Values with exogenous payments

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September 2011

Abstract

The aim of cooperative game theory is to suggest and defend payoffs for the players that depend on a coalition function (characteristic function) describing the economic, social, or political situation. We consider situations where the payoffs for some players are determined exogenously. For example, in many countries lawyers or real-estate agents obtain a regulated fee or a regulated percentage of the business involved. Our aim is to suggest and axiomatize two values with exogenous payments, an unweighted one and a weighted one.

Keywords: Shapley value, exogenous payments, cooperative game theory, cost allocation, real-estate agency
JEL classification: C71

1. Introduction

The aim of cooperative game theory is to suggest and defend payoffs for the players that depend on a coalition function (characteristic function) describing the economic, social, or political situation. In this sense, the players' payoffs are

[0]I would like to acknowledge helpful discussions with, and valuable hints by, André Casajus, Frank Hüttner, Pavel Brendler, and Andreas Tutic. Steffen Birkigt and Martin Kohl provided very able research assistance. Three anonymous referees gave detailed hints with one report being extraordinarily careful.

determined endogenously. However, there are situations in real life where some players' payoffs are exogenous. For example, in many countries lawyers or real-estate agents obtain a regulated fee or a regulated percentage of the business involved. Similarly, civil servants who participate in the production of economic goods in different ways are also paid according to official schedules. Players obtaining exogenous payoffs are called exogenous players while the other players are endogenous.

One additional motivation for developing values that take exogenous payments into account is provided by situations where cost sharing (see Young 1994) occurs in the presence of third parties who also use the resources in question. Examples concern (i) firms whose computing facilities are used by profit centers inside the firm (the endogenous players) but also by other firms (the exogenous players) and (ii) municipalities that build a water-distribution system for themselves (as endogenous players) and also for other towns (the exogenous players) who pay user fees.

In order to address the examples given above (and many others), we aim for values that incorporate the idea of exogenous payments. For example, we are asking the question of how much of the overall cost have to be borne by the endogenous towns after they obtain the fixed user fees by the exogenous ones. Once again, it turns out that a variant of the famous Shapley (1953) value is well suited for that purpose.

In our paper, we introduce two classes of games: exogenous-payments games and weighted exogenous-payments games. The weights determine the burden sharing of the endogenous players with respect to the payments obtained by the exogenous ones. Correspondingly, we introduce two values: the exogenous-payments Shapley value and the exogenous-payments weighted Shapley value. The latter one is not to be confounded with the weighted Shapley value – the weights in the Kalai-Samet weighted Shapley value affect all players' payoffs, depending on the hierarchy level (see Kalai & Samet 1987). For identical weights, the exogenous-payments weighted Shapley value equals the exogenous-payments Shapley value.

Our characterizations use two important axioms. First of all, we demand that the payoffs under the new value actually give the predetermined payoff to the exogenous players, i.e., the realtor's fee to the realtor and the civil-service payments to the civil servants (axiom X). Second, our value obeys the following consistency axiom (axiom C): If the exogenous payments happen to be equal to the payoff determined endogenously (i.e., according to the Shapley value), then the endogenous agents also obtain their Shapley values.

In the next section, we provide the basic definitions and notations. Section 3 introduces the exogenous-payments Shapley value and presents two axiomatizations for this value. We also elaborate on the cost-allocation example. In section 4, we show how to incorporate weights for the endogenous players, i.e., we present and axiomatize the exogenous-payments weighted Shapley value. This value is then applied to the a real-estate agent and the fee he obtains. The final section offers suggestions for future research.

2. Definitions and notation

A TU game (in coalition function form) is a pair (N, v) (often abbreviated by v) where N is a finite set and v a function $2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. The set of all games on N is denoted by $\mathbb{V}_N$. A game (N, v) is convex if for all coalitions S and S' obeying $S \subseteq S'$ we have

$$v(S \cup \{i\}) - v(S) \leq v(S' \cup \{i\}) - v(S')$$

for all players $i \notin S'$. It is concave for $\geq$ rather than $\leq$. v is called inessential if $v(K) = \sum_{i \in K} v(\{i\})$ for all $K \subseteq N$. For $T \neq \emptyset$, $T \subseteq N$, the game u_T is given by $u_T(K) = 1$ for $T \subseteq K$, $u_T(K) = 0$ otherwise. These games are called unanimity games. For $g \in \mathbb{R}$, the coalition function $gu_{\{N\}}$ is abbreviated by just g, i.e., $v := g$ is the coalition function defined by

$$v(K) = \begin{cases} g, & K = N \\ 0, & K \neq N \end{cases} \tag{2.1}$$

$v := 0$ is called the zero game.

A payoff vector x for N is an element of $\mathbb{R}^N$ or a function $N \to \mathbb{R}$. By x_S we mean $\sum_{i \in S} x_i$.

Player $i \in N$ is a null player if

$$v(K \cup \{i\}) = v(K) \text{ for all } K \subseteq N \backslash \{i\}.$$

Two players $i, j \in N$ are called symmetric if for all coalitions K obeying $i \notin K$ and $j \notin K$ we have

$$v(K \cup \{i\}) = v(K \cup \{j\}).$$

A coalition function v is symmetric if a function $f : \mathbb{R} \to \mathbb{R}$ exists such that $v(K) = f(|K|)$ holds for all $K \subseteq N$.

Rules of order r on N are bijective functions $r : N \to N$ where $r(1)$ is to be understood as the first player in the order, $r(2)$ as the second player etc. The set of all rules of order on N is denoted by R. The inverse $r^{-1}(i)$ denotes player i's "position" in the rule of order r. Then, we define $K_i(r) := \{r(1), ..., r(r^{-1}(i))\}$, i.e. $K_i(r)$ is the set of players up to and including player i.

The Shapley (1953) value and other related values make heavy use of the players' marginal contributions MC. For any coalition $S \subseteq N$ and any player $i \in S$ we define

$$MC_i^S(v) := v(S) - v(S \backslash \{i\})$$

and, given some rule of order r from R,

$$MC_i(v, r) := MC_i^{K_i(r)}(v).$$

The Shapley value which we denote by $Sh(N, v) = (Sh_i(N, v))_{i \in N} \in \mathbb{R}^N$ is a solution concept on $\mathbb{V}_N$ and is defined by

$$Sh_i(v, N) = \frac{1}{|N|!} \sum_{r \in R} MC_i(v, r), i \in N.$$

The Shapley value is characterized by the following four axioms:
Efficiency: We have $\varphi_N(N, v) = v(N)$.
Symmetry: For all symmetric players $i, j \in N$, $\varphi_i(N, v) = \varphi_j(N, v)$.
Null player: If $i \in N$ is a null player, then $\varphi_i(N, v) = 0$.
Additivity: For any coalition functions $v', v'' \in \mathbb{V}_N$, and any player i from N,

$$\varphi_i(N, v' + v'') = \varphi_i(N, v') + \varphi_i(N, v'').$$

We now introduce the set of exogenous players $X \subsetneqq N$ (the civil servants or real-estate agents, if you like) and the payments they receive. The other players are called endogenous players (the private sector) and are denoted by $D := N \backslash X$.

We define two games with exogenous payments, an XP game and a weighted XP game (where XP stands for eXogenous Payments):

Definition 2.1. *XP games are tuples*

$$(N, v, X, \pi)$$

where

- (N, v) *is a TU game,*
- X *is a strict subset of* N, *and*
- $\pi \in \mathbb{R}^N$ *is a vector specifying a payoff for every member of* N *and, in particular, for every member of* X.

A weighted XP game is a tuple (N, v, X, π, w) *where* (N, v, X, π) *is an XP game and* $w = (w_i)_{i \in N}$ *a tuple of real numbers obeying* $\sum_{d \in D} w_d \neq 0$.

Note that both π and w are from $\mathbb{R}^N$. While this seems unnecessary for the definition itself, it is helpful when we consider empty and non-empty sets of exogenous players in applications of the consistency axioms or the irrelevance axiom. However, we sometimes abuse notation by writing $(\pi_x)_{x \in X}$ rather than π and $(w_d)_{d \in D}$ rather than w.

