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Why prāṇa is the most excellent among the vital functions, or: the Shapley value in the Upaniṣads

Abstract: This paper deals with the contest of the “vital functions” for superiority in the Brhadāraṇyaka Upaniṣad and the Chāndogya Upaniṣad, in the Aitareya Āraṇyaka and others. The paper concentrates on two particular manners to decide the superiority question, namely (i) singly leaving or entering (breath, sight, etc. leave one after another or enter one after another) and (ii) alternating withdrawal (breath leaves and returns, sight leaves and returns). The paper defends two claims. First, it can be shown that the commentators were aware of the generalizable nature of these two approaches. Second, these two approaches are closely related to the so-called Shapley value developed in cooperative game theory.

Keywords: Shapley value; cooperative game theory; balancing operation; vital functions; superiority

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I. Introduction

Comparisons of the natural body with a political one have been common in many cultures, Egyptian, European, Greek, Indian, and Roman. I will focus on a specific aspect of organic theories, namely disputes about rank order. From the Western point of view, one of Aesop’s fables is most relevant, which deals with the quarrel between the belly and the feet about their relative importance:

The belly and the feet were arguing about their importance, and when the feet kept saying that they were so much stronger that they even carried the stomach around, the stomach replied, “But, my good friends, if I didn’t take in food, you wouldn’t be able to carry anything.”

In the Indian context, one finds the contest of the “vital functions” breath, speech, and the like for superiority. This contest is presented in different versions in the Brhadaranyak Upanisad, the Chandogya Upanisad (both from 7th


3 The dating of this fable is very difficult. It may have originated sometime between 1000 BCE and 100 CE, see Harvey (2007, pp. 4–5). The fragmentary Egyptian version (see Erman (1927, pp. 173–174)) may be older and date from the first half of the first century BCE, see Ilsley Hicks (1963, p. 29).

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5 Daly (1961, p. 148).
to 6th centuries BCE), in the Aitareya Āraṇyaka (6th to 5th centuries BCE) and others. Olivelle (1998) translates \textit{prāṇa} or \textit{karman} by “vital function”. In contrast, breath as one particular member among the other vital forces is called “breath” or “central breath” (\textit{prāṇa} or \textit{madhyamaḥ prāṇah}). I follow Olivelle in this respect.

Indologists have, of course, noted the “Rangstreitfabel” (Ruben (1947)) and the importance of breath (Frauwallner (1997, pp. 41–45)). A detailed discussion of the respiratory term \textit{prāṇa}, in particular in contrast to \textit{apāna} (where the former means exhalation and the latter inhalation), is provided by Bodewitz (1986). Zysk (1993) analyzes the five bodily winds from \textit{prāṇa} down to \textit{vyāna} and the different understanding adopted in \textit{Āyurveda} on the one hand and in \textit{Yoga} on the other hand. Under the heading of “Soul, Body and Person in Ancient India”, Preisendanz (2005) presents the differing and changing, but related, conceptions of \textit{prāṇa}, \textit{asu}, and \textit{ātman}.

As a specific Indian example of the contest, death succeeds in capturing the vital functions speech, sight, and some unspecified others, but not breath in BĀU

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6 This chronology follows Olivelle (1998, p. 12) who cautiously adds “give or take a century or so”. Bronkhorst (2007, pp. 173–262) disputes it and argues that the present form of these Upaniṣads was reached only a few centuries later.
7 See Olivelle (1998, pp. 8, 12-13). The Aitareya Āraṇyaka contains the Aitareya Upaniṣad which does not address the contest of the vital functions.
8 For example, Śāṅkhāyana Āraṇyaka (ŚĀ 9.1-7) and Kauṣītaki Upaniṣad (KauU 2.14).
9 This translational choice seems sensible also in view of Preisendanz (2005, p. 125).
10 Generally, translations are mine unless indicated otherwise.
1.5.21. This fact shows the superiority of breath. See Part I, section A. The contest is explicitly framed as a competition between the vital functions. Without a competition expressis verbis, the superiority of breath also clearly emerges elsewhere.\textsuperscript{11} One can conceive of Aesop’s fable and these Indian tales as presenting \textit{idiosyncratic} solutions to the problem of superiority inasmuch as it is not obvious at all how they might be generalized to apply to other problems of superiority, say, concerning the relationship between people working together in a common joint venture or between the countries of the European Union (EU).

In my mind, the above versions clearly differ from other Indian ones where the vital functions avail themselves of some non-idiosyncratic method to assess their superiority. In this paper, I will concentrate on these \textit{generalizable} approaches. I do not want to offer a definition of “generalizability” in general. However, and with a view to the texts covered in this paper, one may argue that generalizability may refer to some mode or manner that is

(a) a test for something (cf. \textit{parīkṣaṇa})

(b) teachable (cf. \textit{prakāropadeśa}),

(c) applicable beyond the actual application (cf. \textit{yathā loke} and \textit{cetanāvanta īva puruṣāḥ}), and

(d) serves to ward off struggle or competition (cf. \textit{spardhānivāraṇārtham}).\textsuperscript{12}

\textsuperscript{11} For example, ChU 4.3.3 characterizes breath as the “gatherer” (Olivelle 1998, p. 217) into whom the other vital functions pass when a man sleeps.

\textsuperscript{12} All these quotations are contextualized and discussed in Part II, sections B and C.
In the ancient Indian texts, there are two generalizable approaches to the problem of superiority. Both involve the difference a vital function makes. They employ (i) singly leaving or entering and (ii) alternating withdrawal. I want to turn to the “singly leaving or entering” approach first. In AĀ 2.1.4, the superiority of breath is established in two different ways. The vital functions first leave the body one after another, and then they reenter, again serially. Breath is the last to leave and the last to reenter and makes the decisive difference. Turning to the above example of the EU, one may, at least in principle, consider how the remaining countries of the EU would fare if Great Britain and then France, etc., would leave the EU.

The second generalizable approach could be labeled the approach “involving alternating withdrawal” or the “where would you be without me” approach. This approach is seen in BĀU 6.1, ChU 5.1, and ŠĀ 9.1-7. Speech leaves the body and reenters after a while. The remaining functions are then asked how they fared. Then, the same procedure is followed by other vital functions, respectively. It turns out that the departure of breath could not be endured and that, hence, breath is superior. In the example above, a country, like Poland or Portugal, may confront the others with the prospect of leaving the EU. Perhaps the others would fare worse after Poland’s exit than after the exit of Portugal.

