Microeconomics

Household optimum

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Structure

Introduction

- Household theory
  - Budget
  - Preferences, indifference curves, and utility functions
  - **Household optimum**
  - Comparative statics
  - Decisions on labor supply and saving
  - Uncertainty
  - Market demand and revenue
- Theory of the firm
- Perfect competition and welfare theory
- Types of markets
- External effects and public goods

Pareto-optimal review

Overview

- The household's maximization problem
- Marginal willingness to pay versus marginal opportunity cost
- Strictly convex preferences
- Lagrange approach (excursus)
- Concave preferences
- Perfect complements
- Revealed preferences
- The expenditure function

The household's maximization problem

Household optimum =

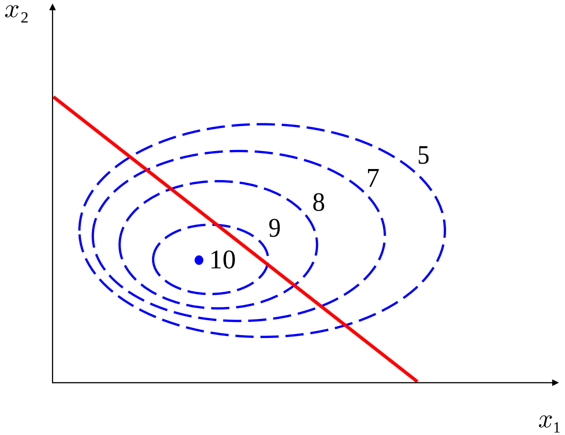
- bundle of goods that
- maximizes utility of the household
- obeying the budget constraint:

$$\max_{x_1, x_2} U(x_1, x_2)$$
$$\text{w. r. t. } p_1 x_1 + p_2 x_2 \leq m$$

or with endowment budget

$$\max_{x_1, x_2} U(x_1, x_2)$$
$$\text{w. r. t. } p_1 x_1 + p_2 x_2 \leq p_1 \omega_1 + p_2 \omega_2$$

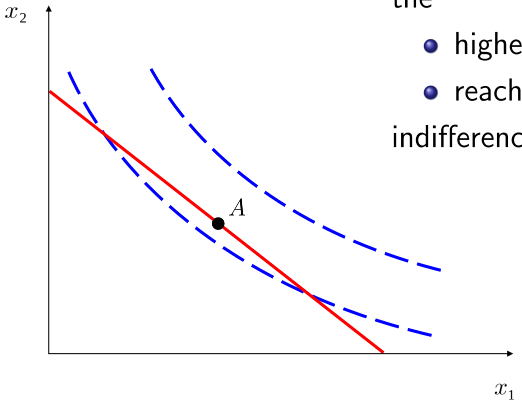
The household's maximization problem



Problem

Which bundle is the household optimum?

The household's maximization problem

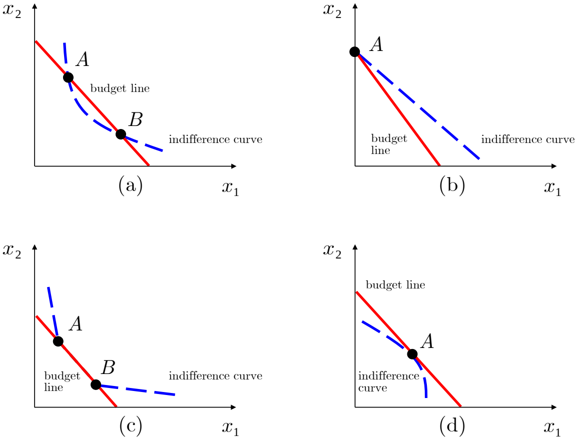


Household optimum:
Choose a bundle of goods that lies on the

- highest (maximization)
- reachable (within the budget set)

indifference curve!