3. The XP Shapley value

3.1. Axioms

An XP value φ assigns a payoff vector to every XP game, $\varphi(N, v, X, \pi) \in \mathbb{R}^N$. Of course, exogenous players should obtain their exogenous payments:

X (exogenous payments): For all $i \in X$, we have $\varphi_i(N, v, X, \pi) = \pi_i$.

Given axiom X, most other axioms focus on the players from D for obvious reasons. Of the following nine axioms, axioms N and M are not fulfilled by the exogenous-payments Shapley value (for short: XP Shapley value).

The most well-known axiomatization for the Shapley value involves the efficiency axiom, the symmetry axiom, the null-player axiom, and the additivity axiom (see the previous section). Efficiency has to hold for all players while symmetry makes sense for endogenous players, only:

E (efficiency): We have $\varphi_N(N, v, X, \pi) = v(N)$.

S (symmetry): For all symmetric players $i, j \in D$, $\varphi_i(N, v, X, \pi) = \varphi_j(N, v, X, \pi)$.

We mention two axioms referring to null players. The null-player axiom N awards the payoff zero to every null player from D while the null-player axiom N-$\emptyset$ demands that a null player obtains the payoff zero if there are no exogenous players in the game:

N (null player): If $i \in D$ is a null player in (N, v), then $\varphi_i(N, v, X, \pi) = 0$.

N-$\emptyset$ (null player for $X = \emptyset$): If $i \in N$ is a null player in (N, v), then $\varphi_i(N, v, \emptyset, \pi) = 0$.

If exogenous players exist, null players cannot, in the present context, have zero payoffs. For example, in the 0-game v (defined by $v(K) = 0$ for all $K \subseteq N$), all players are null players and the endogenous players have to pay π_X for reasons of efficiency. Thus, a null-player axiom is not a reasonable requirement in case of $X \neq \emptyset$. Also, a null-player-out axiom (see Derks & Haller 1999) cannot hold for the value we are to define. If a null player from D is excluded from the game, the other endogenous players have to divide π_X between themselves.

The additivity axiom concerns all the players from $X \cup D$ and refers to payments as well as coalition functions. Thus, if a player is involved in two games, he is to obtain the sum of what he would get in each of them:

A (additivity): For any coalition functions $v', v'' \in \mathbb{V}_N$, any payments π', $\pi'' \in \mathbb{R}^N$ and any player i from N, we obtain

$$\varphi_i(N, v' + v'', X, \pi' + \pi'') = \varphi_i(N, v', X, \pi') + \varphi_i(N, v'', X, \pi'').$$

We now present two axioms (axiom M and axiom BF) each of which is central for a further axiomatization of the Shapley value (see the following subsection). Axiom M states that player i from D is affected by a coalition function only insofar as his marginal contributions are concerned. This holds for the Shapley value but not for the XP Shapley value. The reason is that the players from D pay π to the players from X but enjoy the contributions made by these exogenous players by efficiency.

M (marginalism): Assume two coalition functions v and z from $\mathbb{V}_N$. Let i be a player from D obeying

$$v(S) - v(S \setminus \{i\}) = z(S) - z(S \setminus \{i\})$$

for all $S \subseteq N, i \in S$. Then

$$\varphi_i(N, v, X, \pi) = \varphi_i(N, z, X, \pi).$$

In contrast to axiom M, the XP Shapley value fulfills axiom BF that is due to van den Brink (2001). This axiom says that two players are equally affected by adding a coalition function z (to some given coalition function v) if they are symmetric in (N, z). This property follows from axioms A and S.

BF (Brink fairness): Let i and j be players from D that are symmetric in (N, z). Then

$$\varphi_i(N, v+z, X, \pi) - \varphi_i(N, v, X, \pi) = \varphi_j(N, v+z, X, \pi) - \varphi_j(N, v, X, \pi).$$

Next, we present the shifting property. It says that a player from D does not gain or suffer if a change in π_X is balanced by a corresponding change of v by π_X. In a sense, both π_X and v (see eq. (2.1)), are shifted in the same direction. For example, if a lawyer is responsible for an increase (or a decrease) of the value produced by business partners using his services and if his renumeration is changed by the very same amount, the business partners are not affected. Similarly, a civil servant (working as a policeman or a judge) may be credited with increasing the social product in an economy (by discouraging theft or other harmful activities). Again, if his salary increases by an equal amount, the private-sector agents' payoffs stay the same.

SH (shifting): For all $i \in D$, we have

$$\varphi_i(N, v+\pi_X, X, \pi) = \varphi_i(N, v+\pi'_X, X, \pi')$$

for all $\pi, \pi' \in \mathbb{R}^N$.

It is not difficult to show that axioms X, E, S, and A imply SH. We show later that axioms X, E, N-$\emptyset$, BF, and C also imply SH. While this axiom is not used in any of the axiomatizations that we present, it is an important intermediate step in the second axiomatization presented below.

The final axiom is called consistency and of central importance to all the axiomatizations in this paper: If the players in X obtain what they would obtain if no exogenous payments were made to anybody, the players in D also obtain what they should get without any exogenous players in the game. Differently put, if the players in X (happen to) obtain the value dictated by the axioms for games without exogenous players, so do the other players. Consistency axioms have been surveyed by Thomson (1990) and Driessen (1991).

C (consistency): For any player $i \in D$,

$$\varphi_i\left(N, v, X, (\varphi_x(N, v, \emptyset, \pi))_{x \in X}\right) = \varphi_i(N, v, \emptyset, \pi).$$

3.2. Three XP values

We now present three XP values. The XP Shapley value which will be axiomatized in the next subsection is denoted by $Sh^{X,\pi}$ and given by

$$Sh_i^{X,\pi}(N,v) = \begin{cases} \pi_i, & i \in X \\ Sh_i(N,v) + \frac{1}{|D|}\left(Sh_X(N,v) - \pi_X\right), & i \in D \end{cases}$$

Note that every player in D contributes equally to the exogenous payments. This is not unreasonable given the fact that differences between the D-players clearly show up in the first summand. However, in particular applications, we may have good reasons to argue for different weights and to turn to the weighted XP Shapley value.

We can consider the XP Shapley value as an XP value for XP games (N, v, X, π) or, alternatively, as a solution for TU games on N where X and π enter as parameters. In the second case, we might then ask the question of how exogenous payments influence the endogenous players' payoffs. The formula for the XP Shapley value gives an immediate answer to that question as does SH.

It is easy to see that the XP Shapley value fulfills the axioms X, E, S, N-$\emptyset$, A, BF, C, and, by lemma 3.5 below, axiom SH, too.

For future reference and for the proofs of independence in later sections, we also define the egalitarian value Eg by

$$Eg_i(v,N) = \frac{v(N)}{|N|}, i \in N$$

and the egalitarian value with exogenous payments $Eg^{X,\pi}$ by

$$Eg_i^{X,\pi}(N,v) = \begin{cases} \pi_i, & i \in X \\ Eg_i(N,v) + \frac{1}{|D|}\left(Eg_X(N,v) - \pi_X\right), & i \in D \end{cases}$$

Clearly, this value also fulfills the axioms X, E, S, A (which imply SH), BF, and C while axiom N-$\emptyset$ does not hold.

Axiom C is fulfilled by these two values and plays a central role in all our axiomatizations. It seems a natural requirement. Consider, however, the following alternative. Define a TU game $\left(N\backslash X, p^{v,X,\pi}\right)$ by

$$p^{v,X,\pi}(S) = \begin{cases} v(S \cup X) - \pi_X, & S \neq \emptyset \\ 0, & S = \emptyset \end{cases}$$

For example, X is the set of civil servants in an economy (N, v) and π_X the taxes to be paid for the civil servants. $p^{v,X,\pi}$ is close to coalition functions defined in Aumann & Drèze (1974) and in Peleg (1986). The most important difference is that these authors assume that players from S can choose the players from X they want to use and pay for. Our more simple definition makes sense for the above interpretation.