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13 Translation by Keith (1908, p. 57) and Bodewitz (2002, pp. 73–77).
14 The order is speech, sight, hearing, mind, semen, and breath in BĀU 6.1. In ChU 5.1 and ŠĀ 9.1-7, the order is the same, but without semen.
15 Yet another method of determining superiority is described in PU 2 (Olivelle 1998, p. 461). There breath shows its power by “setting off” (utkram) and then “setting down” (pratiṣṭhā) again. Since the other vital forces speech, mind, sight, and hearing have to follow suit in the case of both movements, they
The aim of this paper is twofold. Firstly, it can be shown that the commentators were aware of the generalizable nature of the “singly leaving or entering” approach and the approach “involving alternating withdrawal”. Secondly, these two approaches are closely related to the so-called Shapley value developed in cooperative game theory.\textsuperscript{16} I will first present the relevant stories of the contest of the vital functions in the next part. In Part III I will then present the Shapley value and discuss the relation between this concept and the literature on contest. Part IV concludes the paper.

\section*{II. The contest among the vital functions}

\subsection*{A. Idiosyncratic approaches}

The following story of a contest from the Bṛhadāraṇyaka Upaniṣad is an example for an idiosyncratic approach, in the sense of not presenting a procedure that may be applicable as a solution to a wide range of problems concerning superiority. BĀU 1.5.21 says:

\begin{quote}
praṇāpatir ha karmāṇi sasrje

tāni śṛṣṭāny anyo ‘nyenāspardhanta

vadiṣyāmy evāham iti vāg dadhre

drakṣyāmy aham iti cakṣuḥ

...

tāni mṛtyuḥ śramo bhūtvopayeme
\end{quote}

thereby acknowledge the superiority of breath. This approach is not covered in this paper.

\textsuperscript{16} Shapley (1953).
tāny āpnot | tāny āptvā mṛtyur avārundha |

...  

athemam eva nāpnod yo 'yaṃ madhyamaḥ prāṇaḥ |
tāni jñātum dadhrire | ayāṃ vai naḥ śreṣṭho

Prajāpati created the vital functions.
Once they were created, they began to compete with each other.

Speech threw out the challenge: “I am going to speak!”

Sight shot back: “I am going to see!”

...

Taking the form of weariness, death took hold of them; it captured and shackled them.

...

The central breath alone, however, death could not capture.

So they sought to know him, thinking: “He is clearly the best among us ...

Likewise, testing how the vital functions respond to being “riddled with evil” can be counted among the idiosyncratic solutions to superiority challenges. Using Olivelle’s (1998, pp. 171-173) translation of ChU 1.2.1-7, the gods, who were fighting the demons, venerate the High Chant successively as breath within the nostrils, speech, sight, hearing, mind, and breath within the mouth. The demons “riddle with evil” (pāpmanā vividhuḥ) these functions from breath within

the nostrils all the way to mind, but they fail to do the same with breath within the mouth.\textsuperscript{18}

Commenting on a part of ChU 1.2.8, Śaṅkara (ChU_Ś, p. 20, ll. 3-6), who lived perhaps sometime between 650 CE and 800 CE\textsuperscript{19}, explains the difference of breath within the nostrils and breath within the mouth:

\begin{quote}
\begin{center}
nanu nāsikyo ’pi prāṇo vāyvātmā yathā mukhyas, tatra nāsikhyah prāṇah pāpmanā viddhaḥ prāṇa eva san na mukhyah katham |

naiṣa doṣah | nāsikyas tu sthānakaraṇavaiguṇyād viddho vāyvātmā ’pi san, mukhyah sthānadevatābalīyastvān na viddha iti yuktam |
\end{center}
\end{quote}

Objection: Breath within the nostrils is also of the nature of wind, like breath within the mouth. How can it be in that regard that breath within the nostrils which is just breath is riddled with evil, but not breath within the mouth.

Answer: This fault does not apply. Due to the bad quality regarding its location and sense organ breath within the nostrils is pierced [riddled with evil] although it is of the nature of wind. [In contrast,] breath within the mouth is not pierced due to its strength of location and [presiding] deity. This is reasonable.

\textsuperscript{18} A very similar sequence is described in BĀU 1.3.1-7. There, using Olivelle’s (1998, pp. 39-41) words, the vital functions speech, breath, sight, hearing, mind, and breath within the mouth have to “sing the High Chant”. The demons “riddle with evil” all these functions except breath within the mouth.

\textsuperscript{19} See Isayeva (1993, pp. 83-87).
B. Singly leaving or entering

Singly leaving (and singly entering) the body is the first generalizable approach. It is described in a story within the first chapter of the second book of the Aitareya Āraṇyaka, addressed by AĀ 2.1. In that chapter, the second section deals with the hymn (uktha). In particular, one finds in AĀ 2.1.2.1:

\[\text{uktham uktham iti vai prajā vadanti} \]
\[\text{tad idam evoktham iyam eva prthivīto hīdам sarvam uttiṣṭhati yad idam kiñ ca \} | \]
People say, ‘Hymn, hymn.’
The hymn is just this earth. For from it all that exists springs.  

Indeed, the hymn is the sky (antarikṣa), yonder heaven (dyau), man (puruṣa), etc. Jumping to the fourth section, Brahman (brahman) enters into man (puruṣa), first into his feet and finally, having worked his way upwards, into the head of man. The entering of the head by Brahman seems to bring to life the head with its vital functions. Then, the head’s vital functions compete for being the hymn (uktha) (AĀ 2.1.4.7-11):

\[\text{tā etāḥ śīrṣañ chrīyaḥ śritāś caksuḥ śrotraṃ mano vāk prāṇaḥ \} | \]
\[\text{śrayante ’smiñ chriyo ya evam etac chirasaḥ śirastvaṃ veda \} | \]
\[\text{tā ahiṃsantāham uktham asmy aham uktham asmīti \} | \]

\[\text{The last digit is added by the current author in line with the danḍas used by Keith.} \]
\[\text{Keith (1909, p. 201).} \]
\[\text{AĀ 2.1.2 with English words from Keith (1909, pp. 201–202).} \]
\[\text{AĀ 2.1.4 with English words from Keith (1909, p. 204).} \]
tā abruvan hantāsmāc charīrād utkrāṃmāma
tad yasmin na utkrānta idaṃ śarīraṃ patsyati tad uktham bhavisyatīti

(10)
vāg udakrāmad avadann aśnan pibann āstaiva

(11)
These delights settled in the head, sight, hearing, mind, speech, breath. (7)
Delights settle on him who knows thus why the head is the head. (8)
They strove together, saying, ‘I am the hymn, I am the hymn.’ (9)
They said, ‘Come, let us leave this body,
then that one of us at whose departure the body falls, will be the hymn.’

