functions, but the superiority of breath in the mouth also clearly shows in the other two examples. We think of Aesop’s fable and of these Indian tales as idiosyncratic solutions to the superiority problem. It is not obvious at all how they might be generalized to other superiority problems, say, between people working together in a common joint venture or between the countries in the EU.

In our mind, these versions differ from others where the vital functions use some non-idiosyncratic way to assess their superiority. In this paper, we concentrate on these generalizable approaches. We do not offer a definition of “generalizability”. However (and with a view to the texts covered in this paper), one may argue that generalizability might refer to

- (a) some method (*prakāra*),
- (b) which is teachable (*prakāropadeśah*),
- (c) which is applicable beyond the actual application (*cetanāvanta iva puruṣāh*), and
- (d) which serves to avoid struggle or competition (*spardhānivāranārtham*).

In the Old-Indian texts, we find two different manners, (i) the sequential approach and, (ii), the withdrawal approach. We first turn to the sequential approach. In KauU 2.14, the vital functions enter the body (which does not stand) one after another.<sup>13</sup> Only when breath enters, the body gets up. In AĀ 2.1.4, the superiority of breath is shown by two different methods. The vital functions first leave the body one after the other and then they reenter. Breath is the last to leave and the last to reenter, respectively, and makes the decisive difference. Turning to the above example of the EU, one may (at least in principle) consider how the remaining EU countries fare if Great Britain and then France etc. leave the EU.

The second generalizable approach could be labeled the withdrawal approach. In PU 2, breath shows its power by setting off and then setting down again. Since the others (here: space, wind, speech, mind, etc.) have to follow suit, they acknowledge breath’s superiority. This version is made symmetric in BĀU 6.1, ChU 5.1, and ŚĀ 9.1-7<sup>14</sup>. Speech leaves the body and reenters after a while. Then, the same procedure is followed by the other vital functions. Each time, the

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<sup>11</sup> Olivelle (1998, p. 41)

<sup>12</sup> Olivelle (1998, p. 217)

<sup>13</sup> Olivelle (1990, pp. 344-345) and Bodewitz (2002, pp. 40-42)

<sup>14</sup> Translations by Keith (1908, p. 57) and Bodewitz (2002, pp. 73-77).

remaining functions are asked how they fared. It turns out that the leaving of breath cannot be endured and that, hence, breath is superior. In the EU example, a country (like Portugal or Germany) may confront the others with the prospect of leaving. Perhaps the others fare worse after Portugal’s exit than after the exit of Germany.<sup>15</sup>

We show that these two approaches are closely related to the so-called Shapley (1953) value from cooperative game theory. In cooperative game theory, the possibilities of alternative groups of “players” to create “worth” are given. The problem is how much each individual player obtains. The Shapley value (and most other concepts from cooperative game theory) admits two different types of definitions, (i) an algorithmic one and (ii) an axiomatic one. An algorithm is a formula or, more generally, some specific manner of algebraic manipulation. The axiomatic approach puts forward some general rules of distribution.

We will present the contest as presented in the *Brāhmaṇas* in the next section. Section III then presents the Shapley value and discusses the relation between that concept and the contest literature. Indeed, (i) the algorithmic definition of the Shapley value builds on rank orders, i.e., on the sequential approach. Also, (ii) one prominent axiom obeyed by the Shapley value amounts to the withdrawal approach. Section IV concludes.

## II. The contest among vital functions

### A. *Idiosyncratic approaches*

The following two contest examples are “idiosyncratic” in the sense of not presenting a solution procedure that might be applicable to a wide range of superiority problems. Consider first BĀU 1.5.21:

*prajāpatir ha karmāṇi sasṛje |*  
*tāni sṛṣṭāny anyo ’nyenāspardhanta |*  
*vadiṣyāmy evāham iti vāg dadhre |*  
 ...  
*tāni mṛtyuḥ śramo bhūtvopayeme |*  
 ...  
*athemam eva nāpnod yo ’yaṃ madhyamaḥ prāṇaḥ |*  
*tāni jñātum dadhrire | ayam vai naḥ śreṣṭho yaḥ ...*

Prajāpati created the vital functions (*prāṇa*).

Once they were created, they began to compete with each other.

Speech threw out the challenge: “I am going to speak!”

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<sup>15</sup> Great Britain (if they were to leave after the 2016 referendum) would not be a good example. The threat of withdrawal is about dividing payoffs under the assumption that the body (here: the European Union) remains intact.

...