On the basis of the above TU game on $N \backslash X$, we define the XP value φ^C by

$$\varphi_i^C(N, v, X, \pi) = \begin{cases} \pi_i, & i \in X \\ Sh_i\left(D, p^{v,X,\pi}\right), & i \in D \end{cases}$$

φ^C violates axiom C in the example of $N = \{1, 2, 3\}$, $v = u_{\{1,2\}}$, $X = \{1\}$, and $\pi_1 = \frac{1}{2} = Sh_1\left(N, u_{\{1,2\}}\right) = Sh_1\left(N, p^{u_{\{1,2\}},\emptyset,\pi}\right) = \varphi_1^C\left(N, u_{\{1,2\}}, \emptyset, \pi\right)$ that leads to

$$\begin{aligned}
& \varphi_2^C\left(N, u_{\{1,2\}}, \{1\}, \varphi_1^C\left(N, u_{\{1,2\}}, \emptyset, \pi\right)\right) \\
= \; & Sh_2\left(D, p^{u_{\{1,2\}},\{1\},\frac{1}{2}}\right) \\
= \; & \frac{1}{2}\left(u_{\{1,2\}}(\{1,2\}) - \frac{1}{2} - 0\right) + \frac{1}{2}\left(u_{\{1,2\}}(\{1,2,3\}) - \frac{1}{2} - \left[u_{\{1,2\}}(\{1,3\}) - \frac{1}{2}\right]\right) \\
= \; & \frac{3}{4} \\
\neq \; & \frac{1}{2} = \varphi_2^C\left(N, u_{\{1,2\}}, \emptyset, \pi\right).
\end{aligned}$$

In this example, player 2 takes all the benefit from the services provided by the civil servant 1, but pays half the taxes. In such-like situations, a violation of consistency makes perfect sense.

It is not difficult to see that φ^C obeys axioms X, E, S, N-$\emptyset$, A, BF, and SH.

3.3. Axiomatization

In order to compare our value with the Shapley value, we note the following theorem:

Theorem 3.1. *Assuming $X = \emptyset$ (in which case N-$\emptyset$ and N are equivalent) and ignoring π in that case, the Shapley value is characterized by the following sets of axioms:*

- *E, S, N, and A (Shapley (1953))*

- *E, S, and M (Young (1985))*
- *E, N, and BF (van den Brink (2001))*

As the above theorem makes clear, we can look for sets of axioms including the axioms E, S, N-$\emptyset$, and A or including E, N-$\emptyset$, and BF. We prepare our two characterizations with a lemma:

Lemma 3.2. *Assuming axiom C and any of the two following axiom sets for solution* φ

- *E, S, N-$\emptyset$, and A or*
- *E, N-$\emptyset$, and BF*

we obtain

$$Sh_i(N,v) = \varphi_i\left(N,v,X,(Sh_x(N,v))_{x\in X}\right)$$

for all players $i \in D$.

Proof. By the above theorem, either one of the set of axioms (E, S, N-$\emptyset$, and A on the one hand or E, N-$\emptyset$, and BF on the other hand) imply

$$\varphi_i(N,v,\emptyset,\pi) = Sh_i(N,v). \tag{3.1}$$

We then find

$$\begin{aligned} Sh_i(N,v) &= \varphi_i(N,v,\emptyset,\pi) \text{ (eq. (3.1))} \\ &= \varphi_i\left(N,v,X,(\varphi_x(N,v,\emptyset,\pi))_{x\in X}\right) \text{ (axiom C)} \\ &= \varphi_i\left(N,v,X,(Sh_x(N,v))_{x\in X}\right) \text{ (eq. (3.1))} \end{aligned}$$

□

Theorem 3.3. *The XP Shapley value is characterized by the axioms X, E, S, N-$\emptyset$, A, and C.*

Proof. We know from the previous subsection that $Sh^{X,\pi}$ fulfills all the axioms mentioned in the theorem. Let φ be an XP value. For $i \in X$, axiom X guarantees $\varphi_i(N, v, X, \pi) = \pi_i$. For $i \in D$, we obtain the desired result by

$$\begin{aligned}
& \varphi_i(N, v, X, \pi) \\
= \; & \varphi_i\left(N, v, X, (Sh_x(N, v))_{x \in X}\right) \\
& + \varphi_i\left(N, 0, X, (\pi_x)_{x \in X} - (Sh_x(N, v))_{x \in X}\right) \text{ (axiom A)} \\
= \; & Sh_i(N, v) + \varphi_i\left(N, 0, X, (\pi_x)_{x \in X} - (Sh_x(N, v))_{x \in X}\right) \text{ (lemma 3.2)} \\
= \; & Sh_i(N, v) + \frac{1}{|D|}\left(Sh_X(N, v) - \pi_X\right) \text{ (axioms E, S)}
\end{aligned}$$

□

Lemma 3.4. *The axioms X, E, S, N-∅, A, and C are independent.*

We relegate the proof to the appendix.

The proof of the axiomatization of theorem 3.6 below rests on the following lemma that we show to hold in the appendix.

Lemma 3.5. *Axioms X, E, N-∅, BF, and C imply axiom SH.*

Theorem 3.6. *The XP Shapley value is characterized by the axioms X, E, N-∅, BF, and C.*

Proof. We need to show that the XP Shapley value is the only value that fulfills the above mentioned axioms. Consider the coalition function $z := \pi_X - Sh_X(N, v)$. Then any two players i and j from D are symmetric in (N, z) and BF implies

$$\varphi_i(N, v + z, X, \pi) - \varphi_i(N, v, X, \pi) = \varphi_j(N, v + z, X, \pi) - \varphi_j(N, v, X, \pi).$$

Fix $i \in D$ and sum this equation for all $j \in D$. Using axioms X and E and hence $\varphi_D(N, v, X, \pi) = v(N) - \pi_X$ and $\varphi_D(N, v + z, X, \pi) = (v + z)(N) - \pi_X$, respectively, we find

$$\varphi_i(N, v, X, \pi) = \varphi_i(N, v + z, X, \pi) + \frac{1}{|D|}\left(Sh_X(N, v) - \pi_X\right).$$

The equations

$$\begin{aligned} Sh_i(N,v) &= \varphi_i\left(N,v,X,(Sh_x(N,v))_{x\in X}\right) \text{ (lemma 3.2)} \\ &= \varphi_i\left(N,v-Sh_X(N,v)+\pi_X,X,(\pi_x)_{x\in X}\right) \text{ (lemma 3.5)} \\ &= \varphi_i(N,v+z,X,\pi) \end{aligned}$$

provide the final bit of our proof.□

Lemma 3.7. *The axioms X, E, N-∅, BF, and C are independent.*

See the appendix for a proof.

3.4. Application: cost allocation with exogenous payments

We adapt the example presented by Young (1994, pp. 1195). Two towns 1 and 2 plan a water-distribution system. Town 1 could build such a system for itself at a cost of 12 million Euro and town 2 would need 8 million Euro for a system tailor-made to its needs. The cost for a common water-distribution system is 16 million Euro. Thus, the coalition, or cost, function $c: 2^{\{1,2\}} \to \mathbb{R}$ is given by

$$\begin{aligned} c(\{1\}) &= 12, c(\{2\}) = 8 \text{ and} \\ c(\{1,2\}) &= 16. \end{aligned}$$

The Shapley value for this game is $Sh(\{1,2\},c) = (10,6)$. Town 1, for example, has to pay 10 rather than 12 million Euro.

We now assume that a third town 3 needs water-distribution services, too. Let us define a new cost function $\hat{c}: 2^{\{1,2,3\}} \to \mathbb{R}$ that obeys $c = \hat{c}|_{\{1,2\}}$ by

$$\begin{aligned} \hat{c}(\{1\}) &= 12, \hat{c}(\{2\}) = 8, \hat{c}(\{3\}) = 4, \\ \hat{c}(\{1,2\}) &= 16, \hat{c}(\{1,3\}) = 12, \hat{c}(\{2,3\}) = 8 \text{ and} \\ \hat{c}(\{1,2,3\}) &= 18. \end{aligned}$$

The Shapley value of that game is $Sh(\{1,2,3\},\hat{c}) = (10,6,2)$. The coalition function for $\hat{c}$ is chosen such that players 1 and 2 obtain the same Shapley values as in the game $(\{1,2\},c)$ above.