(10)
Speech went out, yet [the body, while] speechless, indeed remained [still] eating and drinking. (11)

The sequence of leaving is the following: speech, sight, hearing, and mind. Finally breath leaves the body (AĀ 2.1.4.15):

prāṇa udakrāmat tatprāṇa utkrānte ’padyata

Breath went out. When that breath went out, [the body] fell.

24 Keith (1909, p. 205). The formula in AĀ 2.1.4.8 ya evam ... veda is quite common in the Brāhmaṇas. Olivelle (see, for the specific case of arka, BĀU 1.2.1 and Olivelle (1998, p. 37)) would translate as “who knows the name and nature of head in this way.” It is conceivable to rearrange tā abruvan in AĀ 2.1.4.10 so that it immediately follows uktham asmīti (AĀ 2.1.4.9). I like to stick to Keith’s translation which is in line with the commentary quoted below.

25 While there is nothing wrong with the translation by Keith (1909, p. 205) (“Speech went forth, yet (the body) remained, speechless, eating, and drinking”), I chose to bring out the apparent meaning more clearly by some small changes.

26 My translation is a bit more literal than the one by Keith (1909, p. 205). Apparently, there is an absolute locative here (prāṇa(e) utkrānte). If tat and prāṇa are two separate words, tat might be understood in the sense of tataḥ yielding
The commentary ascribed to Sāyaṇa dating, perhaps, to the 14th century CE\textsuperscript{27} uses the “eating and drinking” from AĀ 2.1.4.11 to argue quite specifically why breath is the winner (AĀ\textsubscript{Sā}, p. 111, ll. 11-15):

\textit{vākcakṣuh śrotamanasām ekaikasmin utkrānte sati tattadindri-yasādhyavāpāramātram lupyate, na tu śarīraṁ patati, kiṁtv annapāne svīkurvan yathāpūrvam abhavad eva ... tac charīraṁ prāṇa utkrānte sati patitam abhūd, na tv aśnāti nāpi pibati}

When speech, sight, hearing, and mind depart individually, only the effective operation of the respective organs is taken away, but the body does not fall. But taking in food and drink, it indeed remained as before. … When breath departs, this body fell and did neither eat nor drink.

While the victory of breath must have been obvious to the vital functions, they reaffirm the result by resolving on entering the body rather than leaving it. The sequence of entering is the same as the sequence of leaving. The result is as expected and, this time, the conclusion is accepted as is described in AĀ 2.1.4.20, 24:

\textit{vāk prāviśat aśayad eva} | (20)

\textsuperscript{27} See Burnell (1873, pp. vi–xv). Slaje (2010) deals with the very doubtful authorship of Sāyaṇa with respect to commentaries on the four Veda-Saṃhitās, but he does not offer an opinion on whether Sāyaṇa might be the author of the Aitareya Āraṇyaka (Slaje (2010, p. 395, fn. 48)).
prāṇaḥ prāviṣat, tatprāṇe prapanna udatiṣṭhat tad uktham abhavat | (24)

Speech entered, [the body] lay still. (20)²⁸

When that breath entered, it [the body] arose and it [breath] became that
hymn. (24)²⁹

In KauU 2.14³⁰, the vital functions enter the body (which is supine) one after an-
other. Only after breath has entered, the body is able to get up.³¹ No leaving se-
quence is described in that Upaniṣad. The procedure of singly entering together
with raising the body is of particular relevance in relation to the “etymology”
given in BĀU 5.13.1:

uktham | prāṇo vā uktham | prāṇo hīdam sarvam utthāpayati | …

Uktha. The uktha (“Ṛgvedic hymn”), clearly, is breath, for breath raises
up (utthā) this whole world.³²

²⁸ Keith (1909, p. 205). Current author replaced round brackets by square brack-
ests.
²⁹ See the third from last footnote for tatprāṇe. Concerning tad uktham abhavat,
the Sāyaṇa commentary offers a little twist: “tat prāṇasvarūpam utthānahetut-
vād uktham abhūt (AĀ_Sā, p. 112, ll. 12-13): “Tat means the specific nature of
breath. By arising, it became the hymn.”
³⁰ athāto niḥśreyasādānam | etā ha vai devatā ahaṁśreyase vivadamānā asmāc
charīrād uccakramuḥ | tad dārubhūtam śiśye | athainad vāk praviveṣa (sic, HW)|
tad vācā vadac chiśya eva | ... athainat prāṇaḥ praviveṣa | tat tata eva
samuttasthau | …
For the purpose of this paper, AĀ 2.1.4.9-10 above in this section is central. Sāyaṇa points to the purpose of the discussion, namely to state the superiority of breath (prāṇasya śraṣṭhyam)\(^{33}\). He then explains (AĀ₃₃₄₅_Sā)\(^{34}\):

\[
tāḥ pūrvoktāḥ śriyaḥ cakṣurādirūpās tadabhimāninyo devatā ahiṃsanta parasparaspardhārūpāṁ hiṃsām akurvan |
spardhāviśayo vispaṣṭam ucyate—aham uktham asmi |
ukthe cakṣuḥsvarūpasya mamaiva dṛṣṭiḥ kartavyety evaṃ cakṣurdevatā vakti |
…
\]

\[
tāḥ spardhamānā devatāḥ spardhānivāraṇārtham samayaviśeṣaṃ paras-param abruvan
\]

The previously mentioned delights whose handsome form is sight and so on [referring to śriyaḥ ... cakṣuḥ etc. in the mūla text], who are proud [presiding] deities with regard to that [tad, namely sight and so on], strove together, i.e., they committed violence in the form of mutual competition. The object of competition is clearly expressed [when they say]: I am the hymn.

With respect to the hymn the deity of sight says in this manner: “The faculty of seeing can be performed by me alone who has the specific nature of sight.”

…

In order to ward off [this kind of] competition, these competing deities mutually told\(^{35}\) a particular agreement [or: agreed on a particular treaty].

\[\]

\(^{33}\) AĀ₃₃₄₅_Sā, p. 110, just before quoting AĀ 2.1.4.9-10, l. 8 from bottom.

\(^{34}\) p. 110, l. 3 from bottom till p. 111, l. 2.

