Decision theory and probability theory
Pascal's wager and pre-modern Indian lotteries

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Overview

Probability

Roughly, a probability p is defined on subsets (called events) of a set E or on propositions

Twofold interpretation:

- frequency of occurrence (tossing a die, ...)
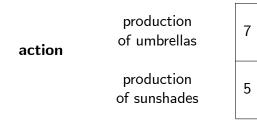- degree of belief without statistical background

Two properties:

- normalization:
  for all events/propositions F: $0 = p(\varnothing) \leq p(F) \leq 1 = p(E)$
- additivity:
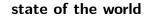 for mutually exclusive events/propositions F and G:
  $p(F \cup G) = p(F) + p(G)$

Decision theory

action	production of umbrellas	7
	production of sunshades	5

- a set of actions A
- a set of consequences C (profits)

Decision theory with states of the world

state of the world

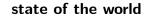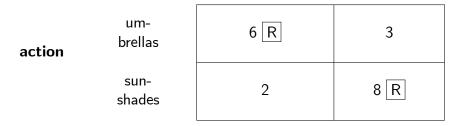|		bad weather	good w.
action	um-brellas	6	3
	sun-shades	2	8

- a set of states of the world W
- $a \in A$ and $w \in W$ determine consequence

Decision theory: best responses

state of the world

		bad weather	good w.
action	um-brellas	6 R	3
	sun-shades	2	8 R

R = best response to state of the world

Decision theory: lotteries

state of the world

	bad w., $\frac{1}{3}$	good w., $\frac{2}{3}$
um-brellas	6	3
sun-shades	2	8

action

$$L_{\text{umbrella}} = \left[6, 3; \frac{1}{3}, \frac{2}{3}\right]$$

$$E\left(L_{\text{umbrella}}\right) = \frac{1}{3} \cdot 6 + \frac{2}{3} \cdot 3 = 4$$

European and Indian probability and decision theory

Ian Hacking (The emergence of probability, 2006):

> *Europe:*
> *The decade around 1660 is the birthtime of probability. ...*
> *Pascal made the first application of probabilistic reasoning*
> *to problems other than games of chance, and thereby*
> *invented decision theory.*
> *India:*
> *... a good deal of Indian probability lore is at present*
> *unknown to us.*

—> C. K. Raju (Probability in Ancient India, 2010)

Pascal's wager

No formal argument, but reconstructible.

		state of the world	
		God exists	God does not exist
action	belief in God	eternal bliss	pious life in vain
	no belief in God	eternal damnation	hedonist life

Hacking: Pascal presents valid arguments for believing in God:

- If a pious life might still be a happy one, then belief in God is a dominant action.
- If a hedonist life is better than the pious one (in God's absence), belief in God is the safer action.

Hitopadeśa: 800-950 CE

> *A wise man should think about knowledge and money as if he were immune to old age and death; but he should perform his duties as if Death had already seized him by the hair.*

With respect to the first decision, we propose the actions

$$
\begin{array}{rcl}
\text{invest} & = & \text{save money/increase knowledge,} \\
\text{do not invest} & = & \text{spend money/do not labour for education}
\end{array}
$$

and the states of the world

$$
\begin{array}{c}
\text{short life,} \\
\text{long life.}
\end{array}
$$

Decision models in the Hitopadeśa

Investment decision

		state of the world	
		short life	long life
action	invest	no use for cap./knowl.	long use of cap./knowl. R
	do not invest	enjoym. money/leis. R	material pov./ spiritual pov. **VB**

R = best response to state of the world
VB = very bad outcome of poverty (material and spiritual)

state of the world

		short life	long life
action	dharma now	good karma $\boxed{R}$	good karma, little enjoym. in youth
	dharma later	bad karma **VB**	good karma, some enjoym. $\boxed{R}$ in youth

No contradiction!

> *A wise man should think about knowledge and money as if he were immune to old age and death; but he should perform his duties as if Death had already seized him by the hair.*

Action versus state of the world, effort versus fate

Hitopadeśa (800-950 CE):

> *Just as a cart cannot move on one wheel,*
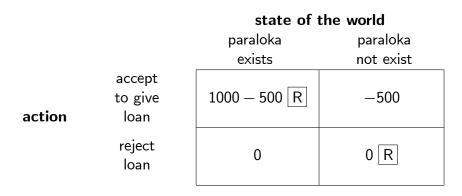> *so fate itself cannot be fulfilled without human effort.*

Mahābhārata (300-500 CE):

> *Just as seed will be fruitlessly sown without a field,*
> *so 'divine [power]' will not succeed without human activity.*

The loan lottery

The birth-story of Brahma (4th c. CE):

- a king does not believe in the afterworld (*paraloka*) and holds other *Cārvāka* views
- the former Buddha = a Brahma deity (a god)
    - is convinced that good and bad deeds have happy and unhappy results in the next life,
    - feels compassion, and
    - is intent on converting the king to virtuous attitudes and behavior.

