A cooperative game theory of town walls

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Abstract

This paper is a contribution to economic geography that leans on cooperative game theory. In particular, we envision towns (that are protected by walls) versus country side where such protection is missing. An extension also provides a basic theory of when trading routes will be protected.

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JEL classification: C71, N70

1. Introduction

This paper's aim is a theory of town walls. The archetypal function of a town wall is urban defence. Indeed, the word “town” is cognate with German “Zaun”, meaning fence, and Irish “dun” – castle or fortress – which show up in many place names from the Irish port Dun Laoghaire (near Dublin) to the German town of Kempten (going back to Cambo-dunum).

However, urban defence was not the only function of town walls. Creighton & Higham (2005, pp. 21-62) mention status, identity, civic pride, escape from fetid streets, and marking the limits of borough law. Indeed, town walls were not even necessary since the majority of British medieval towns "were established

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and prospered without the need for defences". Also, even if walls existed, "it was not always the case that a completely enclosed perimeter was achieved or even intended" (see Creighton & Higham 2005, pp. 22).

These caveats notwithstanding, we assume that defence is the main purpose of town walls. Protection is provided against outsiders and – so we like to argue – against wrongdoers inside, also. That is, a town wall encompasses an area where economic activity and the fruits of cooperation can be enjoyed without fear of expropriation. That is, a town is a non-Hobbesian state of affaires. In a Hobbesian state of nature (see the following quotation from Hobbes 1998, p. 84),

> there is no place for industry; because the fruit thereof is uncertain: and consequently no culture of the earth; no navigation, nor use of the commodities that may be imported by sea; ... ; and the life of man, solitary, poor, nasty, brutish, and short.

Hobbes' answer to this disagreeable state of affairs is a social contract which establishes an authority as a price for a peaceful living. Returning to our theme, we try to answer two related questions:

- Under what circumstances are town walls built?
- Under what circumstances is a secure trading route between two towns realized?

We assume that town walls or trading routes can be built whenever the aggregate willingness to pay exceeds the cost of doing so. Thus, our paper is quite naive from the point of view of public choice.

This paper belongs to economic geography. The seminal paper in this area is Krugman (1991) who uses a monopolistic-competition model to analyse the distribution of agriculture and manufacture in geographic space.

In contrast, we use cooperative game theory and envision that geographic space consists of countryside and towns. The idea is that cooperation flourishes in towns while expropriation due to insufficient law enforcement is typical for the country side.

Thus, town versus countryside can be seen as a metaphor for a more general concept. For example, in our framework, an orderly country like Switzerland as a whole is seen as protected by a wall. In contrast, lawlessness (in some form or other) may well reign in big cities. From the point of view of cooperative game theory, our paper complements the microeconomic literature on "gated communities and the economic geography of crime" to quote the title of Helsley & Strange's (1999) paper.

We generate several interesting results. While the more or less Hobbesian state of affairs in the country side has an effect on whether or not a wall is build, the direction is a priori unclear. We find that complementarity between town and country production is of central importance.

The paper is organized as follows. Section 2 provides some basic concepts of cooperative game theory. We then present two different sets of models – one set with discrete players (section 3) and a second one with continua of players (section 4). Section 5 concludes the paper.

2. Definitions and notation

2.1. The discrete case: coalition functions and payoff vectors

A transferable-utility game (in coalition function form) is a pair (N, v) (often abbreviated by v) where $N = \{1, 2, ..., n\}$ is a finite set and v a function $2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. $v(K)$ is called coalition K's worth. The set of all games on N is denoted by G_N. For $T \subseteq N$, we define $v|_T : 2^N \to \mathbb{R}$ by $v|_T(K) := v(K \cap T)$. Note the domain of $v|_T$.

We mention some important classes of coaliton functions.