\(^{35}\) The particular inclusion of tā abruvan in the commentary lends support to the translation of AĀ 2.1.4.10 above.
According to the last sentence, Sāyaṇa acknowledges that the agreement (i.e., singly leaving or singly entering) is done for the purpose of warding off competition (spardhāṅivāraṇārthaṃ). Thus, the competition that consists in simply insisting on one’s superiority (aham uktham asmi) is warded off in favour of a competition by way of a controlled experiment. To the commentator’s mind, this experiment amounts to a generalizable manner of deciding the superiority question. This is aspect (d) of generalizability mentioned in the introduction.

C. Alternating withdrawal

BĀU 6.1.7-8 most clearly brings out the approach involving alternating withdrawal:

\[
\begin{align*}
te \ heme \ prāṇā \ ahamāśreyase \ vivadamānā \ brahma \ jagmuḥ | \\
tad \ dhocuḥ \ ko \ no \ vasiṣṭha \ iti | \\
tad \ dhovāca \ yasmin \ va \ utkrānta \ idaṁ \ śarīram \ pāpīyo \ manyate \ sa \ vo \ vasiṣṭha \ iti || (7) \\
vāg \ ghoccakrāma | sā saṁvatsaram \ proṣyāgatyovāca \ katham \ aṣakata \\
madṛte \ jīvitum \ iti | \\
te \ hocuḥ \ yathā \ kalā \ avadanto \ vācā \ prāṇantaḥ \ prāṇena \ paśyantaś \ ċakṣuṣā \\
śrīvantaḥ \ śrotreṇa \ vidvāṃso \ manasā \ prajāyamānā \ retasaivam \ ajīvīṣmeti | \\
praviveśa \ ca \ vāk || (8)
\end{align*}
\]

Once these vital functions were arguing about who among them was the greatest. So they went to brahman and asked: “Who is the most excellent of us?” He replied: “The one, after whose departure you consider the body to be the worst off, is the most excellent among you.” (7)
So speech departed. After spending a year away, it came back and asked: “How did you manage to live without me?” They replied: “We lived as the dumb would, without speaking with speech, but breathing with the breath, seeing with the eye, hearing with the ear, thinking with the mind, and fathering with semen.” So speech reentered. (8)

After speech has left and reentered, the very same procedure is followed by sight, hearing, mind, and semen. When breath is about to leave, the other vital functions realize the serious consequences (BĀU 6.1.13-14):

\[
\text{atha ha prāṇa utkramiṣyan yathā mahāsuhayah saindhavah}
\]
\[
pāḍvīśaśaṅkūn saṃvrhed evaṃ haivemān prāṇān saṃvavarha |
\]
\[
te hocur mā bhagava utkramīḥ |
\]
\[
na vai śaksyāmas tvadrte jīvitum iti |
\]
\[
tasya me baliṃ kuruteti |
\]
\[
tatheti ||(13)
\]
\[
sā ha vāg uvāca yad vā aham vasiṣṭhāsmi tvaṃ tad vasiṣṭho ’sīti | … (14)
\]

Then, as the breath was about to depart, it strongly pulled on those vital functions, as a mighty Indus horse would strongly pull on the stakes to which it is tethered. They implored: “Lord, please do not depart! We

36 Olivelle (1998, p. 143). The compound ahamśreyase in BĀU 6.1.7 could be in dative (consonantal stem ahamśreyas) or in locative (thematic stem ahamśreyasa). According to PW, vivad is usually employed with locative of the disputed subject matter. Dative is understood by Śaṅkara who glosses ahamśreyase by aham śreyān ity etasmai prayojanāya (BĀU_Ś, p. 416, l. 13). He uses the similar expression ahamśreṣṭhatāyai vivadantah in the commentary on the Chāndogya Upaniṣad (ChU_Ś, p. 265, l. 16).

37 This first sentence is taken from Olivelle (1998, p. 145) with the important exceptions that “uprooted” (Olivelle) has been replaced by “strongly pulled on” and similarly “would uproot” (Olivelle) by “would strongly pull on”. Wezler
will not be able to live without you.” He told them: “If that's so, offer a tribute to me.” “We will.” they replied. (13)

So speech declared: “As I am the most excellent, so you will be the most excellent.” … (14)38

Apparently, breath’s threat of withdrawal is more damaging to speech than the corresponding threat of speech is to breath. This very fact is the basis for breath’s demand for a tribute.

Commenting on a part of BĀU 6.1.13, Śaṅkara (BĀU_Ś, p. 417, ll. 17-20) explains:

\[
\begin{align*}
te \ vāgādayo \ hocur \ he \ bhagavo \ bhagavan \ motkramīr \\
yasmān \ na \ vai \ šakṣyāmas \ tvadrte \ tvāṃ \ vinā \ jīvitum \ iti \\
yady \ evaṃ \ mama \ šreṣṭhatā \ vijñātā \ bhavadbhir \ aham \ atra \ śreṣṭhas \\
tasya \ u \ me \ mama \ baliṁ \ karaṁ \ kuruta \ karaṁ \ prayaccheti \\
ayaṃ \ ca \ prāṇasamvādaḥ \ (sic, \ HW) \ kalpito \ viduṣah \\
śreṣṭhaparīkṣaṇaprakāropadeśāḥ \ | \\
anena \ hi \ prakāreṇa \ vidvān \ ko \ nu \ khalv \ atra \ śreṣṭha \ iti \ parīkṣaṇaṁ \\
karoṭi \ |
\end{align*}
\]

They, i.e., speech and the others, implored: “Oh Lord (using an alternative form of the vocative), please do not depart! For we will not be able to live

(1982/1983) has examined samvrh in BĀU 6.1.13 and the parallel saṃkhid in ChU 5.1.12 in astounding detail. While Olivelle’s translation closely follows most previous translations, Wezler’s arguments against “uproot” are convincing. Among other arguments, Wezler discusses the meanings of the prefix sam. Importantly for this paper, breath does not leave the body or “uproot” the other vital functions, but just threatens to do so.

without you (glossing tvadrte by tvāṃ vinā).” [Breath replies:] “If my superiority is recognised by you in this manner, I am the best here. If that is indeed so, “If that's so, offer a tribute (bali glossed by kara (tax)) to me (me glossed by mama), i.e., pay a tax.”

And this agreement of the vital functions is imagined on the part of a learned person as a teaching of a mode of testing superiority. For in this manner a learned person performs the test of who, indeed, is the best here.