Taking the form of weariness, death took hold of them.

...

The central breath alone, however, death could not capture.

So they sought to know him, thinking: “He is clearly the best among us

...”<sup>16</sup>

...

Likewise, to “riddle with evil” (see the introduction) can also be counted among the idiosyncratic solutions to superiority challenges.

### **B. The sequential approach**

In contrast to the idiosyncratic approaches, the sequential and the withdrawal approaches offer a generalizable manner to assess superiority. We first turn to the sequential approach and focus on AĀ 2.1.4<sup>17</sup>:

*tā ahimsantāham uktham asmy aham uktham asmīti |*

*tā abruvan hantāsmāc charīrād utkrāmāma*

*tad yasmin na utkrānta idaṃ śarīraṃ patsyati tad ukthaṃ bhaviṣyatīti |*

They strove together, saying, ‘I am the hymn, I am the hymn.’

They said, ‘Come, let us leave this body,

then that one of us at whose departure the body falls, will be the hymn.’<sup>18</sup>

The commentary ascribed to Sāyaṇa from, perhaps, 14. century CE<sup>19</sup> explains *ahimsanta* as *parasparaspardhārūpāṃ hiṃsām akurvan* (AĀ\_Sā, p. 110): The debate is violence in the form of mutual competition. But then, Sāyaṇa (AĀ\_Sā, p. 111) remarks:

*tāḥ spārdhamānā devātāḥ spardhānivāranārthaṃ*

*samayaviśeṣaṃ parasparam abruvan*

In order to avoid this competition, these competing deities came to a particular understanding.

Thus, according to the commentator Sāyaṇa, the competition which consists in simply insisting on one’s superiority (*aham uktham asmi*) is avoided in favour of a competition by way of a controlled experiment. In our mind, this experiment is a generalizable manner of deciding the superiority question. Turning, again, to AĀ 2.1.4, the vital functions leave the body one after another:

*vāg udakrāmad avadann aśnan pibann āstaiva*

<sup>16</sup> Olivelle (1998, p. 57)

<sup>17</sup> All the following quotations from the Aitareyāranyaka are from AĀ 2.1.4.

<sup>18</sup> Keith (1909, p. 205)

<sup>19</sup> See Burnell (1873, pp. vi-xv). But compare Slaje (2010) who deals with the very doubtful authorship of Sāyaṇa with respect to commentaries on the four Veda-Saṃhitās. Note, in particular, Slaje (2010, p. 395, fn.48).

Speech went forth, yet (the body) remained, speechless, eating and drinking.<sup>20</sup>

The sequence of leaving is speech, sight, hearing, mind, and finally breath:

*prāṇa udakrāmat tatprāṇa utkrānte 'padyata*

Breath went forth, when breath went out, (the body) fell.<sup>21</sup>

Then, they start quarreling again, but this time resolve on entering the body one after another. The sequence of entering is the same as before. The result is as expected:

*prāṇaḥ prāviśat tat prāṇe prapanna udatiṣṭhat tat uktham abhavat*

Breath entered, when breath entered, that [the body] arose, and that [breath] became the hymn.<sup>22</sup>

We will see the importance of the leaving and entering sequences later.

### C. The withdrawal approach

For the withdrawal approach, we cite BĀU 6.1:

<sup>7</sup>*te heme prāṇā ahaṁśreyase<sup>23</sup> vivadamānā brahma jagmuḥ |*

*tad dhocuḥ ko no vasiṣṭha iti |*

*tad dhovāca yasmin va utkrānta idaṁ śarīraṁ pāpīyo manyate sa vo vasiṣṭha iti |*

<sup>8</sup>*vāg ghocakrāma sā saṁvatsaram proṣyāgatyo vāca katham aśakata madṛte jīvitum iti |*

*Te hocuḥ yathā kalā avadanto vācā prāṇantaḥ prāṇena paśyantaś cakṣuṣā śṛṅvantāḥ śrotreṇa vidvāṁso manasā prajāyamānā retasaivam ajīviṣmeti |*

*praviveśa ca vāk*

<sup>7</sup>Once these vital functions (*prāṇa*) were arguing about who among them was the greatest. So they went to *brahman* and asked: “Who is the most excellent of us?” He replied: “The one, after whose departure you consider the body to be the worst off, is the most excellent among you.”