- A game v is called positive if $v(K) \geq 0$ holds for all $K \subseteq N$.
- A game v is called monotonic if $v(K \cup \{i\}) \geq v(K)$ holds for all coalitions K and for all players $i \in N$.
- A game (N, v) is symmetric if there exists a function $f : \mathbb{N} \to \mathbb{R}$ such that $v(K) = f(|K|)$ for all coalitions $K \subseteq N$. In that case, we say that f represents v.
- A game v is convex if for all coalitions K and K' obeying $K \subseteq K'$ we have

  $$v(K \cup \{i\}) - v(K) \leq v(K' \cup \{i\}) - v(K')$$

  for all players $i \notin K'$.

Lemma 2.1. *Let v be a symmetric game represented by f. Convexity of v implies*

$$\frac{f(k)}{k} \leq \frac{f(k+1)}{k+1}$$

for $1 \leq k \leq n-1$.

The proof can be found in the appendix.

Player $i \in N$ is a null player (with respect to some $v \in G_N$) if

$$v(K \cup \{i\}) = v(K) \text{ for all } K \subseteq N.$$

Two players $i, j \in N$ are called symmetric (for v) if for all coalitions K obeying $i \notin K$ and $j \notin K$ we have

$$v(K \cup \{i\}) = v(K \cup \{j\}).$$

Let $i \in N$ be a player from N and let $v \in G_N$ be a coalition function on N. Player i's marginal contribution with respect to a coalition K that includes player i is denoted by $MC_i^K(v)$ and given by

$$MC_i^K(v) := v(K) - v(K \backslash \{i\}).$$

A payoff vector x for N is an element of $\mathbb{R}^n$ or a function $N \to \mathbb{R}$.

2.2. The discrete case: the Shapley algorithm

A value on G_N associates with every game $v \in G_N$ a payoff vector $x \in \mathbb{R}^n$. In this paper, we use the Shapley (1953) value. It can be defined by way of an algorithm (this subsection) or through axioms (next subsection). The idea of the algorithmic definition is to look at all the permutations of players and then attribute to players their average marginal contribution.

Bijective function $\rho : N \to N$ are called rank orders or permutations on N. The set of all permutations on N is denoted by RO_N. The set of all players "up to and including player i under rank order ρ" is denoted by $K_i(\rho)$ and given by

$$\rho(j) = i \text{ and } K_i(\rho) = \{\rho(1), .., \rho(j)\}.$$

Player i's marginal contribution with respect to rank order ρ is denoted by $MC_i^\rho(v)$ and given by

$$MC_i^\rho(v) := MC_i^{K_i(\rho)}(v) = v(K_i(\rho)) - v(K_i(\rho) \setminus \{i\}).$$

The Shapley value Sh is given by

$$Sh_i(v) = \frac{1}{n!} \sum_{\rho \in RO_N} MC_i^{\rho}(v), i \in N.$$

2.3. The discrete case: the Shapley axiomatization

Values φ may or may not obey the following axioms:

E (efficiency): $\sum_{i \in N} \varphi_i(v) = v(N)$.

S (symmetry): For all symmetric players $i, j \in N$, $\varphi_i(v) = \varphi_j(v)$.

N (null player): If $i \in N$ is a null player, then $\varphi_i(v) = 0$.

A (additivity): For any coalition functions $v', v'' \in G_N$, and any player i from N,

$$\varphi_i(v' + v'') = \varphi_i(v') + \varphi_i(v'').$$

L (linearity): For any coalition functions $v', v'' \in G_N$, any player i from N, and any real number $\alpha \in \mathbb{R}$,

$$\varphi_i(\alpha v' + v'') = \alpha\varphi_i(v') + \varphi_i(v'').$$

Theorem 2.2 (Shapley axiomatization). *The Shapley formula is the unique value that fulfills the symmetry axiom, the efficiency axiom, the null-player axiom, and the additivity, or linearity, axiom.*

2.4. The continuous case: vector-measure games

Sometimes continua of agents are a good model (a town with many inhabitants) and also easy to work with. Consider two groups of agents, town dwellers (an interval $T \subseteq \mathbb{R}$) and county dwellers (interval C). The set of all agents is $N := T \cup C$, where T and C are disjunct intervals. (It is obvious how to generalize to the m-group case.)

