WHY *PRĀṆA* IS THE MOST EXCELLENT AMONG THE VITAL FUNCTIONS,
OR: THE SHAPLEY VALUE IN THE *UPANIṢADS*

Harald Wiese, University of Leipzig

- Introduction

- The contest among vital functions

- The Shapley value

- Conclusion

INTRODUCTION I

- Comparisons of the natural body with a political one are commonplace.

- Śukra Nīti: The kingdom as an organism of seven limbs, where the king is the head, the Minister the eye …

- Here: rank order disputes between vital functions (breath, speech, …) (similar to Aesop's fable).

- Rank order disputes may be resolved by

  - ad-hoc (idiosyncratic, non-generalizable) arguments

  - systematic (non-idiosyncratic, generalizable) arguments,

INTRODUCTION II

Idiosyncratic solutions in the *Upaniṣad*s:

- In BĀU 1.5.21, death succeeds in capturing the vital functions with the exception of breath. This fact shows breath's superiority.

- In BĀU 1.3.1-7, the vital functions (speech, breath, sight, hearing, mind, breath in the mouth) have to sing the High Chant. The demons "riddle with evil" the functions from speech to mind, but they fail to do the same with breath in the mouth.

- In ChU 4.3.3, breath is characterized as the "gatherer" into whom the other vital functions pass when a man sleeps.

INTRODUCTION III

Non-idiosyncratic = generalizable to other problems, in particular

- some method (*prakāra*),

- which is teachable (*prakāropadeśaḥ*),

- which is applicable beyond the actual application (*cetanāvanta iva puruṣāḥ*), and

- which serves to avoid struggle or competition (*spardhānivāranārtham*).

Two claims made here:

- Systematic manners to resolve rank order disputes clearly present in the *Upaniṣad*s

- These manners foreshadow the Shapley value (1953)

THE CONTEST AMONG VITAL FUNCTIONS: SEQUENTIAL I

Aitareyāraṇyaka:

They strove together, saying, 'I am the hymn, I am the hymn.'

They said, 'Come, let us leave this body,

then that one of us at whose departure the body falls, will be the hymn.'

Sāyaṇa comments:

tāḥ spārdhamānā devātāḥ **spardhānivāranārthaṃ** *samayaviśeṣaṃ parasparam abruvan*

In order to avoid this competition, these competing goddesses came to a particular understanding.

THE CONTEST AMONG VITAL FUNCTIONS: SEQUENTIAL II

Aitareyāraṇyaka:

The vital functions leave the body one after another:

vāg udakrāmad avadann aśnan pibann astaiva

Speech went forth, yet (the body) remained, speechless, eating and drinking.

The sequence of leaving is speech, sight, hearing, mind, and finally breath:

prāṇa udakrāmat tatprāṇa utkrānte 'padyata

Breath went forth, when breath went out, (the body) fell.

THE CONTEST AMONG VITAL FUNCTIONS: SEQUENTIAL III

Aitareyāraṇyaka:

Then, they start quarreling again, but this time resolve on entering the body one after another. The sequence of entering is the same as before. The result is as expected:

prāṇaḥ prāviśat tat prāṇe prapanna udatiṣṭhat tat uktham abhavat

Breath entered, when breath entered, that [the body] arose, and that [breath] became the hymn.

THE CONTEST AMONG VITAL FUNCTIONS: WITHDRAWAL I

Bṛhadāraṇyaka Upaniṣad:

Once these vital functions (*prāṇa*) were arguing about who among them was the greatest.
So they went to *brahman* and asked: "Who is the most excellent of us?"
He replied: "The one, after whose departure you consider the body to be the worst off,
                is the most excellent among you."

So speech departed.
After spending a year away, it came back and asked: "How did you manage to live without me?"
They replied: "We lived as the dumb would, without speaking with speech,
            but breathing with the breath,
            seeing with the eye,
            hearing with the ear,
            thinking with the mind, and
            fathering with semen."
So speech reentered.