Towns 1 and 2 manage to obtain a fee of 3 million Euro from town 3 for allowing town 3 to participate. Thus, we have an XP game $(\{1,2,3\},\hat{c},X,\pi)$ with $X := \{3\}$ and $\pi_3 = 3$. The XP Shapley value is

$$Sh^{X,\pi}(\{1,2,3\},\hat{c}) = \left(10+\frac{1}{2}(2-3), 6+\frac{1}{2}(2-3), 3\right) = \left(\frac{19}{2},\frac{11}{2},3\right)$$

Town 3 is happy to enter into this agreement with towns 1 and 2 because of its stand-alone costs of 4. However, town 3 would be even better off under the Shapley value $Sh(\{1,2,3\},\hat{c})$ while towns 1 and 2 benefit from the agreement (compare $Sh^{X,\pi}(\{1,2,3\},\hat{c})$ with $Sh(\{1,2,3\},\hat{c})$ or $Sh(\{1,2\},c)$).

This simple example does not discuss why towns 1 and 2 succeed in making town 3 the exogenous player. One way to discuss that question would be to embed the cooperative game into a non-cooperative coalition-formation game.

Why is the XP Shapley value a suitable tool to allocate costs? In the above example of a water-distribution system, axiom X is a very obvious requirement and the need to allocate the joint cost $c(N)$ (axiom E) is an immediate consequence of balanced budgets. Without additional information that demands differentiation between the endogenous players, axiom S also imposes itself. A town that does not need any (water-distribution) services should not have to contribute if no exogenous players are present (axiom N-$\emptyset$). Axiom A is easy to justify with respect to the exogenous-payoff vectors. With respect to the coalition functions, we follow Young (1994, pp. 1213) and argue that breaking up the overall costs into different cost categories (such as operating cost and capital cost) should not affect the allocation. Axiom C claims that if the exogenous user (town 3 in the example) pays a fee that is just his (Shapley!) payoff in the TU game, the endogenous users (towns 1 and 2) should also be attributed their Shapley fees. To us, this seems a very sensible requirement.

For cost allocation within firms, standard cost accounting text books implicitly argue for axiom A by giving several reasons why joint costs should be allocated (see, for example, Bhimani, Horngren, Datar & Foster 2008, p. 172). Of course, the joint cost to be allocated among the endogenous players (profit centers, for examples) is $c(N) - \pi_X$, only. In terms of the cost-accounting literature, we could consider the service rendered to exogenous users as a by-product the sale of which can be accounted for by a reduction of cost – this is one out of four ways to account for by-products mentioned by Bhimani et al. (2008, p. 187, exhibit 6.16). However, standard text books do not mention the Shapley method of cost allocation.

4. The weighted XP Shapley value

4.1. Axiomatization

The XP Shapley value can be extended to incorporate weights for the players from D. A weighted XP value ρ assigns a payoff vector to every weighted XP game, $\rho(N, v, X, \pi, w) \in \mathbb{R}^N$. The weighted XP Shapley value (which is not an XP version of the weighted Shapley value due to Kalai & Samet (1987)) is given by

$$Sh_i^{X,\pi,w}(N,v) = \begin{cases} \pi_i, & i \in X \\ Sh_i(N,v) + \frac{w_i}{\sum_{d\in D} w_d}\left(Sh_X(N,v) - \pi_X\right), & i \in D \end{cases}$$

It can be axiomatized on the basis of (obvious variations of) the axioms X, E, N-$\emptyset$, A, and C from the first axiom set.

Xw (exogenous payments): For all $i \in X$, we have $\rho_i(N, v, X, \pi, w) = \pi_i$.

Ew (efficiency): We have $\rho_N(N, v, X, \pi, w) = v(N)$.

Nw-$\emptyset$ (null player for $X = \emptyset$): If $i \in N$ is a null player in (N, v), then $\rho_i(N, v, \emptyset, \pi, w) = 0$.

Aw (additivity): For any coalition functions $v', v'' \in \mathbb{V}_N$, any payments π', $\pi'' \in \mathbb{R}^N$ and any player i from N, we obtain

$$\rho_i(N, v' + v'', X, \pi' + \pi'', w) = \rho_i(N, v', X, \pi', w) + \rho_i(N, v'', X, \pi'', w).$$

Cw (consistency): For any player $i \in D$,

$$\rho_i\left(N, v, X, \left(\rho_x(N, v, \emptyset, \pi, w)\right)_{x\in X}, w\right) = \rho_i(N, v, \emptyset, \pi, w).$$

The symmetry axiom has to take the weights into account:

Sw (symmetry): For all symmetric players $i, j \in D$ obeying $w_i = w_j$,

$$\rho_i(N, v, X, \pi, w) = \rho_j(N, v, X, \pi, w).$$

Additionally, we need two more axioms that refer to the weight structure. Axiom IR states that the exogenous payments and the weights are not relevant for a player i if there are no exogenous players:

IR (irrelevance): For all $i \in D$ and all $\pi, \pi' \in \mathbb{R}^N$, $w, w' \in \mathbb{R}^N$, we have

$$\rho_i(N, v, \emptyset, \pi, w) = \rho_i(N, v, \emptyset, \pi', w').$$

Axiom W ensures that the ratio of weights is equal to the ratio of payoffs in a zero game. It is similar to the "weighting of treatments" axiom by Haeringer (1999).

W (weighing): For all players $i, j \in D$,

$$w_i \rho_j (N, 0, X, \pi, w) = w_j \rho_i (N, 0, X, \pi, w) .$$

Theorem 4.1. *The weighted XP Shapley value is characterized by the axioms Xw, Ew, Nw-∅, Aw, Cw, Sw, IR, and W.*

Proof. $Sh^{X,\pi,w}$ fulfills all the above axioms. Turning to uniqueness, axiom Xw ensures $\rho_i (N, v, X, \pi, w) = \pi_i$ for all $i \in X$. Note that IR and S imply weight-independent symmetry in case of $X = \emptyset$. Assume two symmetric players $i, j \in D$ that do not (necessarily) obey $w_i = w_j$. We then have

$$\begin{aligned} \rho_i (N, v, \emptyset, \pi, w) &= \rho_i (N, v, \emptyset, \pi, (1, ..., 1)) \text{ (axiom IR)} \\ &= \rho_j (N, v, \emptyset, \pi, (1, ..., 1)) \text{ (axiom Sw)} \\ &= \rho_j (N, v, \emptyset, \pi, w) \text{ (axiom IR)} \end{aligned}$$

We now closely follow the proof of lemma 3.2 to show that axioms Ew, Nw-∅, Aw, Cw, Sw, and IR imply

$$Sh_i (N, v) = \rho_i \left(N, v, X, (Sh_x (N, v))_{x \in X}, w\right) .$$

Proceeding as in the proof of theorem 3.3, we easily find

$$\rho_i (N, v, X, \pi, w) = Sh_i (N, v) + \rho_i \left(N, 0, X, (\pi_x)_{x \in X} - (Sh_x (N, v))_{x \in X}, w\right) .$$

We now apply axiom W:

$$\begin{aligned} & \rho_i (N, v, X, \pi, w) \\ = \; & Sh_i (N, v) + \frac{\sum_{d \in D} w_d}{\sum_{d \in D} w_d} \rho_i \left(N, 0, X, (\pi_x)_{x \in X} - (Sh_x (N, v))_{x \in X}, w\right) \\ = \; & Sh_i (N, v) + \frac{1}{\sum_{d \in D} w_d} \sum_{d \in D} w_d \rho_i \left(N, 0, X, (\pi_x)_{x \in X} - (Sh_x (N, v))_{x \in X}, w\right) \\ = \; & Sh_i (N, v) + \frac{1}{\sum_{d \in D} w_d} \sum_{d \in D} w_i \rho_d \left(N, 0, X, (\pi_x)_{x \in X} - (Sh_x (N, v))_{x \in X}, w\right) \text{ (axiom W)} \\ = \; & Sh_i (N, v) + \frac{w_i}{\sum_{d \in D} w_d} \sum_{d \in D} \rho_d \left(N, 0, X, (\pi_x)_{x \in X} - (Sh_x (N, v))_{x \in X}, w\right) \\ = \; & Sh_i (N, v) + \frac{w_i}{\sum_{d \in D} w_d} (Sh_X (N, v) - \pi_X) \text{ (axiom Ew)} \end{aligned}$$

□

Lemma 4.2. *The axioms Xw, Ew, Nw-$\emptyset$, Aw, Cw, Sw, IR, and W are independent.*

Again, the proof is relegated to the appendix.