Four maximization problems



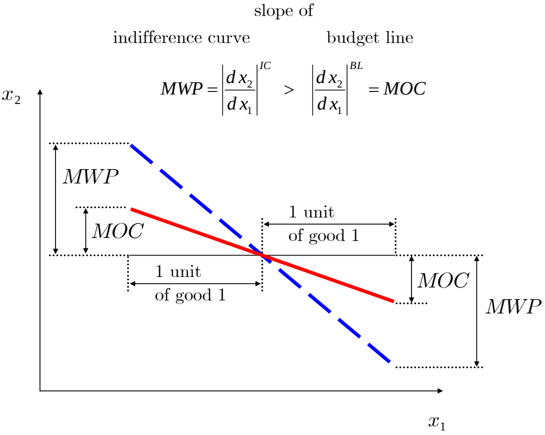
Problem

Assuming monotonic preferences, are bundles A or B optimal?

Marginal willingness to pay MWP versus marginal opportunity cost MOC

- Marginal **willingness to pay**:
  If the household consumes one additional unit of good 1, how many units of good 2 **can** she forgo such that she is indifferent between the two bundles?
- Marginal **opportunity cost**:
  If the household consumes one additional unit of good 1, how many units of good 2 **must** she forgo due to the budget constraint?

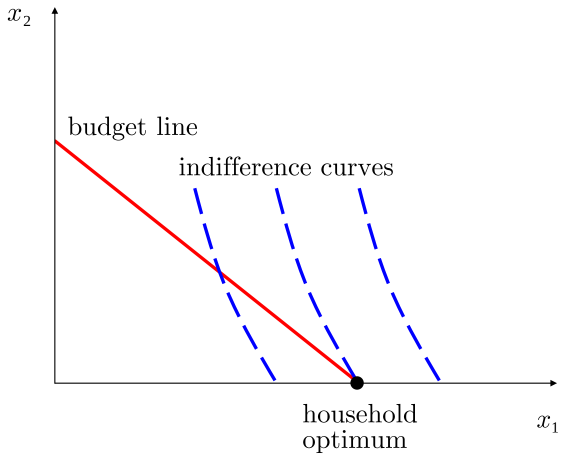
MWP versus MOC



Therefore: Utility increases if the household increases consumption of good 1.

MWP versus MOC

$MRS > MOC \Rightarrow$ increase x_1 (if possible)



MWP versus MOC

Alternatively: the household wants to maximize $U\left(x_1, \frac{m}{p_2} - \frac{p_1}{p_2}x_1\right)$

- Consumption of one additional unit of good 1 yields
  - an increase in utility by $\frac{\partial U}{\partial x_1}$
  - a decrease in x_2 by $MOC = \left|\frac{dx_2}{dx_1}\right| = \frac{p_1}{p_2}$ and hence

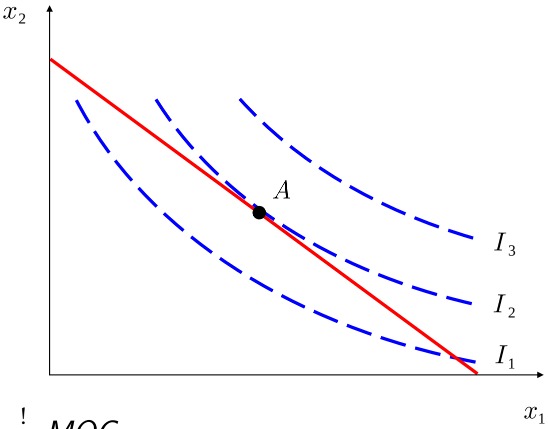
    a decrease in utility by $\frac{\partial U}{\partial x_2}\left|\frac{dx_2}{dx_1}\right|$ (chain rule)
- Increase x_1 as long as

$$\underbrace{\frac{\partial U}{\partial x_1}}_{\text{marginal utility of increasing } x_1} > \underbrace{\frac{\partial U}{\partial x_2}\left|\frac{dx_2}{dx_1}\right|}_{\text{marginal cost of increasing } x_1}$$

$$\text{or } MRS = \frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}} > \left|\frac{dx_2}{dx_1}\right| = MOC$$