We begin with the definition of an appropriate vector measure game. Let λ be the Lebesgues-Borel measure on $\mathbb{R}$. By $\mathcal{B}$ we mean the set of Borel sets of $N = T \cup C$. μ_T and μ_C, defined by $\mu_T(S) := \lambda(S \cap T)$ and $\mu_C(S) := \lambda(S \cap C)$, respectively, are measures on $(N, \mathcal{B})$ with $t := \mu_T(T) = \mu_T(N)$ and $c := \mu_C(C)$.

Let $\mu = (\mu_T, \mu_C)$ be the vector of these measures. We define the vector-measure game

$$v := F \circ \mu : \mathcal{B} \to \mathbb{R}$$

where F is often addressed as a production function. Given a coalition $S \in \mathcal{B}$, $\mu(S)$ specifies how to devide S among the two groups and $F : \mathbb{R}^2 \to \mathbb{R}$ yields the worths. We assume $F(0,0) = 0$ and that F is continuously differentiable.

Note the following definitions:

- A vector-measure game $v = F \circ \mu$ is called positive if $F \geq 0$ holds.
- A vector-measure game $v = F \circ \mu$ is monotonic if $\frac{\partial F}{\partial \mu_T} \geq 0$ and $\frac{\partial F}{\partial \mu_C} \geq 0$ hold.
- A vector-measure game $v = F \circ \mu$ is symmetric if there exists a function $f : \mathbb{N} \to \mathbb{R}$ such that $F(\mu_T(K), \mu_C(K)) = f(\lambda(K))$ for all coalitions $K \subseteq N$. In that case, we say that f represents v.

2.5. The continuous case: the Shapley value

We cannot apply the (discrete) Shapley value but need to introduce the continuous Shapley value proposed by Aumann & Shapley (1974, p. 23) (see also Neyman 2002, pp. 2141). For $S \in \mathcal{B}$, it is given by

$$Sh_S(v) = \lambda(S \cap T) \int_0^1 \left. \frac{\partial F}{\partial \mu_T} \right|_{(q\mu_T(N), q\mu_C(L))} dq + \lambda(S \cap C) \int_0^1 \left. \frac{\partial F}{\partial \mu_C} \right|_{(q\mu_T(N), q\mu_C(L))} dq$$

Thus, $Sh_S(v)$ is the aggregate Shapley value for all the agents in S. By definition, the Shapley value of an individual agent is zero.

The analogue of a player's marginal contribution in the discrete Shapley formula is the derivative of the coalition's worth with respect to the measure. This derivative is evaluated at (qt, qc). Thus, the formula looks at coalitions on the diagonal only. Remember that we have a continuum of agents. If we take a subset of agents "by chance", it is likely that the composition in this subset (how many agents of town dwellers, how many country dwellers) does not deviate much from the composition in the overall population (see Aumann & Shapley 1974, pp. 23).

It is possible to address the average payoff $\frac{Sh_T(v)}{\lambda(T)}$ as a town dweller's payoff and similarly for $\frac{Sh_C(v)}{\lambda(C)}$.

With the appropriate translations, the axioms mentioned in the discrete case hold for the continuous case as well. However, we do not present these axioms in detail but refer the reader to the above-mentioned literature.

3. Discrete models

3.1. A town

Imagine a town inhabited by the discrete set $N = T$ of townspeople. By definition, the town does not have any law-enforecement problems and we model the inhabitants' productive capabilities by the coalition function $v : 2^N \to \mathbb{R}$. Then, every inhabitant $i \in N = T$ obtains the payoff $Sh_i(v)$.