4.2. Application: buying a house in the presence of a realtor

4.2.1. The model

We now turn to the application of the weighted XP Shapley value to a very simple housing market. The three agents are a seller of a house S, a buyer B, and a real-estate agent A. (Thus, this subsection contributes to intermediation theory, Spulber (1999) being the standard reference.) We assume that the seller's reservation price s is below the buyer's willingness to pay b. Thus, the gains from trade are positive, $b - s > 0$.

In many real-world markets, the realtor charges a fee π_A which is a fraction f of the house price p for her service, $\pi_A = fp$. We assume that A is the only exogenous player. This payoff to the realtor π_A is payable by the buyer and the seller in proportions $w_S = 0$ and $w_B = 1$, respectively. These are the weights introduced in the previous section and we assume that they are given exogenously.

The seller and the buyer need the realtor to come into contact. Therefore, the TU game (N, v) is given by $N = \{S, B, A\}$ and

$$v(K) = \begin{cases} b - s, & K = N, \\ 0, & \text{otherwise} \end{cases}$$

Summarizing, we are concerned with the weighted XP game

$$(\{S, B, A\}, v, \{A\}, \pi_A, (w_S, w_B))$$

The guiding question for our application concerns the fee fraction f chosen by the realtor in order to maximize π_A. We approach this problem by considering the following three-stage game:

- At the first stage, the realtor decides on f.
- At the second stage, the seller and the buyer decide whether they will indeed do business with each other. If not, the game ends with a payoff of 0 for every player.

- At the third stage, the seller and the buyer engage in a bargaining process, the outcome of which is determined by the weighted XP Shapley value.

In order to defend the weighted XP Shapley value for the model at hand, let us examine the axioms. If the realtor A is not cheated out of the fee he demands, she should should obtain π_A (axiom Xw). Assuming that no third parties like tax authorities are involved, axiom Ew is a natural requirement. A seller or buyer who does not have positive gains from trade with any buyer or seller, respectively, and who does not engage a real-estate agent, will surely expect a payoff of 0 (axiom Nw-$\emptyset$). Similar to the justification of the XP Shapley value, axiom Aw is an obvious requirement with respect to the exogenous-payoff vectors while the justification with respect to the coalition functions is as easy or difficult as for the Shapley value itself (compare the discussion of alternative axiomatizations by Winter 2002). Axiom Cw makes a reasonable claim: If the real-estate agent happens to obtain the payoff dictated by the game $(\{S, B, A\}, v)$ (without exogenous players!), so do the two trading partners. Axiom Sw is certainly a reasonable assumption but does not apply directly because the weights differ in our model. Axiom IR is easy to justify: If the real-estate agent does not obtain an exogenous payoff, it does not matter how much she would obtain and by whom, otherwise. In our model, axiom W boils down to

$$\begin{aligned}
w_S Sh_B^{\{A\},\pi_A,(w_S,w_B)}(N,0) &= w_S \left[Sh_B(N,0) + \frac{w_B}{w_S + w_B} (Sh_A(N,0) - \pi_A) \right] \\
&= w_S(-\pi_A) = 0 = w_B \cdot 0 \\
&= w_B \left[Sh_S(N,0) + \frac{w_S}{w_S + w_B} (Sh_A(N,0) - \pi_A) \right] \\
&= w_B Sh_S^{\{A\},\pi_A,(w_S,w_B)}(N,0).
\end{aligned}$$

Thus, if there were no gains from trade ($b = s$), the buyer alone would have to pick up the realtor's fee.

Our model belongs to the growing number of hybrid noncooperative-cooperative models which, following Brandenburger & Stuart (2007) (who use the core rather than the Shapley value or the weighted XP Shapley value), can also be called biform games. In our example, the first two stages (setting f and deciding on whether to trade) form an extensive game where the payoffs are calculated by way of cooperative means at the third stage. We follow the usual procedure of backward induction.

4.2.2. The third stage: bargaining

We abbreviate $Sh^{\{A\},\pi_A,(0,1)}(N,v)$ by ξ. For the three agents S, B, and A, we obtain the following weighted XP Shapley value

$$\begin{aligned}
\xi &= (\xi_S, \xi_B, \xi_A) \\
&= \left(\frac{b-s}{3} + \frac{0}{0+1}\left(\frac{b-s}{3} - \pi_A\right), \frac{b-s}{3} + \frac{1}{0+1}\left(\frac{b-s}{3} - \pi_A\right), \pi_A\right) \\
&= \left(\frac{b-s}{3}, \frac{b-s}{3} + 1\cdot\left(\frac{b-s}{3} - \pi_A\right), \pi_A\right) \\
&= \left(\frac{b-s}{3}, \frac{2}{3}(b-s) - \pi_A, \pi_A\right)
\end{aligned}$$

So far, the realtor's fee π_A is exogenous so that we could apply our formula. However, a specific house price p^* (an "equilibrium" house price, if you like) is implicit in the above payments. Indeed, the seller's rent is $p - s = \xi_S$ so that we obtain

$$\begin{aligned}
p^* &= \xi_S^*(f) + s = \frac{b-s}{3} + s \\
&= \frac{2}{3}s + \frac{1}{3}b
\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}
\xi_B^*(f) &= b - p^* - fp^*, \\
\pi_A^*(f) &= fp^*.
\end{aligned}$$

Thus, the payments to the realtor are partly endogenized at fp^* (f will be endogenized at the first stage).

4.2.3. The second stage: do they have a deal

The seller is willing to sell his house if $\xi_S \geq 0$ holds which is true by $b - s > 0$. The buyer will buy this house if $b - p^* - fp^* \geq 0$ or

$$f \leq \frac{b - p^*}{p^*}$$

or (use eq. (4.1))

$$f \leq \frac{b - \left(\frac{b-s}{3} + s\right)}{\frac{b-s}{3} + s} = \frac{2(b-s)}{2s+b}$$

hold. For any $f \geq 0$, the realtor is happy to help in the deal. Thus, the deal can be struck for any fee percentage f obeying

$$0 \leq f \leq \frac{2(b-s)}{2s+b}.$$

4.2.4. The first stage: setting f

Obviously, the real-estate agent maximizes her profit by letting

$$f^* = \frac{2(b-s)}{2s+b}$$

As expected, we find $\frac{df^*}{db} > 0$ and $\frac{df^*}{ds} < 0$.

Finally, we obtain $f^* p^* = \frac{2(b-s)}{2s+b}\left(\frac{2}{3}s + \frac{1}{3}b\right) = \frac{2}{3}b - \frac{2}{3}s$ and the payoffs are

$$\left(\frac{b-s}{3}, 0, \frac{2}{3}b - \frac{2}{3}s\right)$$

in the usual order.

4.2.5. Comment

One may question the usefulness of the weighted XP Shapley value in the example presented above. After all, while π is exogenous at the third stage, i.e., when applying this value, it becomes endogenous after all. The Shapley value of the symmetric (!) TU game $(\{S, B, A\}, v)$ is $\left(\frac{b-s}{3}, \frac{b-s}{3}, \frac{b-s}{3}\right)$. In contrast, the weighted XP Shapley value

- clearly reflects the different weights as they are often given in real-estate markets (the German example is described by Hagemann 2006),
- allows to dissect the realtor's payoff as the product of the fee percentage f (chosen by the realtor himself) and the house price p (as the bargaining outcome between seller and buyer that may in more complicated models depend on f).

5. Issues for future research

The basic idea underlying this paper can be extended in two different manners. First of all, XP values disobeying consistency, such as value φ^C (subsection 3.2), can be axiomatized and applied.

Second, using the Shapley value with exogenous payoffs as a general model, we obtain a class of values where the Shapley value is replaced by the egalitarian value (subsection 3.2), the (normalized) Banzhaf value (see Dubey & Shapley (1979) and van den Brink & van der Laan (1998)), or others. Thus, let φ be a value on $\mathbb{V}_N$ and $\varphi^{X,\pi}$ the XP value given by

$$\varphi_i^{X,\pi}(N,v) = \begin{cases} \pi_i, & i \in X \\ \varphi_i(N,v) + \frac{1}{|D|}\left(\varphi_X(N,v) - \pi_X\right), & i \in D \end{cases}$$

$\varphi^{X,\pi}$ fulfills the axioms X, C and SH (because of $|D| + |X| = |N|$) and, if φ obeys efficiency, symmetry, null player, or additivity, $\varphi^{X,\pi}$ obeys the axioms E, S, N-$\emptyset$, or A, respectively. If additivity and symmetry hold, $\varphi^{X,\pi}$ fulfills axiom BF.