Strictly convex preferences

Cobb-Douglas utility function



1. $MRS \overset{!}{=} MOC$
2. $p_1 x_1 + p_2 x_2 \overset{!}{=} m$

Strictly convex preferences

Cobb-Douglas utility function

For Cobb-Douglas utility functions $U(x_1, x_2) = x_1^a x_2^{1-a}$ the ratio of marginal utilities is given by

$$MRS = \frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}} = \frac{a x_1^{a-1} x_2^{1-a}}{(1-a) x_1^a x_2^{-a}} = \frac{x_2}{x_1} \frac{a}{1-a}$$

Two equations with two unknowns yield the household optimum:

$$\begin{aligned} x_1^* &= a \frac{m}{p_1} \\ x_2^* &= (1-a) \frac{m}{p_2} \end{aligned}$$

Strictly convex preferences

Cobb-Douglas utility function

Problem

Olaf's preferences are represented by the utility function
$U(D, C) = \sqrt{DC}$

- $D =$ the number of visits to a nightclub per month
- $C =$ the number of visits to concerts per month

The budget is given by

- $p_D = 2$/visit to nightclub,
- $p_C = 4$/visit to concert and
- $m = 64$.

Is it possible to consider the utility function $(U(D, C))^2 = DC$ instead of $U(D, C) = \sqrt{DC}$?
Derive the quantities consumed by Olaf!

Strictly convex preferences

Cobb-Douglas utility function

- Gossen's first law

$$\frac{\partial \frac{\partial U}{\partial x_1}}{\partial x_1} = \frac{\partial^2 U}{(\partial x_1)^2} < 0$$

The marginal utility decreases with respect to each unit that is additionally consumed. **Interpretation is possible only in the sense of cardinal utility theory!**

- Gossen's second law

$$\frac{MU_1}{p_1} = \frac{MU_2}{p_2}$$

Makes sense also concerning ordinal utility theory

Lagrange approach (excursus)

Constraint as equation

- is a procedure to solve optimization problems with constraints.
- Assumption: monotonic and convex preferences
- Maximize

$$U(x_1, x_2)$$

with respect to the constraint

$$m - (p_1x_1 + p_2x_2) = 0$$

- For $m - p_1x_1 - p_2x_2 \geq 0$ the so-called Kuhn-Tucker procedure is appropriate.

Lagrange approach (excursus)

Lagrange function

$$L(x_1, x_2, \lambda) = U(x_1, x_2) + \lambda(m - p_1 x_1 - p_2 x_2)$$

- Increasing x_1
  - has a direct (positive) effect on utility, $\frac{\partial U}{\partial x_1} > 0$,
  - and an indirect (negative) effect: The reduced budget surplus

$$\frac{\partial(m - p_1 x_1 - p_2 x_2)}{\partial x_1} = -p_1 < 0$$

  is translated in reduced utility by $\lambda > 0$, in total therefore $\lambda(-p_1)$.

- Optimum: $\frac{\partial U}{\partial x_1} \stackrel{!}{=} \lambda p_1$.

$$\frac{\partial L(x_1,x_2,\lambda)}{\partial x_1} = \frac{\partial U(x_1,x_2)}{\partial x_1} - \lambda p_1 \overset{!}{=} 0,$$
$$\frac{\partial L(x_1,x_2,\lambda)}{\partial x_2} = \frac{\partial U(x_1,x_2)}{\partial x_2} - \lambda p_2 \overset{!}{=} 0,$$
$$\frac{\partial L(x_1,x_2,\lambda)}{\partial \lambda} = m - p_1x_1 - p_2x_2 \overset{!}{=} 0.$$

Hence, Gossen's second law:

$$\frac{\frac{\partial U(x_1,x_2)}{\partial x_1}}{p_1} \overset{!}{=} \lambda \overset{!}{=} \frac{\frac{\partial U(x_1,x_2)}{\partial x_2}}{p_2}$$

and $MRS \overset{!}{=} MOC$.

Lagrange approach (excursus)

Utility of income – interpretation

- λ can be interpreted as marginal utility of income

$$\lambda = \frac{dU}{dm}.$$

- This yields the optimality condition

$$\frac{\partial U\left(x_1, x_2\right)}{\partial x_1} \overset{!}{=} \frac{dU}{dm} p_1.$$

- Nevertheless: U is no function of m ...