<sup>8</sup>So speech departed. After spending a year away, it came back and asked: “How did you manage to live without me?” They replied: “We lived as the dumb would, without speaking with speech, but breathing with the breath, seeing with the eye, hearing with the ear, thinking with the mind, and fathering with semen.” So speech reentered.<sup>24</sup>

<sup>20</sup> Keith (1909, p. 205)

<sup>21</sup> Keith (1909, p. 205)

<sup>22</sup> After Keith (1909, p. 205).

<sup>23</sup> This compound could be in dative or locative. Dative seems more likely in view of *ahaṁśreṣṭhatāyai* found in ChU\_Ś (p. 252) and cited below.

<sup>24</sup> Olivelle (1998, p. 143)

This version from the BĀU is very close to one found in ChU 5.1. Śaṅkara (perhaps sometime between 650 CE and 800 CE<sup>25</sup>) offers a long comment on this setting. In particular, he ponders this objection (ChU\_Ś, p. 252):

*nanu katham idaṃ yuktaṃ cetanāvanta iva puruṣā ahaṃśreṣṭhatāyai vivadanto 'nyonyaṃ spardherann iti*

How can this be logical that [the vital functions] compete against each other by arguing about who among them was the greatest, as reasonable humans [would].

While Śaṅkara's answer is not helpful for our purpose, it needs to be noted that he thought humans the most obvious contenders in such fights for superiority.<sup>26</sup>

Returning to BĀU, after speech has left and reentered, the very same procedure is followed by sight, hearing, mind, and semen. When breath is about to leave, the other vital functions beg:

<sup>13</sup>*mā bhagava utkramīh |  
na vai śakyāmas tvadr̥te jīvitum iti |  
tasyo me baliṃ kuruteti |  
tatheti |*

<sup>14</sup>*sā ha vāg uvāca yad vā ahaṃ vasiṣṭhāsmi tvam tad vasiṣṭho 'sīti |  
yad vā ahaṃ pratiṣṭhāsmi tvam tad pratiṣṭho 'sīti cakṣuḥ |*

<sup>13</sup>“Lord, please do not depart! We will not be able to live without you.”

He told them: “If that's so, offer a tribute to me.” “We will,” they replied.

<sup>14</sup>So speech declared: “As I am the most excellent, so you will be the most excellent.”

Sight declared: “As I am the firm base, so you will be the firm base.”<sup>27</sup>

The tribute (*bali*) offered to the best (*śreyas*) is a familiar topic. As ŚB 11.2.6.14 (p. 842) states:

*śreyase pāpīyān baliṃ hared vaiśyo vā rājñe baliṃ hared*  
an inferior brings tribute to his superior, or a man of the people brings  
tribute to the king<sup>28</sup>

While the contest runs similarly in ChU 5.1, breath does not explicitly demand a tribute, i.e., BĀU 6.1.13 is “missing”. Nevertheless, Śaṅkara's commentary on ChU 5.1.13-14 (where the other vital functions offer their tributes similar to BĀU 6.1.14) says (ChU\_Ś, p. 251):

*atha hainaṃ vāgādayaḥ prāṇasya śreṣṭhatvaṃ kāryenāpādayanta āhur  
balim iva haranto rājñe viśaḥ*

<sup>25</sup> See Isayeva (1993, pp. 83-87).

<sup>26</sup> See also Śaṅkara's words *yathā loke puruṣāḥ* (ChU\_Ś, p. 254) that he uses in his answer to similar objections.

<sup>27</sup> Olivelle (1998, p. 145)

<sup>28</sup> After Eggeling (1900, p. 38)

Speech and the rest, establishing, by their action, the *superiority* of Breath, said to him—making offerings like the people to their King.<sup>29</sup>

Thus, the reason behind the tribute may lie in the fact that the competition of the vital functions serves as a “political allegory where the superiority of *prāṇa* in relation to the other vital functions is likened to the supremacy of the king among his rivals and ministers” (Black 2007, p. 122). While this is certainly true,<sup>30</sup> the tribute can be seen as serving a specific purpose in the context of our approach (see section III C).

We have argued in the introduction that the threat of withdrawal is generalizable to other situations. This was clear to Śaṅkara<sup>31</sup> (BĀU\_Ś, p. 417):

*ayam ca prāṇasamvādaḥ kalpito viduṣaḥ*

*śreṣṭhaparīkṣanaprakāropadeśaḥ |*

*anena hi prakāreṇa vidvān ko nu khalvatra śreṣṭha iti parīkṣaṇam karoti |*

And this agreement of the vital functions is imagined<sup>32</sup> by the wise as a teaching of a manner to test superiority.