This version of the story in the BĀU is very close to one found in ChU 5.1. While breath does not explicitly demand a tribute, the other vital functions offer their tributes in ChU 5.1.13-14 similar to BĀU 6.1.14. Śaṅkara (ChU_Ś, p. 165, l. 8) comments:

\[ \text{atha hainam vāgādayaḥ prāṇasya śreṣṭhatvam kāryenāpādayanta āhur balim iva haranto rājne viśah } \ldots \]

Speech and the rest, establishing, by their action, the superiority of Breath, said to him—making offerings like the people to their King …

Indeed, the tribute (bali) offered to the best (śreyas) is a familiar topic. As ŚB 11.2.6.14 (p. 842) states:

\[ \ldots śreyase pāpiyān baliṃ hared vaiśyo vā rājne baliṃ haret \ldots \]

\[ \ldots \text{an inferior brings tribute to his superior, or a merchant brings tribute to the king } \ldots \]

\[ ^{39} \text{Jha (2005, p. 225).} \]
Thus, the reason behind the tribute may lie in the fact that the competition of the vital functions serves as a “political allegory where the superiority of prāṇa in relation to the other vital functions is likened to the supremacy of the king among his rivals and ministers” (Black 2007, p. 122). While this is certainly true,\(^{40}\) the tribute can also be seen as serving a specific purpose in the context of the approach taken in this paper (see section III C).

Now, turning to the main topic of our paper, with the last two sentences of the above commentary on BĀU 6.1.13 (ayam ca prāṇasamvādaḥ ... iti parīkṣaṇaṃ karoti), Śaṅkara explains the agreed-upon withdrawal. He makes abundantly clear that he considers the threat of withdrawal a generalizable procedure. In particular, he talks about a test (parīkṣaṇa, see (a) in the introduction) and a method that is teachable (prakāropadeśa, (b)).

Similarly (also with the words kalpito viduṣaḥ), Śaṅkara comments on the purpose of this method in his Chāndogya-Upaniṣad commentary (ChU_Ś, p. 167, ll. 3-4):

\[
\text{vāgādīnāṃ ceha.saṃvādaḥ kalpito viduṣo ’nvayavyatirekābhhyāṃ prāṇaśreṣṭhatānirdhāraṇārtham}\\
\text{yathā loke puruṣāḥ anyonyam ātmānāḥ śreṣṭhatāyai vivadamānāḥ kañcid}\\
\text{guṇaviśeṣābhijñāṃ prcchanti ko naḥ śreṣṭho guṇair iti}
\]

And this agreement by speech and so on is imagined on the part of a learned person in order to determine the superiority of breath with the

\(^{40}\) See also Rau (1957, p. 34) and Bodewitz (1992, p. 57).
method of concomitant presence and concomitant absence\textsuperscript{41}, as people in
the world who mutually dispute about their own superiority ask somebody
who is knowledgeable about special qualities: “who of us is the best in
terms of qualities”?

The second sentence in the quotation above (\textit{yathā loke ...}) points to the wider
applicability of the approach involving alternating withdrawal, just as suggested
by (c) in the introduction. Consider a second piece of evidence where Śaṅkara
(ChU_Ś, p. 165, ll. 16-17) presents the following objection against this method:

\begin{quote}
\begin{flushleft}
\textit{nanu katham idaṃ yuktaṃ cetanāvanta iva puruṣā ahamśreṣṭhatāyai viva-
danto 'nyonyaṃ spardherann iti |}
na hi cakṣurādīnāṃ vācaṃ pratyākhyāya pratyekam vadanam sambhavati
tathā 'pagamo dehāt punah praveśo brahmagamanam pṛāṇastutir vopa-
padyate |}
\end{flushleft}
\end{quote}

How could this be logical, namely that [the vital functions] can compete
against each other by arguing about who among them is the greatest, as
conscious humans would. For speaking one by one is not possible for
sight and so on, excepting speech. Likewise, departing from the body,
entering again, going to Brahman, or praising breath are not reasonable.

\textsuperscript{41} Jha (2005, p. 227) translates \textit{anvayavyatirekābhyām} as “by means of negation
and affirmation”. Halbfass (1991, pp. 162–177) analyzes Śaṅkara’s use of this
term. For the application at hand, “concomitant presence” (p. 170) means that
the presence of breath goes together with the presence of life. In contrast, “con-
comitant absence” refers to the simultaneous absence (or rather threat of ab-
sence) of breath and (threatening) loss of life.
While Śaṅkara’s reply is not helpful for the present purpose, it needs to be noted that he considered conscious humans (cetanāvantah puruṣāḥ) the most obvious contenders in such fights for superiority, just in line with (c) in the introduction. Thus Śaṅkara presupposes a wider applicability of this method.

III. The Shapley value

A. Basic definitions

Before linking the Shapley value to the pre-modern Indian contest of the vital functions, a short tutorial is called for. The Shapley value belongs to the realm of cooperative game theory. This theory presupposes $n$ players, collected in a set $N = \{1, 2, \ldots, n\}$, and a so-called coalition function $w$. The players are supposed to “cooperate” in any economic, political, or social venture. Coalition functions are meant to reflect the “production” possibilities of groups of players. Production is to be understood in a wide sense and may refer to economic production in a narrow sense, but also point to other social or political frameworks.

A subset $K$ of $N$ is called a coalition. $N$ itself is called the grand coalition. To each coalition, the coalition function attributes a “worth” $w(K)$. The worth stands for the economic, social, political, or other gain that the particular group of players can achieve (“create”) by cooperating. A worth can only be created if at least one player is present, i.e., the empty set $\emptyset$ creates the worth zero,

\[42\] See, for example, chapter 9 in the textbook by Myerson (1991).
\( w(\emptyset) = 0 \). To simplify the notation, I write \( w(i) \) instead of \( w(\{i\}) \) for the worth created by player \( i \) (or for the worth of the one-man coalition that hosts only player \( i \)), \( w(1, 2) \) instead of \( w(\{1, 2\}) \) for the worth created by the two players 1 and 2, and \( w(K \cup i) \) instead of \( w(K \cup \{i\}) \).

The aim of cooperative game theory is to specify payoffs for the players. These specifications are called “solution concepts”. Several solution concepts, i.e., possibilities of how to determine the payoffs, have been explored. For each solution concept, cooperative game theory uses two different approaches to arrive at payoff vectors from coalition functions: (i) The algorithmic approach applies some algebraic manipulations of the coalition functions in order to derive payoff vectors. (ii) The axiomatic approach suggests general rules of distribution. The most famous solution concept is the Shapley value. The two different approaches are presented in the next sections.