6. Appendix

Proof of lemma 3.4:

In order to show that the axioms X, E, S, N-$\emptyset$, A, and C are independent, we leave out one axiom at a time and show that the remaining axioms are fulfilled by a value different from the XP Shapley value.

- Disregarding X and π, the (Shapley) value φ^X defined by $\varphi_i^X(N, v, X, \pi) = Sh_i(N, v)$ violates axiom X but clearly obeys the remaining axioms E, S, N-$\emptyset$, A, and C.
- Axiom E is necessary because the XP value φ^E defined by

$$\varphi_i^E(N, v, X, \pi) = \begin{cases} \pi_i, & i \in X \\ \frac{1}{2} Sh_i(N, v) + \frac{1}{|D|}\left(\frac{Sh_X(N,v)}{2} - \pi_X\right), & i \in D \end{cases}$$

is not efficient but the axioms X, S, N-$\emptyset$, A, and C are obviously fulfilled.

- The necessity of axiom S can be seen from the XP value φ^S defined by

$$\varphi_i^S(N, v, X, \pi) = \begin{cases} \pi_i, & i \in X \\ Sh_i^{X,\pi}(N, v), & i \in D, 1 \notin D \\ Sh_i(N, v) + (Sh_X(N, v) - \pi_X), & i \in D, 1 \in D, i = 1 \\ Sh_i(N, v), & i \in D, 1 \in D, i \neq 1 \end{cases}$$

While axiom S does not hold (player 1 alone pays off the exogenous players), it is easy to see that the axioms X, E, N-$\emptyset$, and C are fulfilled. For axiom A, consider coalition functions $v', v'' \in \mathbb{V}_N$, exogenous payoff π', $\pi'' \in \mathbb{R}^N$ and any player i from N. The axiom obviously holds for exogenous players and for players that obtain the Shapley value. For player $i = 1 \in D$, additivity is confirmed by

$$\begin{aligned} & \varphi_i^S(N, v' + v'', X, \pi' + \pi'') \\ = \; & Sh_i(N, v' + v'') + (Sh_X(N, v' + v'') - (\pi' + \pi'')_X) \\ = \; & Sh_i(N, v') + Sh_i(N, v'') + (Sh_X(N, v') + Sh_X(N, v'') - \pi'_X - \pi''_X) \\ = \; & \varphi_i^S(N, v', X, \pi') + \varphi_i^S(N, v'', X, \pi''). \end{aligned}$$

- If axiom N-$\emptyset$ were missing, we could put forward the XP value $\varphi^{\text{N-}\emptyset}$ given by $\varphi_i^{\text{N-}\emptyset}(N, v, X, \pi) = Eg_i^{X,\pi}(N, v)$ (see subsection 3.2).

- With regard to axiom A, we use an idea due to Hiller (2011, pp. 54) and consider the XP value φ^A defined by

$$\varphi_i^A(N,v,X,\pi) = \begin{cases} \pi_i, & i \in X \\ Sh_i^{X,\pi}(N,v), & i \in D, 1 \notin D \vee 2 \notin D \vee X = \emptyset \vee \pi = \left(Sh_x^{\emptyset,\pi}(N,v)\right)_{x\in X} \\ & \vee Sh_1^{X,\pi}(N,v) \neq 3 \vee Sh_2^{X,\pi}(N,v) \neq 5 \\ & \vee Sh_j^{X,\pi}(N,v) \in \{3,5\} \text{ for some } j \in D\backslash\{1,2\} \\ 5, & i = 1 \in D, 2 \in D, X \neq \emptyset, \pi \neq \left(Sh_x^{\emptyset,\pi}(N,v)\right)_{x\in X}, \\ & Sh_1^{X,\pi}(N,v) = 3, Sh_2^{X,\pi}(N,v) = 5, \\ & Sh_j^{X,\pi}(N,v) \notin \{3,5\} \text{ for all } j \in D\backslash\{1,2\} \\ 3, & i = 2 \in D, 1 \in D, X \neq \emptyset, \pi \neq \left(Sh_x^{\emptyset,\pi}(N,v)\right)_{x\in X}, \\ & Sh_1^{X,\pi}(N,v) = 3, Sh_2^{X,\pi}(N,v) = 5, \\ & Sh_j^{X,\pi}(N,v) \notin \{3,5\} \text{ for all } j \in D\backslash\{1,2\} \end{cases}$$

This value does not fulfill axiom A. Consider the games with $N = \{1,2,3\}$, $X = \{3\}$, $\pi_3 = 2$ and the inessential coalition functions v and z on N given by $v(\{1\}) = 4$, $v(\{2\}) = 6$, $v(\{3\}) = 0$ and $z(\{1\}) = z(\{2\}) = 1, z(\{3\}) = 0$ for all $i \in N$. We then have

$$\begin{aligned} Sh_1^{\{3\},2}(N,v+z) &= 5 + \frac{1}{2}(0-2) = 4 \neq 3, \\ Sh_2^{\{3\},2}(N,v+z) &= 7 + \frac{1}{2}(0-2) = 6 \neq 5, \\ Sh_1^{\{3\},2}(N,v) &= 4 + \frac{1}{2}(0-2) = 3, \\ Sh_2^{\{3\},2}(N,v) &= 6 + \frac{1}{2}(0-2) = 5 \text{ and} \\ Sh_1^{\{3\},0}(N,z) &= 1 + \frac{1}{2}(0-0) = 1 \end{aligned}$$

and hence

$$\varphi_1^A(N,v+z,X,\pi) = 4 \neq 5+1 = \varphi_1^A(N,v,X,\pi) + \varphi_1^A(N,z,X,0)$$

The other axioms are fulfilled. In particular, $X = \emptyset$ prompts the XP Shapley payoffs (second line) so that axiom N-$\emptyset$ still holds. Also, the payoff exchange happens only for players 1 and 2 that are not symmetric to other players in D (axiom S is fulfilled). Axiom C also holds because of $X = \emptyset \vee \pi = \left(Sh_x^{\emptyset,\pi}(N,v)\right)_{x\in X}$ in the second line.

- The independence of axiom C has already be shown in subsection 3.2.

Proof of lemma 3.5:
Note that axioms E, N-$\emptyset$, and BF imply $\varphi_i(N, v, \emptyset, \pi) = Sh_i(N, v)$. For $z \in \mathbb{V}_N$ defined by $z := \sum_{x \in X} (\pi_x - Sh_x(v + \pi_X)) u_{\{x\}}$, the player $\hat{x} \in X$ obtains

$$\begin{aligned}
\varphi_{\hat{x}}(N, v + \pi_X + z, \emptyset, \pi) &= Sh_{\hat{x}}(N, v + \pi_X + z) \text{ (axioms E, N-}\emptyset\text{, BF)} \\
&= Sh_{\hat{x}}(N, v + \pi_X) + \sum_{x \in X} (\pi_x - Sh_x(v + \pi_X)) Sh_{\hat{x}}\left(N, u_{\{x\}}\right) \\
&= Sh_{\hat{x}}(N, v + \pi_X) + (\pi_{\hat{x}} - Sh_{\hat{x}}(v + \pi_X)) \\
&= \pi_{\hat{x}}.
\end{aligned}$$

Note that players i and j from $N \backslash X$ are symmetric with respect to (N, z). Hence, we find

$$\begin{aligned}
& \varphi_i(N, v + \pi_X, X, \pi) - \varphi_j(N, v + \pi_X, X, \pi) \\
= \; & \varphi_i(N, v + \pi_X + z, X, \pi) - \varphi_j(N, v + \pi_X + z, X, \pi) \text{ (axiom BF)} \\
= \; & \varphi_i\left(N, v + \pi_X + z, X, (\varphi_x(N, v + \pi_X + z, \emptyset, \pi))_{x \in X}\right) \\
& -\varphi_j\left(N, v + \pi_X + z, X, (\varphi_x(N, v + \pi_X + z, \emptyset, \pi))_{x \in X}\right) \text{ (see above)} \\
= \; & \varphi_i(N, v + \pi_X + z, \emptyset, \pi) \\
& -\varphi_j(N, v + \pi_X + z, \emptyset, \pi) \text{ (axiom C)} \\
= \; & Sh_i(N, v + \pi_X + z) - Sh_j(N, v + \pi_X + z) \\
= \; & Sh_i(N, v) - Sh_j(N, v) \text{ (additivity and symmetry).}
\end{aligned}$$