This overly simplistic model assumes that law-enforcement comes without any cost.

3.2. The countryside

The countryside is populated by the set $N = C$ of country dwellers. For reasons of constant fighting, disencouragement of investment etc., the ideal coalition function v cannot be realized. Instead, we work with the coalition function $hv : 2^N \to \mathbb{R}$. We assume $0 \leq h \leq 1$ where a low value of h corresponds to Hobbes's (1998) state of nature. Country dweller $i \in C$ obtains the Shapley payoff $Sh_i(hv) = hSh_i(v)$.

3.3. A village without walls

A settlement in the countryside is a village that does not possess, as yet, a wall. Building a wall pays if its costs $Cost\,(\text{wall})$ are sufficiently small. Indeed, the wall transforms a countryside coalition function hv into the town coalition function v so that the wall will be build if and only if

$$\begin{aligned} Cost\,(\text{wall}) &\leq \underbrace{v(N) - hv(N)}_{\text{Additional product}} = (1-h)\,v(N) \\ \Leftrightarrow \quad h &\leq \frac{v(N) - Cost\,(\text{wall})}{v(N)} = 1 - \frac{Cost\,(\text{wall})}{v(N)} \end{aligned}$$

holds. This reasoning presupposes that public-good problems can be solved costlessly.

Note that the wall is build if h is sufficiently small (and the cost of the wall is smaller than the ideal overall product $v(N)$).

3.4. A town within the countryside

We now assume that a town with inhabitants T is surrounded by countryside with inhabitants C. Thus, we have $N = T \cup C$ and $T \cap C = \emptyset$ with cardinalities

n, t, and c and typical individual members i, τ and γ, respectively. We assume (T, C)-superadditivity, i.e., $v(K) \geq v(K \cap T) + v(K \cap C)$ for all $K \subseteq N$.

We now need a coalition function v_{TC} that takes cooperation between town and country into account while still assuming the problems associated with lawlessness in the countryside. We suggest to define this coalition function by

$$\begin{aligned} v_{TC}(K) &= v|_T(K \cap T) + h\left[v(K) - v|_T(K \cap T)\right] \\ &= v(K \cap T) + h\left[v(K) - v(K \cap T)\right] \\ &= hv(K) + (1-h)\, v|_T(K \cap T) \end{aligned}$$

By (T, C)-superadditivity, we have

$$v_{TC}(K) \geq v(K \cap T) + hv(K \cap C)$$

so that cooperation between town and countryside pays.

Lemma 3.1. *For a symmetric game v that is represented by f, we obtain*

$$Sh_i(v_{TC}) = \begin{cases} h\frac{f(n)}{n} + (1-h)\frac{f(t)}{t}, & i \in T \\ h\frac{f(n)}{n}, & i \in C \end{cases}$$

The Shapley payoffs result from the axioms L, S, and N.

The town and countryside dwellers' Shapley values for symmetric positive and convex games show the positive effects (see lemma 2.1) of cooperation between town people and cooperation between all the agents. A town person obtains the mean of the average productivity in the overall population and the average productivity in the town where the weights are h and $1-h$, respectively.

Assume that a group T of countryside dwellers gets together to build a wall. They will do so if

$$\begin{aligned} & Cost\,(\text{wall}) \leq \underbrace{tSh_\tau(v_{TC}) - ht\frac{f(n)}{n}}_{\text{Additional product}} = (1-h)\, f(t) \\ \Leftrightarrow\quad & h \leq \frac{f(t) - Cost\,(\text{wall})}{f(t)} = 1 - \frac{Cost\,(\text{wall})}{f(t)} \end{aligned}$$

holds.

Thus, the more settlers can be accomodated within the town wall, the easier financing becomes if monotonicity holds. However, since a town wall for a large number of people can be expected to be more expensive than for a small number, the interesting question is the number of people for whom

$$\frac{Cost\,(\text{wall}) - (1-h)\,f\,(t)}{t}$$

is maximized.