**B. The algorithmic approach**

The algorithm of the Shapley value builds on the players’ “marginal contributions”. A player’s marginal contribution is the worth of a coalition with him minus the worth of this coalition without him, i.e., the difference he would make. In the following I will focus on two players; for further details and the general case, the reader is referred to the footnotes and the appendix. Player 1 has two
marginal contributions, the first with respect to the empty set $\emptyset$, where his marginal contribution is $w(1) - w(\emptyset)$, the second with respect to the other player, where his marginal contribution is $w(1, 2) - w(2)$.\textsuperscript{43}

Player 1’s Shapley value is the average of his marginal contributions, taken over all sequences of the two players. For two players, there are just two sequences, player 1 may be first, amounting to sequence $(1, 2)$, or second, amounting to sequence $(2,1)$. Thus, the players’ Shapley values\textsuperscript{44} are

\begin{equation}
(1) \quad Sh_1 = \frac{1}{2} (w(1) - w(\emptyset)) + \frac{1}{2} (w(1, 2) - w(2))
\end{equation}

and

\begin{equation}
(2) \quad Sh_2 = \frac{1}{2} (w(2) - w(\emptyset)) + \frac{1}{2} (w(1, 2) - w(1))
\end{equation}

The procedure of the vital functions’ singly leaving the body or entering into it (see section II B) is closely related to the algorithmic approach of defining the Shapley value. In A\textsuperscript{A} 2.1.4, the sequence of the vital functions that enter the body is speech ($sp$), sight ($si$), hearing ($h$), mind ($m$), and finally breath ($b$).

\textsuperscript{43} For a general player set $N$, the marginal contribution ($MC$) of player $i$ with respect to a coalition $K$ that does not contain that player is defined by the difference $MC_i(K) = w(K \cup i) - w(K)$.

\textsuperscript{44} For a general player set $N$, let $S$ be the set of sequences. For $n$ players, there exist $n! = 1 \cdot 2 \cdot \ldots \cdot n$ different sequences. Let $K_i(s)$ denote the set of players in the sequence $s$ up to but not including player $i$. Then, player $i$’s Shapley value is $Sh_i = \frac{\sum_{s \in S} MC_i(K_i(s))}{n!}$.
One might now consider the player set $N = \{b, sp, si, h, m\}$ and worths for each coalition consisting of one or several vital functions. These worths can in principle be summarized in a coalition function. While numerical values are not mentioned in the examined Indian texts, it seems clear from the text that the “worths” increase the more vital functions are involved. A body with speech, sight, and hearing would be “more alive” than a body with just two of these functions.

In general, the payoffs involved in entering and leaving differ for a given sequence of the vital functions. In the special case of just breath ($b$) and speech ($sp$) as players, consider the sequence ($sp, b$). In the entering case, speech’s marginal contribution has to be calculated with respect to the empty set. In the case of leaving, one calculates speech’s marginal contribution with respect to both vital functions. Therefore, AÄ 2.1.4 does not mention both of them without a purpose. In general, AÄ 2.1.4 and the KauU 2.14 do not reflect the Shapley value. Instead, they hint at the payoffs relating to one specific entering sequence, starting from the empty set, and one specific leaving sequence, starting from the grand coalition. The Shapley value for the case of all five vital functions is calculated in Appendix (Section B).

C. The axiomatic approach

For two players, the Shapley value fulfills the following axioms:

Additivity axiom: The sum of the Shapley values equals the worth of the grand coalition, i.e.,
(3) \( Sh_1 + Sh_2 = w(1,2) \)

This property means that (i) all the players “work together”, i.e., the grand coalition forms, and that (ii) the Shapley value distributes the worth of the grand coalition among the players.

Equal-damage axiom: If player 1 withdraws\(^{45}\) from the game, the damage to player 2 in terms of his Shapley payoff equals the damage that player 1 suffers should player 2 withdraw, i.e.,

(4) \( Sh_2 - w(2) = Sh_1 - w(1) \)

Consider the left side of the equation. If player 1 withdraws, player 2 does not obtain the Shapley value \( Sh_2 \) anymore, but the Shapley value of the game of which he is now the only player. In that game he obtains the worth \( w(2) \) of his one-man coalition.

Equations (3) and (4) lead to the Shapley values in equations (1) and (2) above where \( w(\emptyset) = 0 \) should be noted.\(^{46}\) Cooperative game theorists then say that the

\( ^{45} \)Withdrawal means that the player set is reduced by withdrawing players and that the worths of the remaining players stay the same.

\( ^{46} \)For more than two players, the corresponding equality reads \( \sum_{i \in N} Sh_i = w(N) \) (compare equation (3)) and the following version of equation (4): Consider any subset \( K \) of \( N \). On the basis of \( K \) as the new grand coalition, a \( K \)-game can be defined where the coalitions in that game have the same worth as in the original game. Consider two players \( i \) and \( j \) that are members of \( K \). If player \( i \) withdraws from the \( K \)-game, player \( j \)’s change in his Shapley payoff equals the change that player \( i \) endures should player \( j \) withdraw.
axioms expressed by equations (3) and (4) axiomatize the Shapley value. This means that the Shapley value in its algorithmic form (see section III B) fulfills these axioms and that there is no value different from the Shapley value that also fulfills these axioms. This particular axiomatization has been introduced by Myerson (1980).

Myerson’s axiom is related to the threat of withdrawal addressed in section II C. One may object that the threat uttered by breath (b) is more serious than the threat uttered by speech (sp). Indeed, BĀU 6.1\textsuperscript{47} and ChU 5.1 can be expressed by the inequality

\begin{equation}
(5) \quad w(si, h, m, b) > w(sp, si, h, m)
\end{equation}

or, equivalently,

\begin{equation}
(6) \quad w(sp, si, ..., b) - w(si, ..., b) < w(sp, ..., m, b) - w(sp, ..., m)
\end{equation}

The first inequality says that the body can get up in the presence of breath even if speech is not present, but not the other way around. The second inequality is equivalent and says that the marginal contribution of speech (left side) is smaller than the marginal contribution of breath (right sight). Or, differently put, the damage of withdrawal that breath can inflict in terms of worth is larger than the corresponding damage that speech or the other vital functions can inflict.