For by this manner the wise performs the test of who, indeed, is the best here.

### III. The Shapley value

#### A. Basic definitions

The Shapley value belongs to the realm of cooperative game theory. This theory presupposes  $n$  players that are collected in a set  $N = \{1, 2, \dots, n\}$  and a so-called coalition function  $w$ . A subset  $K$  of  $N$  is also called a coalition.  $N$  itself is called the grand coalition. To each coalition, the coalition function attributes a “worth”  $w(K)$ . The worth stands for the economic, social, political, or other gain that the particular group of players can achieve. A worth can only be created if at least one player is present, i.e., the empty set  $\emptyset$  creates the worth zero,  $w(\emptyset) = 0$ . For

<sup>29</sup> Jha (2005, p. 225)

<sup>30</sup> See also Rau (1957, p. 34) and Bodewitz (1992, p. 57).

<sup>31</sup> Śaṅkara’s wording in his comment for the Chāndogya Upaniṣad (ChU\_Ś, p. 254) is less telling. He states *vāgādīnāṃ ceha samvādaḥ kalpito viduṣo ’nvayavyatirekābhyām prāṇaśreṣṭhatānirdhāraṇartham* And this agreement of speech and so on is imagined by the wise in order to determine the superiority of breath by connecting and separating.

Our text shows the obvious error °*rekāmyām*. This word is spelled correctly in Āgaṣe (1894, 6.21 and 6.23, p. 2009). Jha (2005, p. 227) translates as “by means of negation and affirmation”. While this term is normally used in logic, in the present context connecting and separating refer to the functions being within and leaving the body.

<sup>32</sup> The same word is used in the commentary for the Chāndogya Upaniṣad. Śaṅkara guards against the objections raised before (ChU\_Ś, p. 252) that the whole procedure is unrealistic because (i) people, not vital functions, argue for superiority (this objection has been quoted in the main text), (ii) the senses cannot talk (with the exception of speech), and (iii) leaving and reentering the body is not possible.

ease of notation, we write  $w(i)$  instead of  $w(\{i\})$ ,  $w(1, 2)$  instead of  $w(\{1, 2\})$ , and  $w(K \cup i)$  instead of  $w(K \cup \{i\})$ .

The aim of cooperative game theory is to specify payoffs for the players. These payoffs depend on the coalition function. In the main text, we assume just two players 1 and 2. The case with more than two players and the general case are explained in the footnotes and in the appendix. A solution function  $\varphi$  defines, for each coalition function  $w$ , payoffs  $\varphi_1(w)$  and  $\varphi_2(w)$ .<sup>33</sup>

Cooperative game theory uses two different approaches for arriving at payoff vectors from coalition functions. (i) The algorithmic approach applies some algebraic manipulations on the coalition functions in order to derive payoff vectors. For example, each player might obtain the worth of his one-man coalition plus 5. This solution function would be described by  $\varphi_1(w) = w(1) + 5$  and  $\varphi_2(w) = w(2) + 5$ .<sup>34</sup>

(ii) The axiomatic approach suggests general rules of distribution. One axiom might stipulate that the worth of the grand coalition  $\{1, 2\}$  is distributed among the players:  $\varphi_1(w) + \varphi_2(w) = w(1, 2)$ .<sup>35</sup>

A second axiom might demand payoff equality. These two axioms together define a specific solution function, namely the one given by  $\varphi_1(w) = \varphi_2(w) = \frac{w(1, 2)}{2}$ .<sup>36</sup>

## B. The algorithmic approach

The Shapley value's algorithm builds on the players' "marginal contributions". A player's marginal contribution is the worth of a coalition with him minus the worth of the coalition without him, i.e., the difference he makes. In the two-player case, player 1 has two marginal contributions, the first with respect to the empty set  $\emptyset$  (the marginal contribution is  $w(1) - w(\emptyset)$ ), the second with respect to  $\{2\}$  (with marginal contribution  $w(1, 2) - w(2)$ ).<sup>37</sup>

Player 1's Shapley value is the average of his marginal contributions, taken over all sequences (rank orders) of the two players. For two players, there are just two

<sup>33</sup> For a general player set  $N$ , a solution function defines a payoff vector  $\varphi(w)$  that has the entries  $\varphi_1(w), \dots, \varphi_n(w)$ .

<sup>34</sup> For a general player set  $N$ , the solution function is given by  $\varphi_i(w) = w(i) + 5$ ,  $i \in N$ .