Interestingly, the building of the wall does not depend on the number of people living in the country side. The wall increases the overall product from $hf\,(n)$ to $hf\,(n) + (1-h)\,f\,(t)$. This increase benefits the future town dwellers, exclusively.

3.5. Several towns (secure trading routes)

We now proceed to a different setting and assume m towns already supplied with walls. Let T_j be the population in town j $(j = 1, ..., m)$ and $N = \cup_{j=1}^{m} T_j$ where the T_j are pairwise disjunct. Travelling from one town to the other is hazardous. We propose to work with the coalition function $v_{T_1...T_m}$ defined by

$$\begin{aligned}
v_{T_1...T_m}\,(K) &= \sum_{j=1}^{m} v|_{T_j}\,(K \cap T_j) + h\left[v\,(K) - \sum_{j=1}^{m} v|_{T_j}\,(K \cap T_j)\right] \\
&= \sum_{j=1}^{m} v\,(K \cap T_j) + h\left[v\,(K) - \sum_{j=1}^{m} v\,(K \cap T_j)\right] \\
&= hv\,(K) + (1-h)\sum_{j=1}^{m} v\,(K \cap T_j)
\end{aligned}$$

In case of symmetry, we can rewrite this function as

$$\begin{aligned}
v_{T_1...T_m}\,(K) &= hf\,(|K|) + (1-h)\sum_{j=1}^{m} f\,(|K \cap T_j|) \\
&= \sum_{j=1}^{m} f\,(|K \cap T_j|) + h\left[f\,(|K|) - \sum_{j=1}^{m} f\,(|K \cap T_j|)\right]
\end{aligned}$$

Lemma 3.2. *If v is symmetric, we find*

$$Sh_i\,(v_{T_1...T_m}) = h\frac{f\,(n)}{n} + (1-h)\,\frac{f\,(t_j)}{t_j}, i \in T_j.$$

Similarly to the previous subsection, a town dweller obtains the mean of average productivities, in this case between the overall population and his own town.

Since we already have town walls, the interesting question to pose is whether it pays to link towns by a secure highway or by building a wall around both. Consider two towns 1 and 2. They will secure the trading route in between if

$$\begin{aligned}
Cost\,(\text{trading route}) &\leq \underbrace{(1-h)\,f\,(t_1+t_2)-[(1-h)\,f\,(t_1)+(1-h)\,f\,(t_2)]}_{\text{Additional product}} \\
&= (1-h)\,(f\,(t_1+t_2)-[f\,(t_1)+f\,(t_2)]) \\
&\Leftrightarrow\ h \leq \frac{(f\,(t_1+t_2)-[f\,(t_1)+f\,(t_2)])-Cost\,(\text{common wall})}{f\,(t_1+t_2)-[f\,(t_1)+f\,(t_2)]} \\
&= 1-\frac{Cost\,(\text{common wall})}{f\,(t_1+t_2)-[f\,(t_1)+f\,(t_2)]}
\end{aligned}$$

Thus, superadditivity of v is a necessary condition for securing the trading route.

4. A continuous model for the town within the countryside

4.1. Wall building without town-country complementarity

The Shapley values obtained in subsection 3.4 are the same as those that we find for the continuous analogue. Indeed, let us proceed with the continuous two-group case explained in section 2. We then obtain

$$v_{TC}\,(K)=hF\,(\mu_T\,(K)\,,\mu_C\,(K))+(1-h)\,F\,(\mu_T\,(K\cap T)\,,\mu_C\,(K\cap T))$$

in general and

$$v_{TC}\,(K)=hf\,(\lambda\,(K))+(1-h)\,f\,(\lambda\,(K\cap T))$$

in case of symmetry.