\textsuperscript{47} For the present purpose, it is a minor aspect that BĀU 6.1 enumerates six vital functions, among them also semen.
At first sight, this inequality seems to contradict equation (4), which can be rewritten in the following manner:

\[
(7) \quad Sh_b(w \text{ with all players}) - Sh_b(w \text{ with all players except } sp) \\
= Sh_{sp}(w \text{ with all players}) - Sh_{sp}(w \text{ with all players except } b)
\]

How can it be explained that breath’s leaving the body exerts such great damage as seen on the right-hand side of equation (6), but that the threat of withdrawal is balanced by equation (7)?

This seeming puzzle is “solved” in BĀU 6.13 where breath tells the other vital functions: “If that’s so [i.e., if I, leaving the body, can exert more damage than you], offer a tribute to me.” Apparently, the tribute is a positive entity. After they reply with “We will,” breath’s Shapley value includes the bali. Now, after having turned over the tribute to breath within the body, i.e., in the grand coalition, speech does not suffer more from breath’s leaving the body than breath would suffer from speech’s exit.

The mechanism that is at work here has been explained by the sociologist Emerson (1962). He presented a simple and intriguing theory of power and dependence. According to him, whenever a person is more dependent on another one, the relation is unbalanced and calls for “balancing operations”. It is best to illustrate this by the following example taken from Emerson’s paper. Consider two

\footnote{48 For other examples, see Wiese (2009).}
children A and B. They take turns in playing their favorite games. Their relationship is balanced. Now, one of these two children (child B, say) finds another playing buddy C. B is therefore less dependent on A than before and the relationship of A and B has become unbalanced. As a consequence, B can impose her favorite game on A more often than before. While B still has available the buddy C, not available to A, the relationship between B and A has become balanced once again because A gives in to B’s wishes more often than before. As a consequence, under different terms, the relationship has become balanced once again. In sum, Emerson has convincingly argued that social-exchange situations tend to be “balanced” in the long run.

IV. Conclusion

While the Āranyakas and the utilized Upaniṣads (being post-Vedic, but pre-classical texts) are normally considered to deal with esoteric, religious, and philosophical matters, Black (2007) focuses on the social questions and questions of power that are also involved. The thesis of this paper is that in some of its versions the ancient Indian motif of the contest among the vital functions employs generalizable procedures and that this was obvious to the commentators. In contrast, Aesop’s related fable belongs to what I have termed idiosyncratic approaches. I am not aware of any pre-modern solutions to the problem of superiority that were developed outside India and proceed along these generalizable lines.

Turning to pre-modern Indian texts on the problem of superiority, the controversy about daiva versus puruṣakāra known especially from the Mahābhārata
comes to mind. MBh XIII.6 deals with the question of whether divine or human activity is superior. MBh XIII.6.7 presents the following simile:

\begin{quote}
\textit{yathā bījaṁ vinā kṣetram uptaṁ bhavati nisphalam}
\textit{tathā puruṣakāreṇa vinā daivaṁ na sidhyati}
\end{quote}

Just as seed will be fruitlessly sown without a field, so ‘divine [power]’ will not succeed without human activity.

Here, the idea of “where would you be without me” is clearly present. In this example, let $N = \{b, kṣ\}$, with $b$ for $bīja$ and $kṣ$ for $kṣetra$, and let the one-player worths be given by $w(b) = w(kṣ) = 0$. Then the Shapley values for $bīja$ and $kṣetra$ are the same and reflect the idea that both ‘divine [power]’ and human activity are necessary for success.

A second, but more difficult example, might be found in the Arthaśāstra. In the framework of the seven-membered theory of a state, Kauṭilya (KAŚ 6.1.1) enumerates:

\begin{quote}
\textit{svāmyamātyajanapadadurgakośadaṇḍamitrāṇi prakṛtayāḥ}
\end{quote}

Lord, minister, countryside, fort, treasury, army, and ally are the constituent elements.

\begin{enumerate}
\item Slaje (1998) presents a translation and detailed discussion of MBh XIII.6.
\item Slaje (1998, p. 31).
\item In YSm I.347 the fact that a chariot cannot move with only one wheel is aduced as a simile for the same problem.
\item However, according to YSm I.68-69, when a brother-in-law is employed to make a sonless married woman pregnant, a “son born according to this procedure belongs to the owner of the field” (Olivelle (2019)), i.e., the husband.
\item Olivelle (2013, p. 271).
\end{enumerate}
The constituent elements enumerated here are adduced in this specific order for a certain reason: Kauṭilya argues in detail why, in the order given above, “a calamity affecting each previous one is more serious”.\textsuperscript{54} If we, somewhat loosely, identify “a member withdraws” with “a calamity affects a member”, Kauṭilya hints at the approach involving alternating withdrawal here.

It seems that the generalizable procedures advocated in the above Āranyakas and Upaniṣads were not so well understood by later readers that their use would automatically have come to (their) mind. Thus, further examples for the application of these generalizable procedures are difficult to find.

However, various superiority problems without the application of the generalizable procedures demonstrated in the late Vedic literature can be found easily. Just consider the Ṛgvedic Hymn of the Man (\textit{puruṣasūkta}) or Manu on the rank order of creatures (MDh 1.96-97).\textsuperscript{55}

Second, there is a whole class of superiority questions in the Upaniṣadic literature that are “solved” by similar mechanisms. For example, some item A is superior to another item B if

- A is “the essence of” B as in \textit{prthivyā āpo rasah}\textsuperscript{56} (“the essence of the earth is the waters”)\textsuperscript{57}

\textsuperscript{54} Olivelle (2013, p. 331).
\textsuperscript{55} Taking the Indian case as a starting point, Dumont (1980) analyzes hierarchy and considers man as “homo hierarchicus”. Halbfass (1991, pp. 347–405) discusses the pre-modern Indian history of thought of varṇa.
\textsuperscript{56} ChU 1.1.2
\textsuperscript{57} Olivelle (1998, p. 171).
• A is “higher than” B as in manasas tu parā buddhiḥ⁵⁸ (“higher than the mind is the intellect”)⁵⁹
• B is “woven back and forth on” A as in kasminn u khalu prajāpatilokā otāś ca protāś ca⁶⁰ (“On what, then, are the worlds of Prajāpati woven back and forth?”)⁶¹

Although these mechanisms are similar in that B rests on A/B is lower than A etc., I argue that they are not truly generalizable. After all, rather specific arguments (not given in the text) would be needed in order to justify why “the worlds of Prajāpati … are woven back and forth on … the worlds of brahma”⁶². Similarly, what specific factor might make “the intellect … higher than the mind”⁶³? However, a certain closeness of the ideas presented here and those underlying alternating withdrawal must not be denied. In fact, if the worlds of Prajāpati are woven back and forth on the worlds of brahman, it seems that the former would be “nowhere” without the latter and in that sense the latter’s threat of withdrawal should indeed be very serious.