Lagrange approach (excursus)

Indirect utility function

$$V(p_1, p_2, m) := U(x_1(p_1, p_2, m), x_2(p_1, p_2, m)).$$

$x_1(p_1, p_2, m)$ is optimal consumption of good 1 at prices p_1 and p_2 and income m.

Problem

Determine the indirect utility function for a Cobb-Douglas utility function $U(x_1, x_2) = x_1^a x_2^{1-a}$ $(0 < a < 1)$!

Lagrange approach (excursus)

Marginal utility of income – formally correct

- $V(p_1, p_2, m) := U(x_1(p_1, p_2, m), x_2(p_1, p_2, m))$ yields

$$\begin{aligned} \frac{\partial V}{\partial m} &= \frac{\partial U}{\partial x_1}\frac{\partial x_1}{\partial m} + \frac{\partial U}{\partial x_2}\frac{\partial x_2}{\partial m} \\ &= (\lambda p_1)\frac{\partial x_1}{\partial m} + (\lambda p_2)\frac{\partial x_2}{\partial m} \\ &= \lambda\left(p_1\frac{\partial x_1}{\partial m} + p_2\frac{\partial x_2}{\partial m}\right) \end{aligned}$$

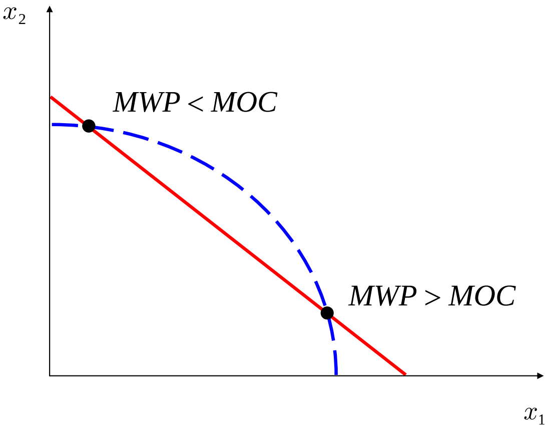
- $p_1 x_1(p_1, p_2, m) + p_2 x_2(p_1, p_2, m) = m$ leads to

$$p_1\frac{\partial x_1}{\partial m} + p_2\frac{\partial x_2}{\partial m} = 1$$

hence, more exactly

$$\frac{\partial V}{\partial m} = \lambda.$$

Concave preferences



- $MRS > MOC \Rightarrow$ increase x_1 (if possible)
- $MRS < MOC \Rightarrow$ decrease x_1 (if possible)

Concave preferences

specific example

$$U(x_1, x_2) = x_1^2 + x_2^2$$

Increasing consumption of good 1 increases utility if

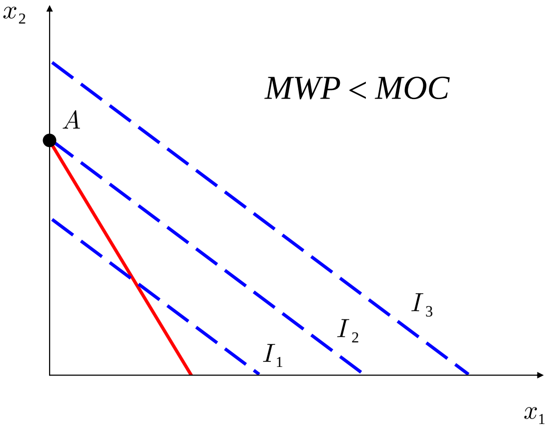
$$\frac{x_1}{x_2} = \frac{2x_1}{2x_2} = \frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}} = MRS > MOC = \frac{p_1}{p_2}$$

holds. Hence, corner solutions:

$$x^*(m, p) = \begin{cases} \left(\frac{m}{p_1}, 0\right), & p_1 < p_2 \\ \left\{\left(\frac{m}{p_1}, 0\right), \left(0, \frac{m}{p_2}\right)\right\}, & p_1 = p_2 \\ \left(0, \frac{m}{p_2}\right), & p_1 > p_2 \end{cases}$$