Lemma 4.1. *Let v be a symmetric vector-measure game represented by f. For a symmetric game v we obtain*

$$\begin{aligned}
Sh_T\,(v_{TC}) &= h\frac{t}{n}f\,(n)+(1-h)\,f\,(t)\ \text{ and} \\
Sh_C\,(v_{TC}) &= h\frac{c}{n}f\,(n)
\end{aligned}$$

Again, the lemma is a result of the linearity axom, the symmetry axiom and the null-player axiom, applied to continuous games. (We spare the reader the concrete details as well as the exact definitions of the axioms in the continuous case.)

Both the discrete and the continuous case do not reflect the complementarity that one might expect to reign between town production (craft) and country production (agriculture). We deal with this issue in the two following subsections.

4.2. Wall building for the Cobb-Douglas production function

We now propose the production function F given by

$$F(t,c) = t^{\alpha}(hc)^{\beta}, \; 0 < \alpha, \beta.$$

F is homogeneous of degree $d := \alpha + \beta$ where constant returns to scale are reflected by $d = 1$. In contrast to the discrete models, the vector-measure game building on Cobb-Douglas functions highlights the complementarity between the town and the country dwellers. Indeed, if $t = 0$ or $c = 0$, we have $F(t,c) = 0$. Intuitively, the town needs the countryside for the vital supply of food. Inversely, the country needs the town to provide a minimum of law-enforcement. Thus, "no wall" implies the zero vector-measure game.

Lemma 4.2. *The production function F yields the town inhabitants and country dwellers aggregate Shapley payoffs*

$$\begin{aligned} Sh_T(v) &= \frac{\alpha}{\alpha+\beta} t^{\alpha}(hc)^{\beta} \text{ and} \\ Sh_C(v) &= \frac{\beta}{\alpha+\beta} t^{\alpha}(hc)^{\beta}. \end{aligned}$$

The proof can be found in the appendix.

Assume that a group T of countryside dwellers get together to build a wall at a cost $Cost\,(\text{wall})$. The agents in T may build a wall if

$$\begin{aligned} Cost\,(\text{wall}) &\leq \underbrace{Sh_T(v) - 0}_{\text{Additional product}} \\ \Leftrightarrow \quad h &> \left(\frac{Cost\,(\text{wall})}{\frac{\alpha}{\alpha+\beta} t^{\alpha} c^{\beta}} \right)^{\frac{1}{\beta}} \end{aligned}$$

Thus, building a wall pays if h is sufficiently large (!). The modelling chosen now reflects the complementarity between town and country production. The larger the town or the country population, the more worthwhile a town wall. For $t \geq 1$, a productivity increase of town production (larger α) increases the chances of building a wall. However, the appendix shows that a larger β has ambiguous effects. Indeed, unless the wall's cost are relatively low or the town population very large, a larger β also increases the wall's chances.

4.3. Wall building in the general case

We found hat h has to be sufficiently small for wall building in the model without town-country complementarity (subsection 4.1) while h has to be sufficiently large in the complementary (Cobb-Douglas) model (subsection 4.2).

We now put the two vector-measure games together to revisit the question of when to build a town wall. Thus, assuming that v is symmetric with respect to the non-complementary part, we have the vector-measure game

$$v_{TC}(K) = (1-\rho)(hf(\lambda(K)) + (1-h) f(\lambda(K \cap T))) + \rho \mu_T(K)^{\alpha} (h\mu_C(K))^{\beta}$$

We consider ρ ($0 \leq \rho \leq 1$) a measure of complementarity.

The results presented in the two previous subsections and linearity immediately yield the following lemma.