Returning to the main topic of this article, I have shown that the approach of singly leaving or entering described in pre-modern Indian texts gets close to the algorithmic definition of the Shapley value. Furthermore, the approach involving alternating withdrawal is not far from Myerson’s axiomatic definition of the

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⁵⁸ KU 3.10
⁶⁰ BĀU 3.6
Shapley value. I have attempted to show in which respect the Indian thinkers would have had to take a few extra steps if they were to arrive at the Shapley value, defined algorithmically or axiomatically. The main difference is this: The Shapley value produces numerical figures, whereas in the Indian context superiority is only about rank order.

One may, of course, surmise that arguments of the sort “where would you be without me” are commonplace in mankind. In modern times, Emerson argued for balancing operations that bring initially unbalanced social situations into balance. In the contest of the vital functions, the *bali* serves as such a “balancing mechanism”. The balanced situation itself is implicit in the Shapley value. However, it was only Myerson who realized this property.

When the economist Myerson provided another axiomatization for the Shapley value of 1953 in 1980, he was not aware of the paper by the sociologist Emerson published already in 1962. The latter, for his part, did not acknowledge the Shapley value. Not surprisingly, none of these modern-day scholars took their Indian forerunners into account.

### Appendix

For the player set $N = \{sp, si, h, m, b\}$ and the coalition of vital functions other than breath $V = \{sp, si, h, m\}$, assume the coalition function $w$ defined by $w(v) \geq 0$ for all $v \in N$ and
$w(K) = \begin{cases} 
\sum_{v \in V \cap \overline{K}} w(v), & b \text{ not in } K \\
 w(b) + \alpha \sum_{v \in V \cap K} w(v), & b \text{ in } K 
\end{cases}$

for every subset $K$ of $N$. Let $\alpha \geq 1$ which amounts to the superadditivity of $w$, i.e., $w(N) \geq w(K) + w(N \setminus K)$ for every subset $K$ of $N$. For this coalition function, the following assertions hold:

A) Along the sequence $(sp, si, h, m, b)$ the marginal contributions are
- $w(v)$ for each vital function $v$ from $V$ and
- $w(b) + (\alpha - 1) \sum_{v \in V} w(v)$ for $b$.

B) Along the sequence $(sp, si, h, m, b)$ the marginal damages (or: along the sequence $(b, m, h, si, sp)$ the marginal contributions) are
- $\alpha w(v)$ for each vital function $v$ from $V$ and
- $w(b)$ for $b$.

If $\alpha$ takes the special value of 1, the payoffs are the same for the entering and the leaving sequence.

C) The Shapley values for the above coalition function are:
- $Sh_v = \frac{1 + \alpha}{2} w(v)$ for the vital functions $v \in V$
- $Sh_b = w(b) + \frac{\alpha - 1}{2} \sum_{v \in V} w(v)$ for breath.

Proof:
Speech (and any other vital functions from $V$) has the same chance of entering before breath (with the marginal contribution being $w(sp)$) or entering after breath (with the marginal contribution being $\alpha w(sp)$). This explains the Shapley
values for the vital functions from $V$. The Shapley value distributes the worth of the grand coalition among the players. Hence, breath gets the rest.

D) If speech withdraws from the game, the Shapley values for the remaining players are

- $\text{Sh}_v = \frac{1+\alpha}{2} w(v)$ for the vital functions $v \in \{si, h, m\}$
- $\text{Sh}_b = w(b) + \frac{\alpha-1}{2} \sum_{v \in \{si,h,m\}} w(v)$ for breath.

Proof: If speech has withdrawn, the other players’ payoffs are derived similar as in C.

E) If breath withdraws from the original game, the Shapley values are $\text{Sh}_v = w(v)$ for the vital functions $v \in \{sp, si, h, m\}$.

Proof: If breath has withdrawn, the vital functions $sp, si, h, m$ receive their one-man worth in each sequence and hence as the Shapley value.

F) Before the contest, each vital function has obtained $\frac{1}{5}$ of the body’s proper functioning of $w(b) + \alpha \sum_{v \in V} w(v)$. After the contest, breath obtains the bali, which is implicitly defined by

$$\frac{w(b) + \alpha \sum_{v \in V} w(v)}{5} + \text{bali} = \text{Sh}_b = w(b) + \frac{\alpha - 1}{2} \sum_{v \in V} w(v)$$

and hence explicitly by

$$\text{bali} = \frac{4}{5} w(b) + \left[\frac{3}{10} \alpha - \frac{1}{2}\right] \sum_{v \in V} w(v)$$
By solving \( bali > 0 \) for \( \alpha \), the tribute is found to be positive if 
\[
\alpha > \frac{5}{3} - \frac{8}{3} \frac{w(b)}{\sum_{v \in V} w(v)}
\]
holds.

**Abbreviations**

AĀ Aitareya Āraṇyaka (Keith 1909)
AĀ_Sā Commentary on Aitareya Āraṇyaka by Śāyaṇa (Deo 1992)
BĀU Brhadāraṇyaka Upaniṣad (Olivelle 1998)
BĀU_Ś Commentary on Brhadāraṇyaka Upaniṣad by Śaṅkara (Shastri 1986)
ChU Chāndogya Upaniṣad (Olivelle 1998)
ChU_Ś Commentary on Chāndogya Upaniṣad by Śaṅkara (Shastri 1982)
KauU Kauṭītaki Upaniṣad (Olivelle 1998)
KAŚ Kauṭiliya's Arthaśāstra (Kangle 1969)
KU Kaṭha Upaniṣad (Olivelle 1998)
MBh Mahābhārata (Sukthankar 1927-1959)
MDh Mānava Dharmaśāstra (Olivelle 2005)
PU Praśna Upaniṣad (Olivelle 1998)
PW Sanskrit-Wörterbuch in kürzerer Fassung (Böhtlingk 2009)
ŚĀ Śāṅkhāyana Āraṇyaka (Apte 1922)
ŚB Śatapatha Brāhmaṇa (Weber 1855)
YSm Yājñavalkya Smṛti (Olivelle 2019)
l. line
ll. lines
Publication bibliography