The loan lottery

The king is not convinced and comes up with a clever proposal:
"Well. great seer!

> If the next world is not a bogey man for children,
> and if you think I should believe in it,
> then give me five hundred nishkas
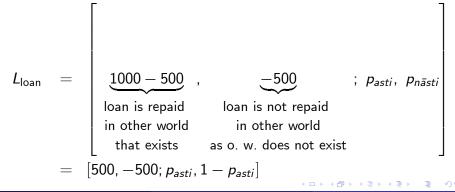> and I'll return you a thousand in another life!"

The loan lottery

	state of the world	
	paraloka exists	paraloka not exist
action accept to give loan	$1000 - 500$ R	-500
reject loan	0	0 R

The loan lottery

If (!) the Buddha-to-be uses
- probabilities and
- expected value

then ...

$$L_{\text{loan}} = \left[\underbrace{1000 - 500}_{\substack{\text{loan is repaid} \\ \text{in other world} \\ \text{that exists}}} , \underbrace{-500}_{\substack{\text{loan is not repaid} \\ \text{in other world} \\ \text{as o. w. does not exist}}} ; p_{asti}, p_{n\bar{a}sti} \right]$$

$$= [500, -500; p_{asti}, 1 - p_{asti}]$$

The loan lottery

$$
\begin{aligned}
E\left(L_{\text{loan}}\right) &= p_{asti} \cdot 500 + (1 - p_{asti}) \cdot (-500) \\
&= -\underbrace{500}_{\substack{\text{loan given} \\ \text{in both cases}}} + p_{asti} \cdot \underbrace{1000}_{\substack{\text{repayment} \\ \text{if o. w. exists}}} > 0
\end{aligned}
$$

if

$$
p_{asti} > \frac{500}{1000} = \frac{1}{2}
$$

This seems a good test of whether the god himself believes in the other world. If he assumes a probability larger than $\frac{1}{2}$, he should accept the lottery.
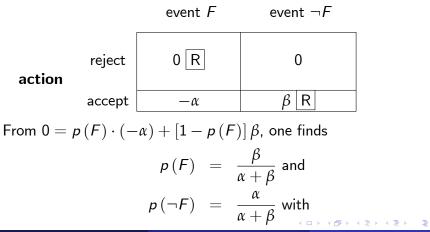
The loan lottery

Who would harrass you for a thousand nishkas
when you lie in hell, senseless, sick with pain,
brought there by your own actions
caused by the evil of your false views?

...

In the next world, where nihilists [nāstika] live
a thick darkness and icy wind tortures
people by tearing through their very bones.
What prudent man would go there to get money?''

Probability theory?

For any event F, $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta > 0$ can be found such that the agent is indifferent between rejecting and accepting:

state of the world

	event F	event $\neg F$
reject	0 R	0
accept	$-\alpha$	β R

action

From $0 = p(F) \cdot (-\alpha) + [1 - p(F)]\beta$, one finds

$$p(F) = \frac{\beta}{\alpha + \beta} \text{ and}$$

$$p(\neg F) = \frac{\alpha}{\alpha + \beta} \text{ with}$$

Conclusion

- Pascal may have been the inventor of decision theory for Europeans. Indian sources can claim priority by 1000 years or so.
- The lottery offered by the *Cārvāka* king may well be the world-first invention of using a lottery in order to find out about the strength of a decision maker's belief.
- Immodestly, one may try to defend the claim that the birth-story of Brahma shows the world-first quantitatively defined probability.

Normalization?

$[-\alpha, \beta]$ amounts to probability $p(F)$!

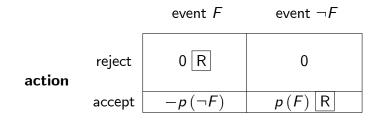By $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta > 0$, we find

$$0 \leq p(F) = \frac{\beta}{\alpha + \beta} \leq 1$$

- $p(F) = 0$ is compatible with indifference only for $\beta = 0$.
- $p(F) = 1$ is compatible with indifference only for $\alpha = 0$ (and $\beta > 0$).

Normalization?

If $[-\alpha, \beta]$ defines a lottery and hence the probability $\frac{\beta}{\alpha+\beta}$, the same holds for $[-5\alpha, 5\beta]$ or any multiplication with a non-zero constant, in particular with $\frac{1}{\alpha+\beta}$

Then, for any event F, $0 \leq p(F) \leq 1$ can be found such that the agent is indifferent between rejecting and accepting:

state of the world

	event F	event $\neg F$
reject	0 $\boxed{\text{R}}$	0
accept	$-p(\neg F)$	$p(F)$ $\boxed{\text{R}}$

action

Additivity?

- Three
    - events F, G, and H with
    - lotteries $[-p\,(\neg F)\,,p\,(F)]$, $[-p\,(\neg G)\,,p\,(G)]$, and $[-p\,(\neg H)\,,p\,(H)]$
- Events mutually exclusive and $F \cup G \cup H = E$ and hence $H = \neg\,(F \cup G)\,.$
- As in $p\,(F) + p\,(\neg F) = 1$, we obtain

$$p\,(F \cup G) + p\,(H) = 1$$

Additivity?