Lemma 4.3. *The above vector-measure game yields the aggregate Shapley payoffs*

$$\begin{aligned} Sh_T(v) &= (1-\rho)\left[h\frac{t}{n}f(n) + (1-h) f(t)\right] + \rho\frac{\alpha}{\alpha+\beta}t^{\alpha}(hc)^{\beta} \text{ and} \\ Sh_C(v) &= (1-\rho) h\frac{c}{n}f(n) + \rho\frac{\beta}{\alpha+\beta}t^{\alpha}(hc)^{\beta}. \end{aligned}$$

Hence, the prospective town dwellers will build a wall if

$$\begin{aligned} Cost\,(\text{wall}) &\leq (1-\rho)\left[h\frac{t}{n}f(n) + (1-h) f(t)\right] + \rho\frac{\alpha}{\alpha+\beta}t^{\alpha}(hc)^{\beta} - (1-\rho) h\frac{t}{n}f(n) \\ &= (1-\rho)(1-h) f(t) + \rho\frac{\alpha}{\alpha+\beta}t^{\alpha}(hc)^{\beta} \qquad (4.1) \end{aligned}$$

holds.

Lemma 4.4. *The wall-building condition holds for one of five different outcomes:*

1. *no* $h \in [0, 1]$,
2. *an interval* $[0, h^-]$ *with* $h^- < 1$,
3. *an interval* $[h^+, 1]$ *with* $h^+ > 0$,
4. *the union of two disjunct intervals* $[0, h^-]$ *and* $[h^+, 1]$ *with* $h^- < h^+$ *or*
5. *for the whole interval* $[0, 1]$.

Increasing the complementarity parameter ρ *cannot lead to outcome* 2 *rather than outcome* 3 *or* 4*. Decreasing the complementarity parameter* ρ *cannot lead to outcome* 3 *rather than outcome* 2 *or* 4.

Thus, the wall-building condition is fulfilled for one or two intervals comprising 0 or 1. If the vector-measure game becomes more complementary (a higher ρ), 3 becomes more likely than 2. The proof can be found in the appendix.

5. Conclusions

According to our knowledge, this paper is the first to model town walls and trading routes in a formal way and is certainly the first one to do so by means of cooperative game theory. May-be, cooperative game theory is suitable for such a project because it need not be concerned with minute detail of actions, strategies and incentives.

Of course, the theory presented here, is very simplistic and naive in several respects. While hoping that future research will dig deeper and also provide empirical confirmation or refutation, the theoretical analysis shows that

- town walls and secure trading routes can be meaningfully analyzed within a formal (cooperative game theory) model,
- the law-enforcement problems do not unambiguously militate in favor, or against, a town wall, and
- complementarity between town and country production facilitates the building of a wall if country life is relatively peaceful.

6. Appendix

A. Proof of lemma 2.1

Convexity of v implies

$$f(k+1) - f(k) \le f(k+2) - f(k+1)$$

and hence

$$f(k+1) \le \frac{f(k+2) + f(k)}{2}. \tag{A.1}$$

Thus, the claim of the lemma is true for $k = 0$, $f(0+1) \le \frac{f(0+2)+f(0)}{2}$. We now assume that the claim is true for any k obeying $0 \le k \le n-2$. Eq. A.1 then leads to

$$\begin{aligned} f(k+1) &\le \frac{f(k+2) + f(k)}{2} \\ &\le \frac{f(k+2) + k\frac{f(k+1)}{k+1}}{2} \text{ (induction hypothesis).} \end{aligned}$$

Rearranging yields $f(k+1) - \frac{k}{2(k+1)} f(k+1) \le \frac{f(k+2)}{2}$ and hence the desired result:

$$\frac{f(k+1)}{k+1} \le \frac{f(k+2)}{2\left(1 - \frac{k}{2(k+1)}\right)(k+1)} = \frac{f(k+2)}{k+2}$$