<sup>35</sup> For  $n$  players, this axiom is expressed by  $\sum_{i=1}^n \varphi_i(w) = w(N)$ .

<sup>36</sup> For a general player set  $N$ , the solution function is given by  $\varphi_i(w) = \frac{w(N)}{n}$ ,  $i \in N$ .

<sup>37</sup> For a general player set  $N$ , the marginal contribution of player  $i$  with respect to a coalition  $K$  that does not contain that player is defined by  $MC_i(K) = w(K \cup i) - w(K)$ .

sequences, player 1 may be first (sequence (1, 2)) or second (sequence (2,1)). Thus, the players' Shapley values<sup>38</sup> are

$$(1) \quad Sh_1 = \frac{1}{2}(w(1) - w(\emptyset)) + \frac{1}{2}(w(1, 2) - w(2))$$

and

$$(2) \quad Sh_2 = \frac{1}{2}(w(2) - w(\emptyset)) + \frac{1}{2}(w(1, 2) - w(1))$$

The sequential approach (see section II B) gets close to the algorithmic approach of defining the Shapley value. Consider the vital functions that enter the body in a sequential manner. In AĀ 2.1.4, the sequence is speech (*sp*), sight (*si*), hearing (*h*), mind (*m*), and finally breath (*b*). Let us denote the vital functions except breath by  $V = \{sp, si, h, m\}$ .

Of course, a coalition function is not explicitly mentioned in Old-Indian texts. However, it seems clear from the text that the “worths” increase the more vital functions are present. In the appendix, we present a specific coalition function. There, each vital function  $v$  from set  $N = \{sp, si, h, m, b\}$  creates the worth of its one-man coalition  $w(v)$ . Additionally, if breath is present, the worths of the other vital functions is increased by some factor  $\alpha \geq 1$ .

We assume this coalition function but deal with two players only, speech (*sp*) and breath (*b*). Then, the marginal contributions for the entering sequence (*sp, b*) are

- $w(sp)$  for *sp* and
- $w(b) + (\alpha - 1)w(sp)$  for *b*.

Breath is superior to speech if his payoff is highest, i.e., if  $w(b) > (2 - \alpha)w(sp)$  holds. Thus breath's superiority, claimed by AĀ 2.1.4 and KauU 2.14 (in the case of all five vital functions), is true if  $w(b)$  is large relative to  $w(sp)$  and if the “productivity” of speech is enhanced by breath's presence (large  $\alpha$ ). For the case of five vital functions, see the appendix A.

AĀ 2.1.4 also covers the leaving sequence (*sp, b*). The players then should not obtain the marginal contribution, but the damage they do to the body, i.e.,

- $\alpha w(sp)$  for *sp* and
- $w(b)$  for *b*.

Again, breath is superior to speech if  $w(b)$  is large relative to  $w(sp)$ . However, in contrast to the entering sequence, the “productivity” of speech as enhanced by

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<sup>38</sup> For a general player set  $N$ , let  $R$  be the set of rank orders. For  $n$  players, there exist  $n! = 1 \cdot 2 \cdot \dots \cdot n$  different rank orders. Let  $K_i(r)$  denote the set of players in the rank order  $r$  up to but not including player  $i$ . Then, player  $i$ 's Shapley value is  $Sh_i = \frac{\sum_{r \in R} MC_i(K_i(r))}{n!}$ .

breath needs to be small for breath's superiority. In general, the payoffs for the entering and the leaving sequence differ. Therefore, AĀ 2.1.4 does not mention both of them in vain. For the case of five vital functions, see the appendix B.

In the special case of just breath and speech, the average of the payoffs for entering and leaving equals the Shapley value. Indeed, if we look at the entering sequence  $(b, sp)$ , with breath first, we obtain the payoffs put down for the leaving sequence  $(sp, b)$ . In general, however, AĀ 2.1.4 and the KauU 2.14 do not reflect the Shapley value. Instead, they reflect the payoffs for specific entering sequences (starting from the empty set) and leaving sequences (starting from the grand coalition).

### C. *The axiomatic approach*

For any number of players and any coalition function, the Shapley value fulfills these axioms:

- The sum of the Shapley values equals the worth of the grand coalition, i.e., we have

$$(3) \quad Sh_1 + Sh_2 = w(1, 2)$$

in the case of two players. The property means that the grand coalition forms and the Shapley value distributes the worth of the grand coalition among the players.