Thus, the difference $\varphi_i(N, v + \pi_X, X, \pi) - \varphi_j(N, v + \pi_X, X, \pi)$ does not depend on π so that we have

$$\varphi_i(N, v + \pi_X, X, \pi) - \varphi_i(N, v + \pi'_X, X, \pi') = \varphi_j(N, v + \pi_X, X, \pi) - \varphi_j(N, v + \pi'_X, X, \pi')$$

for all π and π' from $\mathbb{R}^N$. We now sum over all $j \in D$ and get

$$\begin{aligned}
& |D| \varphi_i(N, v + \pi_X, X, \pi) - |D| \varphi_i(N, v + \pi'_X, X, \pi') \\
= \; & \sum_{j \in D} \varphi_j(N, v + \pi_X, X, \pi) - \sum_{j \in D} \varphi_j(N, v + \pi'_X, X, \pi') \\
= \; & v(N) + \pi_X - \pi_X - (v(N) + \pi'_X - \pi'_X) \text{ (axiom E, X)} \\
= \; & 0
\end{aligned}$$

and hence axiom SH.

Proof of lemma 3.7:

Again, we present values that fulfill all the axioms X, E, BF, N-$\emptyset$, and C but one.

- Proceeding as in the proof of lemma 3.4, the (Shapley) value ψ^X defined by $\psi_i^X(N,v,X,\pi) = Sh_i(N,v)$ violates axiom X but clearly obeys the remaining axioms E, BF, N-$\emptyset$, and C.

- Axiom E is necessary because the XP value ψ^E (which equals φ^E) defined by

$$\psi_i^E(N,v,X,\pi) = \begin{cases} \pi_i, & i \in X \\ \frac{1}{2}Sh_i(N,v) + \frac{1}{|D|}\left(\frac{Sh_X(N,v)}{2} - \pi_X\right), & i \in D \end{cases}$$

  does not fulfill efficiency but obeys the other axioms.

- For axiom BF, we define the value ψ^{BF} by

$$\begin{aligned} &\psi_i^{BF}(N,v,X,\pi) \\ = &\begin{cases} \pi_i, & i \in X \\ Sh_i^{X,\pi}(N,v), & i \in D, \exists k,j \in N, g \in \mathbb{R}, z \in \mathbb{V}_N : \\ & v = z + g \text{ and } k \text{ and } j \text{ are symmetric for } z \\ & \vee\ \exists k \in N, g \in \mathbb{R}, z \in \mathbb{V}_N : \\ & v = z + g \text{ and } k \text{ is a null player in } z \\ Eg_i^{X,\pi}(N,v), & i \in D, \forall k,j \in N, g \in \mathbb{R}, z \in \mathbb{V}_N : \\ & v \neq z + g \text{ or } k \text{ and } j \text{ are not symmetric for } z, \\ & \forall k \in N, g \in \mathbb{R}, z \in \mathbb{V}_N : \\ & v \neq z + g \text{ or } k \text{ is not a null player in } z \end{cases} \end{aligned}$$

  In order to show that this value does not fulfill axiom BF consider the player set $N = D = \{1,2,3\}$ and the coalition functions $v := 2u_{\{2\}} + 2u_{\{3\}}$ and $z := u_{\{1\}} + u_{\{2\}}$. Players 1 and 2 are symmetric with respect to z, players 2 and 3 are symmetric with respect to v and there are no symmetric players with respect to $v + z - g = u_{\{1\}} + 3u_{\{2\}} + 2u_{\{3\}} - g$ for any $g \in \mathbb{R}$. Also, no player is a null player for $v+z-g$ for any $g\in\mathbb{R}$. We obtain

$$\begin{aligned}
& \psi_1^{BF}(N,v+z,X,\pi)-\psi_1^{BF}(N,v,X,\pi) \\
= \; & Eg_1(v+z)-Sh_1(v) \\
= \; & \frac{1+3+2}{3}-0 \\
\neq \; & \frac{1+3+2}{3}-2 \\
= \; & \psi_2^{BF}(N,v+z,X,\pi)-\psi_2^{BF}(N,v,X,\pi)
\end{aligned}$$

ψ^{BF} obviously fulfills the axioms X, E, and N-$\emptyset$. Axiom C holds true because we have $\psi_x^{BF}(N,v,\emptyset,\pi)=Sh_x(N,v)$ for all $x\in X$ or $\psi_x^{BF}(N,v,\emptyset,\pi)=Eg_x(N,v)$ for all $x\in X$.

- For axiom N-$\emptyset$, consider the egalitarian value with exogenous payoffs introduced in subsection 3.2.
- Turning to axiom C, consult subsection 3.2 once more.

Proof of lemma 4.2:
The axioms Xw, Ew, Sw, Nw-$\emptyset$, Aw, Cw, IR, and W are independent.

- The weighted XP value χ^{Xw} defined by $\chi_i^{Xw}(N,v,X,\pi,w)=Sh_i(N,v)$ violates axiom Xw while obeying the other axioms Ew, Sw, Nw-$\emptyset$, Aw, Cw, and IR. Axiom W holds because the Shapley payoffs for zero games are 0.
- Axiom Ew cannot be discarded because of the weighted XP value χ^{Ew} given by

$$\chi_i^{Ew}(N,v,X,\pi,w)=\begin{cases}\pi_i, & i\in X\\ \frac{1}{2}Sh_i(N,v)+\frac{w_i}{\sum_{d\in D}w_d}\left(\frac{Sh_X(N,v)}{2}-\pi_X\right), & i\in D\end{cases}$$

that does not obey axiom Ew but fulfills the axioms Xw, Sw, Nw-$\emptyset$, Aw, Cw, and IR. Axiom W also holds:

$$\begin{aligned}
w_i\chi_j^{Ew}(N,0,X,\pi,w) &= w_i\left(\frac{1}{2}\cdot 0+\frac{w_j}{\sum_{d\in D}w_d}\left(\frac{0}{2}-\pi_X\right)\right)\\
&= w_j\left(\frac{1}{2}\cdot 0+\frac{w_i}{\sum_{d\in D}w_d}\left(\frac{0}{2}-\pi_X\right)\right)\\
&= w_j\chi_i^{Ew}(N,0,X,\pi,w).
\end{aligned}$$

- For axiom Sw, we remind the reader of the Harsanyi dividends given by

$$\begin{aligned} h_v &: 2^N \to \mathbb{R}, \\ & T \mapsto h_v(T) = \sum_{K \in 2^T \backslash \{\emptyset\}} (-1)^{|T|-|K|} v(K). \end{aligned}$$

  and of

$$v(S) = \sum_{T \subseteq S} h_v(T) = \sum_{T \in 2^N \backslash \{\emptyset\}} h_v(T) u_T(S)$$

  for every coalition coalition function v and every coalition $S \subseteq N$. We define the value A on $\mathbb{V}_N$ by

$$A_i(N, u_T) = \begin{cases} Sh_i(N, u_T), & T \neq \{1,2\} \\ \frac{3}{4}, & T = \{1,2\}, i = 1 \\ \frac{1}{4}, & T = \{1,2\}, i = 2 \\ Sh_i(N, u_T) & T = \{1,2\}, i \notin \{1,2\} \end{cases}$$

  and

$$A_i(N, v) = \sum_{T \in 2^N \backslash \{\emptyset\}} h_v(T) A_i(N, u_T).$$

  This value is efficient, obeys the null-player axiom and additivity, but not symmetry.

  Consider, now, the weighted XP value χ^{Sw} defined by

$$\chi_i^{Sw}(N, v, X, \pi, w) = \begin{cases} \pi_i, & i \in X \\ A_i(N, v) + \frac{w_i}{\sum_{d \in D} w_d} (A_X(N, v) - \pi_X), & i \in D \end{cases}$$

  It is not difficult to see that axiom Sw is violated (use $v := u_{\{1,2\}}$) while axioms Xw, Ew, Nw-$\emptyset$, Aw, Cw, IR, and W obviously hold.