THE CONTEST AMONG VITAL FUNCTIONS: WITHDRAWAL II

Śaṅkara considers humans the most obvious contenders in such fights for superiority:

nanu katham idaṃ yuktaṃ **cetanāvanta iva puruṣā** *ahaṃśreṣṭhatāyai vivadanto 'nyonyaṃ spardherann iti*

How can this be logical that [the vital functions] compete against each other by arguing about who among them was the greatest, as reasonable humans [would].

THE CONTEST AMONG VITAL FUNCTIONS: WITHDRAWAL III

Bṛhadāraṇyaka Upaniṣad:

When breath is about to leave, the other vital functions beg:

mā bhagava utkramīḥ **|**
na vai śakyāmas tvadṛte jīvitum iti **|**
tasyo me baliṃ kuruteti **|**
tatheti **|**

"Lord, please do not depart!
We will not be able to live without you."
He told them: "If that's so, offer a tribute to me."
"We will," they replied.

THE CONTEST AMONG VITAL FUNCTIONS: WITHDRAWAL IV

Śatapatha Brāhmaṇa:

śreyase pāpīyān baliṃ hared vaiśyo vā rājñe baliṃ hared

an inferior brings tribute to his superior, or a man of the people brings tribute to the king

Śaṅkara:

atha hainaṃ vāgādayaḥ prāṇasya śreṣṭhatvaṃ kāryenāpādayanta āhur balim iva haranto rājñe viśaḥ

Speech and the rest, establishing, by their action, the *superiority* of Breath, said to him—making offerings like the people to their King.

THE CONTEST AMONG VITAL FUNCTIONS: WITHDRAWAL V

Śaṅkara on generalizability:

ayaṃ ca prāṇasamvādaḥ kalpito viduṣaḥ
*śreṣṭhaparīkṣaṇa***prakāropadeśaḥ |**
*anena hi **prakāreṇa** vidvān ko nu khalvatra śreṣṭha iti parīkṣaṇaṃ karoti* **|**

And this agreement of the vital functions is imagined by the wise as a teaching of a manner to test superiority.
For by this manner the wise performs the test of who, indeed, is the best here.

THE SHAPLEY VALUE

The generalizable approaches in the *Brāhmaṇa*s are related to the Shapley (1953) value from cooperative game theory:

- Possibilities of alternative groups of "players" to create "worth" are given.

- The problem is how much each individual player obtains.

The Shapley value admits two different types of definitions:

- an algorithmic one

- an axiomatic one.

THE SHAPLEY VALUE: ALGORITHMIC = SEQUENTIAL

Algorithmic definition of the Shapley value = a formula that builds on rank orders.

- For each rank order and each player, the contribution of that player (the difference he makes) is noted.

- Each player obains the average of his contributions.

Similarly, in the sequential approach,
the vital functions note the differences made by speech etc.

The difference made by breath is the largest and breath wins the competition.

THE SHAPLEY VALUE: AXIOMATIC, FOR EXAMPLE WITHDRAWAL I

Axiomatic approach to Shapley value = general rules of how payoffs are to be determined, for example

- All players together should get what they produce together.

- A player who does not make any difference should get nothing.

- If a player 1 withdraws from the game, another player 2's damage in terms of his Shapley payoff equals the damage that player 1 endures should player 2 withdraw, i.e., we have „where would you be without me" equality (Myerson 1980).

THE SHAPLEY VALUE: AXIOMATIC, FOR EXAMPLE WITHDRAWAL II

Breath's threat to withdraw carries more weight than the other function's threat!

True. But: When turning the tribute over to breath within the body, speech does not suffer more from breath's leaving the body than breath suffers from speech's exit.

CONCLUSION

- 650 BCE   Bṛhadāraṇyaka Upaniṣad

- 1953   Shapley value

- 1980   Myerson (without referring to the Upaniṣads)

THE SHAPLEY VALUE: AXIOMATIC APPROACH IV

Economic example:

- 1 seller

- 4 potential buyers

Are the buyers more dependent on the seller than the seller is on any particular buyer? After all, we have the inequality

$$(5') \qquad w(\text{seller with three buyers}) > w(\text{four buyers without seller})$$

Why, then, does the seller's threat of withdrawal not carry more weight than any particular buyer's threat of withdrawal?