- Any player whose marginal contribution is zero with respect to every coalition obtains the Shapley value of zero.
- The payoffs do not depend on the players' names.
- If player 1 withdraws<sup>39</sup> from the game, another player 2's damage in terms of his Shapley payoff equals the damage that player 1 endures should player 2 withdraw, i.e., we have

$$(4) \quad Sh_2 - w(2) = Sh_1 - w(1)$$

in the case of two players. Consider the left side of the equation. If player 1 withdraws, player 2 does not obtain the Shapley value  $Sh_2$  anymore, but the Shapley value of the game of which he is the only player. In that game he obtains the worth  $w(2)$  of his one-man coalition. This is clear from the only rank order that exists in that game (see section III B) as also from the first property in this section.

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<sup>39</sup> Withdrawal means that the player set is reduced by the withdrawing players and that the worths for the remaining players stay the same.

Equations (3) and (4) lead to the Shapley values in equations (1) and (2) above (note  $w(\emptyset) = 0$ ).<sup>40</sup> Cooperative game theorists then say that the axioms expressed by equations (3) and (4) axiomatize the Shapley value. This means, the Shapley value (in its algorithmic form, see section III B) fulfills these axioms and that there is no value different from the Shapley value that also fulfills these axioms. This particular axiomatization is due to Myerson (1980).

Myerson's axiom of balanced contributions (Myerson 1980, pp. 172-173) is clearly related to the threat of withdrawal from section II C. One may object that the threat uttered by breath ( $b$ ) is more serious than the threat uttered by speech ( $sp$ ). Finally, however, this is not the case. BÄU 6.1<sup>41</sup> and ChU 5.1 can be translated into our framework by the inequalities

$$(5) \quad w(si, h, m, b) > w(sp, si, h, m)$$

or, equivalently,

$$(6) \quad w(sp, si, \dots, b) - w(si, \dots, b) < w(sp, \dots, m, b) - w(sp, \dots, m)$$

The first inequality says that the body can get up in the presence of breath even if speech is not present, but not the other way around. The second inequality is equivalent and says that the marginal contribution of speech (left side) is smaller than the marginal contribution of breath (right side). Or, differently put, the damage of withdrawal that breath can inflict (in terms of worth!) is larger than the corresponding damage that speech or the other vital functions can inflict.

This inequality does not contradict eq. (4) which we rewrite in this manner:

$$(7) \quad Sh_b(w \text{ with all players}) - Sh_b(w \text{ with all players except } sp) \\ = Sh_{sp}(w \text{ with all players}) - Sh_{sp}(w \text{ with all players except } b)$$

Indeed, the Shapley values noted in the appendix (see C, D, and E) fulfill eq. (7).

How can we explain that breath's leaving the body exerts a lot of damage (see right-hand side of eq. (6)), but that the threat of withdrawal is balanced by eq. (7)?

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<sup>40</sup> For more than two players, we need  $\sum_{i \in N} Sh_i = w(N)$  (compare eq. (3)) and the following version of eq. (4): Consider any subset  $K$  of  $N$ . On the basis of  $K$  as the new grand coalition, a  $K$ -game can be defined where the coalitions in that game have the same worth as within the original game. Consider two players  $i$  and  $j$  that are members from  $K$ . If player  $i$  withdraws from the  $K$ -game, player  $j$ 's change in his Shapley payoff equals the change that player  $i$  endures should player  $j$  withdraw.

<sup>41</sup> For our purpose, it is a minor aspect that BÄU 6.1 has six vital functions, among them semen.

This seeming puzzle is “solved” in BĀU 6.13 where breath tells the other vital functions: “If that's so [i.e., if I, leaving the body, can exert more damage than you], offer a tribute to me.” After they reply with “We will,” breath’s Shapley value includes the *bali*. Now, when turning the tribute over to breath within the body (in the grand coalition), speech does not suffer more from breath’s leaving the body than breath suffers from speech’s exit.

The mechanism that is at work here can be illustrated by the following example taken from Wiese (2009). Assume a seller of a good that is wanted by four potential buyers. This seller is in a monopoly position and, so it seems, the buyers are more dependent on him than he is on any particular buyer. Indeed, we have the inequality

$$(5') \quad w(\text{seller with three buyers}) > w(\text{four buyers without seller})$$

Why, then, does the seller’s threat of withdrawal not carry more weight than any particular buyer’s threat of withdrawal? Here is the reason for eq. (7) in this context: The seller obtains a very high price in case of 4 potential buyers and a slightly reduced price in case of 3 potential buyers. So indeed, one potential buyer’s withdrawal would not do much harm to the seller. But this potential buyer’s disutility caused by the seller’s withdrawal is small also. In the presence of the seller, this buyer will have a small chance (1/4) of getting the item in question and will also have to pay a high price. Therefore, the buyer does not lose much if the seller withdraws and his chance of getting the item is reduced to zero.