Perfect substitutes

graphical solution



$MRS < MOC \Rightarrow$ reduce x_1 (if possible)

Perfect substitutes

analytical solution

$U(x_1, x_2) = ax_1 + bx_2$, where $a > 0$ and $b > 0$

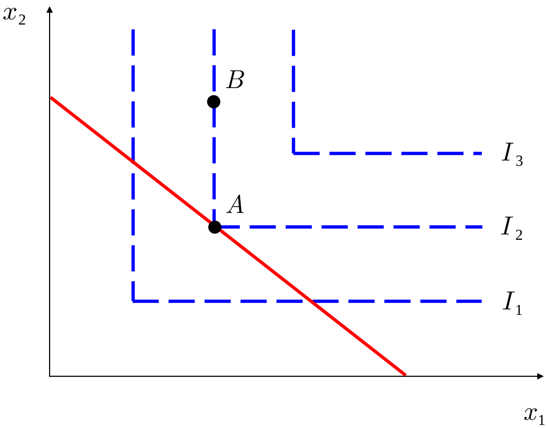
Increasing consumption of good 1 increases utility if

$$\frac{a}{b} = MRS > MOC = \frac{p_1}{p_2}$$

holds. Hence, (almost always) corner solutions:

$$x^*(m, p) = \begin{cases} \left(\frac{m}{p_1}, 0\right), & \frac{a}{b} > \frac{p_1}{p_2} \\ \left\{\left(x_1, \frac{m}{p_2} - \frac{p_1}{p_2}x_1\right) \in \mathbb{R}_+^2 : x_1 \in \left[0, \frac{m}{p_1}\right]\right\}, & \frac{a}{b} = \frac{p_1}{p_2} \\ \left(0, \frac{m}{p_2}\right), & \frac{a}{b} < \frac{p_1}{p_2} \end{cases}$$

Perfect complements



1. Determine L-corner!
2. $p_1 x_1 + p_2 x_2 \stackrel{!}{=} m$

Perfect complements

$$U(x_1, x_2) = \min(ax_1, bx_2)$$

- L-corner

$$ax_1 \overset{!}{=} bx_2 \text{ or}$$
$$x_2 \overset{!}{=} \frac{a}{b}x_1$$

- Budget line

$$m \overset{!}{=} p_1x_1 + p_2x_2 = p_1x_1 + p_2\frac{a}{b}x_1 = x_1\left(p_1 + \frac{a}{b}p_2\right)$$

- For good 1 we obtain

$$x_1^* = \frac{m}{p_1 + \frac{a}{b}p_2}.$$

Household optima

Problem

Determine the household optimum for variable income m and

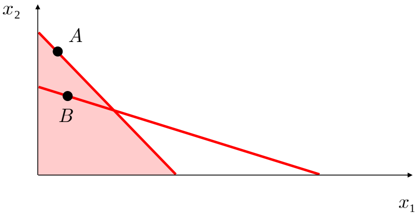
- utility function $U(x_1, x_2) = \frac{1}{2}x_1^2 + x_2^2$ and prices $p_1 = 1$, $p_2 = 2$;
- lexicographic preferences, where good 1 is the important good, and prices $p_1 = 2$, $p_2 = 5$;
- utility function $U(x_1, x_2) = x_1 + 2x_2$ and prices $p_1 = 1$, $p_2 = 3$;
- utility function $U(x_1, x_2) = \min(x_1, 2x_2)$ and prices $p_1 = 1$, $p_2 = 3$!

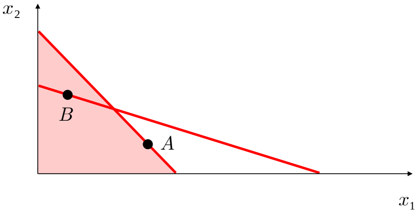
Revealed preferences

- Usual way: budget and preferences $\Rightarrow$ household optimum
- Alternative: actual decisions of households $\Rightarrow$ preferences.