- That Nw-$\emptyset$ is necessary, can be seen from the weighted XP value $\chi^{\text{Nw-}\emptyset}$ given by

$$\chi_i^{\text{Nw-}\emptyset}(N, v, X, \pi, w) = \begin{cases} \pi_i, & i \in X \\ Eg_i(N, v) + \frac{w_i}{\sum_{d \in D} w_d} (Eg_X(N, v) - \pi_X), & i \in D \end{cases}$$

- With regard to axiom Aw, we use the weighted XP value χ^{Aw} defined by

$$\begin{aligned}
&\chi_i^{Aw}(N,v,X,\pi,w)\\
&= \begin{cases}
\pi_i, & i \in X \\
Sh_i^{X,\pi,w}(N,v), & i \in D, 1 \notin D \vee 2 \notin D \vee X = \emptyset \vee \pi = \left(Sh_x^{\emptyset,\pi,w}(N,v)\right)_{x\in X} \\
& \vee Sh_j^{X,\pi,w}(N,v) \in \{3,5\} \text{ for some } j \in D\backslash\{1,2\} \\
& \vee Sh_1^{X,\pi,w}(N,v) \neq 3 \vee Sh_2^{X,\pi,w}(N,v) \neq 5 \vee v = 0 \\
5, & i = 1 \in D, 2 \in D, X \neq \emptyset, \pi \neq \left(Sh_x^{\emptyset,\pi,w}(N,v)\right)_{x\in X}, \\
& Sh_1^{X,\pi,w}(N,v) = 3, Sh_2^{X,\pi,w}(N,v) = 5, \\
& Sh_j^{X,\pi,w}(N,v) \notin \{3,5\} \text{ for all } j \in D\backslash\{1,2\}, v \neq 0 \\
3, & i = 2 \in D, 1 \in D, X \neq \emptyset, \pi \neq \left(Sh_x^{\emptyset,\pi,w}(N,v)\right)_{x\in X}, \\
& Sh_1^{X,\pi,w}(N,v) = 3, Sh_2^{X,\pi,w}(N,v) = 5, \\
& Sh_j^{X,\pi,w}(N,v) \notin \{3,5\} \text{ for all } j \in D\backslash\{1,2\}, v \neq 0
\end{cases}
\end{aligned}$$

  This value does not fulfill axiom Aw. Assume $w = (1,1) \in \mathbb{R}^{\{1,2\}}$ and proceed as in the proof of lemma 3.4 concerning the XP value φ^A. χ^{Aw} obeys axioms Xw, Ew, Sw, and Nw-$\emptyset$. Axiom Cw holds because of $X = \emptyset \vee \pi = \left(Sh_x^{\emptyset,\pi,w}(N,v)\right)_{x\in X}$ in the second line, axiom IR is fulfilled by $X = \emptyset$ in the second line and, finally, axiom W holds because of $v = 0$ in the second line.

- For axiom Cw, we consider a variant of the TU game $\left(N\backslash X, p^{v,X,\pi}\right)$ introduced in subsection 3.2. In particular, we propose the TU game $\left(N\backslash X, p^{v,X,\pi,w}\right)$ given by

$$p^{v,X,\pi,w}(S) = \begin{cases} v(S\cup X) - \frac{\sum_{d\in S} w_d}{\sum_{d\in D} w_d}\pi_X, & S \neq \emptyset \\ 0, & S = \emptyset \end{cases}$$

  and on that basis the weighted XP value χ^{Cw} defined by

$$\chi_i^{Cw}(N,v,X,\pi,w) = \begin{cases} \pi_i, & i \in X \\ Sh_i\left(D, p^{v,X,\pi,w}\right), & i \in D \end{cases}$$

  χ_i^{Cw} violates axiom Cw. Consider the TU game $\left(\{1,2,3\}, u_{\{1,2\}}\right)$, $X = \{1\}$, $\pi_1 = \frac{1}{2}$, and $w = (1,1)$ that lead to $Sh_1\left(N, u_{\{1,2\}}\right) = \frac{1}{2} = Sh_1\left(N, p^{u_{\{1,2\}},\emptyset,\pi,w}\right) =$

$\chi_1^{Cw}\left(N, u_{\{1,2\}}, \emptyset, \pi, w\right)$ and hence to

$$\begin{aligned}
& \chi_2^{Cw}\left(N, u_{\{1,2\}}, \{1\}, \chi_1^{Cw}\left(N, u_{\{1,2\}}, \emptyset, \frac{1}{2}, (1,1)\right), (1,1)\right) \\
= \; & Sh_2\left(D, p^{u_{\{1,2\}}, \{1\}, \frac{1}{2}, (1,1)}\right) \\
= \; & \frac{1}{2}\left(u_{\{1,2\}}\left(\{1,2\}\right) - \frac{1}{1+1}\cdot\frac{1}{2} - 0\right) \\
& + \frac{1}{2}\left(u_{\{1,2\}}\left(\{1,2,3\}\right) - \frac{1+1}{1+1}\cdot\frac{1}{2} - \left[u_{\{1,2\}}\left(\{1,3\}\right) - \frac{1}{1+1}\cdot\frac{1}{2}\right]\right) \\
= \; & \frac{1}{2}\left(1 - \frac{1}{4}\right) + \frac{1}{2}\left(1 - \frac{1}{2} - \left[0 - \frac{1}{4}\right]\right) = \frac{3}{4} \\
\neq \; & \frac{1}{2} = \chi_2^{Cw}\left(N, u_{\{1,2\}}, \emptyset, \frac{1}{2}, (1,1)\right).
\end{aligned}$$

χ^{Cw} obeys the other axioms. In particular, $p^{0,X,\pi,w}$ is an inessential coalition function by $p^{0,X,\pi,w}(S) = -\frac{\sum_{d\in S} w_d}{\sum_{d\in D} w_d}\pi_X$, $S \subseteq D$, so that axiom W follows from

$$w_i \chi_j^{Cw}(N, 0, X, \pi, w) = w_i Sh_j\left(D, p^{0,X,\pi,w}\right) = w_i\left(\frac{-w_j}{\sum_{d\in D} w_d}\pi_X\right)$$

- We now turn to axiom IR. We define the weighted XP value χ^{IR} in three steps:

  1. Let T be any nonempty subset of N and u_T a unanimity game. Then, for any $\alpha \in \mathbb{R}$, the weighted XP value χ^{IR} for $X = \emptyset$ is defined by

  $$\begin{aligned}
  \chi_i^{IR}(N, \alpha u_T, \emptyset, \pi, w) \; &= \; \left\{ \begin{array}{ll} \frac{\alpha}{|M_T|}, & i \in T, w_i = W_T \\ 0, & i \notin T \text{ or } w_i < W_T \end{array} \right. , \\
  \text{where } W_T \; &: = \max_{i\in T} w_i \text{ and } M_T := \{i \in T : w_i = W_T\}
  \end{aligned}$$

  2. As any game v can be written as

  $$v = \sum_{\substack{T\neq\emptyset, \\ T\subseteq N}} \lambda_T u_T$$

for suitably chosen λ_T, we obtain

$$\chi_i^{IR}(N, v, \emptyset, \pi, w) = \sum_{\substack{T \neq \emptyset, \\ T \subseteq N}} \chi_i^{IR}(N, \lambda_T u_T, \emptyset, \pi, w)$$

3. Finally, we define

$$\begin{aligned} &\chi_i^{IR}(N, v, X, \pi, w) \\ =& \begin{cases} \pi_i, & i \in X \\ \chi_i^{IR}(N, v, \emptyset, \pi, w) + \frac{w_i}{\sum_{d \in D} w_d} \left(\chi_X^{IR}(N, v, \emptyset, \pi, w) - \pi_X \right), & i \in D \end{cases} \end{aligned}$$

It is not difficult to show that the axioms mentioned in the lemma are indeed fulfilled. Also, in general, we have $\chi_i^{IR}(N, \alpha u_T, \emptyset, \pi, w) \neq Sh_i^{X,\pi,w}(N, \alpha u_T)$.

- Finally, the weighted XP value χ^W defined by $\chi^W(N, v, X, \pi, w) = Sh^{X,\pi}(N, v)$ violates axiom W while fulfilling the other axioms.

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