We can even calculate the tribute (see appendix F). In BĀU 6.13, the tribute is positive. In the context of our model, this holds if  $\alpha$  is large and if  $w(b)$  is large in comparison with the sum of the other one-man worths.

## IV. Conclusion

While the *Āraṇyakas* and the *Upaniṣads* (being post-Vedic, but pre-classic texts) are normally considered to deal with esoteric, religious, and philosophical matters, Black (2007) focuses on the social and power questions that are also involved. The thesis of this paper is that the Old-Indian contest among vital functions (in some versions) employ generalizable procedures. In contrast, Aesop’s fable belongs to what we have termed idiosyncratic approaches. We are not aware of any non-modern non-Indian solutions to the superiority problem that also proceed along these generalizable lines.

Turning to non-modern Indian texts about the superiority problem, the *daiva* versus *puruṣakāra* controversy from the Mahābhārata comes to mind. MBh XIII.6 deals with the question of whether divine or human activity is superior.<sup>42</sup> MBh XIII.6.7 presents the following simile:

*yathā bījaṃ vinā kṣetram uptaṃ bhavati niṣphalam*

*tathā puruṣakāreṇa vinā daivaṃ na sidhyati*

Just as seed will be fruitlessly sown without a field, so ‘divine [power]’ will not succeed without human activity.<sup>43</sup>

Here, the “where would you be without me” idea is clearly present<sup>44</sup>. In this example, one may be tempted to let  $V = \{b, kṣ\}$ , with  $b$  for *bīja* and  $kṣ$  for *kṣetra*, and  $w(b) = w(kṣ) = 0$ . Then, the Shapley values for *bīja* and *kṣetra* are the same and reflect the idea that both ‘divine [power]’ and human activity are needed for success.

Other examples are difficult to find. In particular, in the framework of the seven-member theory of state, Kauṭilya (KAŚ 6.1.1) enumerates:

Lord, minister, countryside, fort, treasury, army, and ally are the constituent elements.<sup>45</sup>

The constituent elements enumerated in KAŚ 6.1.1 come in this specific order for a reason: Kauṭilya argues in detail why, in the order given above, “a calamity affecting each previous one is more serious”.<sup>46</sup> Kauṭilya presents clever arguments, but they do fall under the rubric of “idiosyncratic approaches”. One may surmise that the generalizable procedures advocated in the *Āraṇyakas* and the *Upaniṣads* were not so wellknown that their use would automatically come to mind.

The sequential approach gets close to the algorithmic definition of the Shapley (1953) value. The withdrawal approach is not far from the axiomatic definition of the Shapley value that is due to Myerson (1980). In the previous sections, we have worked out in which respect the Indian thinkers would have needed a few extra steps if they were to arrive at the Shapley value, algorithmically defined or axiomatically. Also, the Shapley value produces numerical figures. In contrast, the Indian Rangstreitfabel is only about rank orders.

One may, of course, surmise that arguments of the sort “where would you be without me” are common place in mankind. But, perhaps, the Indian literature is the first where the threat of withdrawal is used in a systematic manner. In modern times, Myerson (1980) is not the first to build a theory on that idea. Nearly

<sup>42</sup> Slaje (1990) presents a detailed translation and discussion of MBh XIII.6.

<sup>43</sup> Slaje (1990, p. 31)

<sup>44</sup> As it is also in YSm I.349 where a chariot cannot move with only one wheel.

<sup>45</sup> Olivelle (2013, p. 271)

<sup>46</sup> Olivelle (2013, p. 331)

20 years before Myerson, the sociologist Emerson (1962) wrote the seminal paper in that area, without taking Shapley's work or the Indian sources into account. Emerson's theory is simple and intriguing. According to him, whenever a person is more dependent on another one (or the second has more power over the first than the other way around), the relation is unbalanced and calls for "balancing operations". One kind of balancing operation has been explained for the monopolistic seller: The buyer (who has a low probability of getting the item) has to pay a relatively high price for it. In the context of the vital functions, the *bali* serves as a "balancing mechanism".