$\Longrightarrow$ This preferences are called revealed. (SH 18)

Revealed preferences





Problem

Are the illustrated decisions:

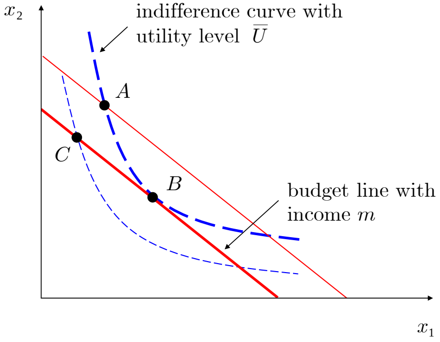
- choose bundle B for the flatter budget line and
- choose bundle A for the steeper budget line

in compliance with monotonicity?

Minimization and maximization problem

- Maximization: Find the bundle of goods that maximizes utility at a given budget line!

- Minimization: Find the bundle of goods that minimizes expenditures necessary to reach a given utility level.



Expenditure function

$$e\left(p_1, p_2, \overline{U}\right) := \min_{\substack{x_1, x_2 \\ \text{w.r.t. } \overline{U} = U(x_1, x_2)}} \left(p_1 x_1 + p_2 x_2\right)$$

Household optimum: compensated demand

$$\chi_1\left(p_1, p_2, \overline{U}\right) \text{ and } \chi_2\left(p_1, p_2, \overline{U}\right)$$

Compensated: In order to maintain the utility level when prices increase, the income needs to be increased as compensation

Expenditure function

function	arguments	optimum
utility function	quantities	$x_1(p_1, p_2, m)$, $x_2(p_1, p_2, m)$
expenditure function	utility level, prices	$\chi_1(p_1, p_2, \overline{U})$, $\chi_2(p_1, p_2, \overline{U})$

Expenditure function

- χ_1 and χ_2 = **Hicksian demand** (compensated demand): bundle of goods that reaches a given utility level with least possible expenditures.
- x_1 and x_2 = **Marshallian** demand: bundle of goods that reaches the highest reachable utility level for a given income.
- Almost always:

$$x_1\left(p_1, p_2, e\left(p_1, p_2, \overline{U}\right)\right) = \chi_1\left(p_1, p_2, \overline{U}\right).$$

The expenditure function

Problem

Determine the expenditure function for a Cobb-Douglas utility function $U(x_1, x_2) = x_1^a x_2^{1-a}$, where $0 < a < 1$! Determine the quantities of both goods that minimize expenditures for a given utility level! Hint: You can use the quantities $x_1^* = a\frac{m}{p_1}$ and $x_2^* = (1-a)\frac{m}{p_2}$. Use the approach $\overline{U} = U(x_1^*, x_2^*)$ and determine the budget $m = e\left(p_1, p_2, \overline{U}\right)$ that is necessary to reach the given utility level $\overline{U}$!

Central tutorial I

Problem D.9.1.
A stock farmer lives on milk (good 1) and bread (good 2). Every morning he milks his cow and obtains 10 liter milk. He takes the milk to the market to trade it for bread. His utility function is given by $U(x_1, x_2) = \ln x_1 + 3 \ln x_2$, where x_1 and x_2 denote the quantities of milk and bread, respectively. The price for milk is 1 dollar/liter and 5 dollar/loaf for bread. How many liter of milk and loaves of bread does the farmer consume in a day?

Problem D.9.2.
A household with the utility function $U(x_1, x_2) = \sqrt{x_1} + 2\sqrt{x_2}$ spends income $m = 16$ on the two goods with prices $p_1 = 1$ and $p_2 = 4$. Determine the household optimum!

Central tutorial II

Problem D.9.3.
$U(x_1, x_2) = \min\left(x_1, \frac{1}{2}x_2\right)$
$m = 15$, $p_1 = 2$ and $p_2 = 4$
Household optimum!

Problem D.9.4.
$U(x_1, x_2) = 2x_1 + 4x_2$
$m = 8$, $p_1 = 2$ and $p_2 = 4$
Household optimum!

Problem D.9.5.
$U(x_1, x_2) = \ln x_1 + x_2$
m, p_1 and p_2, where $\frac{m}{p_2} > 1$
Household optimum!
Expenditure function!