B. Proof of lemma 4.2

The proof follows from

$$\begin{aligned} \left.\frac{\partial F}{\partial \mu_T}\right|_{(qt,qc)} &= \alpha (qt)^{\alpha-1} (hqc)^{\beta} = \alpha q^{\alpha+\beta-1} t^{\alpha-1} (hc)^{\beta}, \\ \left.\frac{\partial F}{\partial \mu_C}\right|_{(qt,qc)} &= (qt)^{\alpha} \beta (hqc)^{\beta-1} h = \beta h q^{\alpha+\beta-1} t^{\alpha} (hc)^{\beta-1} \text{ and} \\ \int_0^1 \left[q^{\alpha+\beta-1}\right] dq &= \frac{1}{\alpha+\beta} \end{aligned}$$

and

$$\begin{aligned}
Sh_T(v) &= \lambda(T\cap T)\int_0^1 \left.\frac{\partial F}{\partial \mu_T}\right|_{(qt,qc)} dq \\
&= t\cdot\frac{\alpha}{\alpha+\beta}t^{\alpha-1}(hc)^\beta = \frac{\alpha}{\alpha+\beta}t^\alpha (hc)^\beta \text{ and} \\
Sh_C(v) &= c\cdot h\frac{\beta}{\alpha+\beta}t^\alpha (hc)^{\beta-1} = \frac{\beta}{\alpha+\beta}t^\alpha (hc)^\beta .
\end{aligned}$$

C. Ambiguous effects of increasing β

The h-threshold can be an increasing and a decreasing function of β. In the table below, compare the second with the third and the fourth with the fifth column.

$Cost$ (wall)	500	500	2000	2000
t	1000	1000	10^7	10^7
c	3000	3000	30000	30000
α	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
β	$\frac{2}{3}$ >	$\frac{1}{2}$	$\frac{2}{3}$ >	$\frac{1}{2}$
threshold ≈	0.075 <	0.333	5.98×10^{-5} >	5.33×10^{-5}

D. Proof of lemma 4.4

By

$$\begin{aligned}
&\frac{\partial\left((1-\rho)(1-h)f(t)+\rho\frac{\alpha}{\alpha+\beta}t^\alpha (hc)^\beta\right)}{\partial h} > 0 \\
\Leftrightarrow\quad & h > \left(\frac{f(t)}{t^\alpha c^\beta}\frac{1-\rho}{\rho}\frac{\alpha+\beta}{\alpha}\right)^{\frac{1}{\beta-1}} \qquad \text{(D.1)}
\end{aligned}$$

we observe: If there is a h^+ such that equations 4.1 and D.1 hold, the wall will be build for any h greater than h^+, too. Similarly, if there is a h^- such that equations 4.1 holds and D.1 does not hold, the wall will be build for any h smaller than h^+, too. Thus, we can have the five different outcomes mentioned in the lemma.

Consider, now, $0\le\rho_\ell<\rho_h\le 1$ and assume that 3 holds at ρ_ℓ and 2 at ρ_h. Then,

(i) $\exists h^+ > 0$ s.t. $Cost\,(\text{wall}) \leq (1-\rho_\ell)(1-h)\,f(t) + \rho_\ell \frac{\alpha}{\alpha+\beta} t^\alpha (hc)^\beta \;\; \forall h \geq h^+$,
(ii) $Cost\,(\text{wall}) > (1-\rho_\ell)(1-h)\,f(t) + \rho_\ell \frac{\alpha}{\alpha+\beta} t^\alpha (hc)^\beta \;\; \forall h < h^+$,
(iii) $\exists h^- < 1$ s.t. $Cost\,(\text{wall}) \leq (1-\rho_h)(1-h)\,f(t) + \rho_h \frac{\alpha}{\alpha+\beta} t^\alpha (hc)^\beta \;\; \forall h \leq h^-$ and
(iv) $Cost\,(\text{wall}) > (1-\rho_h)(1-h)\,f(t) + \rho_h \frac{\alpha}{\alpha+\beta} t^\alpha (hc)^\beta \;\; \forall h > h^-$

It is easy to see that $h = 1$ produces a contradiction between (i) and (iv). Thus, we cannot have outcome 2 at ρ_h and outcomes 3 or 4 at ρ_ℓ. The last claim is shown similarly.

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