## Appendix

For the player set  $N = \{sp, si, h, m, b\}$  and the coalition of vital functions other than breath  $V = \{sp, si, h, m\}$  assume the coalition function  $w$  defined by  $w(v) \geq 0$  for all  $v \in N$  and

$$w(K) = \begin{cases} \sum_{v \in V \cap K} w(v), & b \text{ not in } K \\ w(b) + \alpha \sum_{v \in V \cap K} w(v), & b \text{ in } K \end{cases}$$

for every subset  $K$  of  $N$ . We assume  $\alpha \geq 1$  thus implying superadditivity of  $w$ , i.e.,  $w(N) \geq w(K) + w(N \setminus K)$  for every subset  $K$  of  $N$ . We find:

A) Along the sequence  $(sp, si, h, m, b)$  the marginal contributions are

- $w(v)$  for each vital function  $v$  from  $V$  and
- $w(b) + (\alpha - 1) \sum_{v \in V} w(v)$  for  $b$ .

B) Along the sequence  $(sp, si, h, m, b)$  the marginal damages (or: along the sequence  $(b, m, h, si, sp)$ ) the marginal contributions are

- $\alpha w(v)$  for each vital function  $v$  from  $V$  and
- $w(b)$  for  $b$ .

If  $\alpha$  takes the special value of 1, the payoffs are the same for the entering and the leaving sequence.

C) The Shapley values for the above game are:

- $Sh_v = \frac{1+\alpha}{2} w(v)$  for the vital functions  $v \in V$
- $Sh_b = w(b) + \frac{\alpha-1}{2} \sum_{v \in V} w(v)$  for breath.

Proof: Speech (and the other vital functions from  $V$  have the same chance of entering before breath (the marginal contribution is  $w(sp)$ ) or entering after breath (the marginal contribution is  $\alpha w(sp)$ ). This explains the Shapley values for the vital functions from  $V$ . The Shapley value distributes the worth of the grand coalition (which is 1 in the present case) among the players. Hence, breath gets the rest.

D) If speech withdraws from the game, the Shapley values for the remaining players are

- $Sh_v = \frac{1+\alpha}{2} w(v)$  for the vital functions  $v \in \{si, h, m\}$
- $Sh_b = w(b) + \frac{\alpha-1}{2} \sum_{v \in \{si, h, m\}} w(v)$  for breath.

Proof: If speech has withdrawn, the other players' payoffs are derived similar to C.

E) If breath withdraws from the original game, the Shapley values are  $Sh_v = w(v)$  for the vital functions  $v \in \{sp, si, h, m\}$ .

Proof: If breath has withdrawn, the vital functions  $sp, si, h, m$  receive their one-man worth in each sequence and hence in the Shapley value.

F) Before the contest, each vital function has obtained  $\frac{1}{5}$  of the body's proper functioning of  $w(b) + \alpha \sum_{v \in V} w(v)$ . After the contest, breath obtains the *bali* which is implicitly given by

$$\frac{w(b) + \alpha \sum_{v \in V} w(v)}{5} + bali = Sh_b = w(b) + \frac{\alpha - 1}{2} \sum_{v \in V} w(v)$$

and hence explicitly by

$$bali = \frac{4}{5} w(b) + \left[ \frac{3}{10} \alpha - \frac{1}{2} \right] \sum_{v \in V} w(v)$$

We find that the tribute is positive if  $\alpha > \frac{5}{3} - \frac{8}{3} \frac{w(b)}{\sum_{v \in V} w(v)}$  holds.

## Abbreviations

AĀ	Aitareyāraṇyaka (Keith 1909)
AĀ_Sā	Commentary on Aitareyāraṇyaka by Sāyaṇa (Deo 1992)
BĀU	Bṛhadāraṇyaka Upaniṣad (Olivelle 1998)
BĀU_Ś	Commentary on Bṛhadāraṇyaka Upaniṣad by Shankara (Shastri 1986)
ChU	Chāndogya Upaniṣad (Olivelle 1998)

ChU_Ś	Commentary on Chāndogya Upaniṣad by Shankara (Āgaśe 1934)
KauU	Kauṣītaki Upaniṣad (Olivelle 1998)
KAŚ	Kauṭilya's Arthasastra (Olivelle 2013)
MBh	Mahābhārata (Bhandarkar Oriental Research Institute 1974)
PU	Praśna Upaniṣad (Olivelle 1998)
ŚĀ	Śāṅkhāyana Āraṇyaka (Apte 1922)
ŚB	Śatapatha Brāhmaṇa (Weber 1855)
YSm	Yājñavalkya Smṛti (with Aparārka commentary)

## Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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