- Ramsey-type argument
    - Indifference between rejecting and accepting all three of them.
    - If event H materializes, neither F nor G come to pass, and the agent obtains the payoff

    $$-p\left(\neg H\right) + p\left(F\right) + p\left(G\right).$$

    - By indifference, we then find $0 = -p\left(\neg H\right) + p\left(F\right) + p\left(G\right)$ and

    $$p\left(F \cup G\right) \underset{p(F\cup G)+p(H)=1}{=} 1 - p\left(H\right)$$
    $$\underset{p(H)+p(\neg H)=1}{=} p\left(\neg H\right) \underset{0=-p(\neg H)+p(F)+p(G)}{=} p\left(F\right) + p\left(G\right)$$

- This confirms additivity.

The loan lottery

The god rejects the lottery:

> *"Even in this world, wealth seekers*
> *do not offer money to the wicked,*
> *nor to the greedy, fools or indolents.*
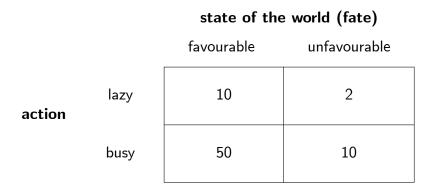> *For whatever goes there comes to ruin.*
>
> *But if they see someone who is modest,*
> *naturally calm and skilled in business,*
> *they will give him a loan, even without witnesses.*
> *For money entrusted to such a man brings reward.*
>
> *The same procedure for giving a loan*
> *should be used for the next world, king.*
> *But it would be improper to entrust money to you;*
> *for your conduct is corrupted by wicked views.*

One should not give up one's efforts, even when acknowledging the role of fate; without effort, one cannot obtain oil from sesame seeds.
And there is another verse on this:
Fortune gravitates towards eminant men who work hard; only cowards say it depends on fate.
Forget about fate and be a man–use your strength!
Then, if you don't succeed inspite of your efforts, what is there to blame?

Decision models in the Hitopadeśa

Fate and human effort

state of the world (fate)

		favourable	unfavourable
action	lazy	10	2
	busy	50	10

A payoff of 10: Then, if you don't succeed inspite of your efforts, what is there to blame?

Understandably, the king is convinced:

> *"My mind almost runs wild with fear*
> *at learning of the punishments in hell.*
> *It practically burns with blazing thoughts*
> *regarding my plight on meeting that fate.*
>
> *Shortsightedly I trod the wrong path,*
> *my mind destroyed by evil views.*
> *Be then my path, recourse of the good!*
> *By my resort and refuge, sage!*
>
> *As you dispelled the darkness of my views*
> *like the rising sun dispels night,*
> *so tell me, seer, the path I should follow*
> *to avoid a bad rebirth after this life."*

The god is prepared to give this advice:

Conquer vice, so difficult to vanquish!
Pass beyond greed, so difficult to overcome!
You will thus reach the gleaming gold-gated city
of the king of heavens, ablaze with fine gems.

May your mind, which once praised evil views,
firmly cherish the creeds valued by good men.
Abandon immoral beliefs proclaimed
by those eager to pleasure fools.

...

With glory as its banner,
pity as its retinue
and tranquility as
its lofty flag, king,
if you travel in this chariot
glittering with wisdom
to benefit others and yourself,
you will certainly not enter hell.

Lotteries against and in favor of God's existence
The hell lottery IV

$$L_{C\bar{a}rv\bar{a}ka} = \left[\underbrace{\text{pleasures in this life, but hell with endless horrors,}}_{\text{other world exists}} \right.$$

$$\left. \underbrace{\text{pleasures in this life}}_{\text{other world does not exist}} ; \ p_{asti}, \ p_{n\bar{a}sti} \right]$$

$$= [-100\,000, \ 10; p_{asti}, p_{n\bar{a}sti}]$$

with expected payoff

$$E\left(L_{C\bar{a}rv\bar{a}ka}\right) = p_{asti} \cdot (-100\,000) + (1 - p_{asti}) \cdot 10$$
$$= 10 - 100\,010\,p_{asti}$$

$$L_{\text{virtue}} = [\text{life of pity, tranquility, and wisdom, no hell; } 1]$$

$$E\left(L_{\text{virtue}}\right) = 2 \text{ (for example)}$$

$$E\left(L_{\text{virtue}}\right) > E\left(L_{C\bar{a}rv\bar{a}ka}\right)$$

$$\Leftrightarrow \quad p_{asti} > \frac{8}{100010} \approx \frac{8}{10000} = 0.0008$$

Thus, the king may not (really) believe in the other world, but prefers to play it safe.

Thus, Blaise Pascal was not the first to present an argument for believing in God.