- The seller obtains a very high price in case of 4 potential buyers and a slightly reduced price in case of 3 potential buyers. So one potential buyer's withdrawal would not do much damage to the seller.

- But this potential buyer's disutility caused by the seller's withdrawal is small also. In the presence of the seller, this buyer will have a small chance (1/4) of getting the item in question and will also have to pay a high price. Therefore, the buyer does not loose much if the seller withdraws and his chance of getting the item is reduced to zero.

THE SHAPLEY VALUE: COALITION FUNCTIONS

Cooperative game theory

- n players collected in a set $N = \{1,2, \dots, n\}$, here $n = 2$

- Any subset of N is called a coalition, here four subsets: $\emptyset, \{1\}, \{2\}, \{1,2\}$

- coalition function w

  To each coalition K, the coalition function attributes a "worth" $w(K)$ where $w(\emptyset) = 0$.

Specific example for vital functions:

- Each vital function v from set $N = \{sp, si, h, m, b\}$ creates the worth of its one-man coalition $w(v)$.

- Additionally, if breath is present, the worths of the other vital functions is increased by some factor $\alpha \geq 1$.

THE SHAPLEY VALUE: PAYOFFS

Aim: Specify payoffs $\varphi_1(w)$ and $\varphi_2(w)$ for the players

- algorithmic approach,
  for example by $\varphi_1(w) = w(1) + 5$ and $\varphi_2(w) = w(2) + 5$.

- axiomatic approach that suggests general rules of distribution,
  for example $\varphi_1(w) + \varphi_2(w) = w(1, 2)$

THE SHAPLEY VALUE: ALGORITHMIC APPROACH I

A player's marginal contribution

    = the difference a player makes

    = worth of a coalition with player minus the worth without him

    player 1 has two marginal contributions,

        the first with respect to the empty set $\emptyset$: $w(1) - w(\emptyset))$,

        the second with respect to $\{2\}$: $w(1,2) - w(2))$.

Player 1's Shapley value is the average of his marginal contributions, taken over all sequences (rank orders) of the two players:

(1) $Sh_1 = \frac{1}{2}\big(w(1) - w(\emptyset)\big) + \frac{1}{2}\big(w(1,2) - w(2)\big)$

and

(2) $Sh_2 = \frac{1}{2}\big(w(2) - w(\emptyset)\big) + \frac{1}{2}\big(w(1,2) - w(2)\big)$

THE SHAPLEY VALUE: ALGORITHMIC APPROACH II

Special case: only speech (sp) and breath (b).

Marginal contributions for the entering sequence (sp, b) are

- $w(sp)$ for speech and
- $w(b) + (\alpha - 1)w(sp)$ for b.

Breath is superior to speech if his payoff is higher, i.e., if $w(b) > (2 - \alpha)w(sp)$ holds.
Thus breath's superiority, claimed by AĀ 2.1.4 (in the case of all five vital functions), is true

- if $w(b)$ is large relative to $w(sp)$ and
- if the "productivity" of speech is enhanced by breath's presence (large α).

AĀ 2.1.4 also covers the leaving sequence (sp, b).
Payoffs for the entering and the leaving sequence differ, but

- Payoffs for the entering sequence (b, sp) equals
- payoffs for the leaving sequence (sp, b).

Shapley value = average

THE SHAPLEY VALUE: AXIOMATIC APPROACH I

- The sum of the Shapley values equals the worth of the grand coalition, i.e., we have

  (3) $Sh_1 + Sh_2 = w(1,2)$

  in the case of two players.

- Any player whose marginal contribution is zero with respect to every coalition obtains the Shapley value of zero.

- The payoffs do not depend on the players' names.

- If a player 1 withdraws from the game, another player 2's damage in terms of his Shapley payoff equals the damage that player 1 endures should player 2 withdraw, i.e., we have

  (4) $Sh_2 - w(2) = Sh_1 - w(1)$

  in the case of two players. (Myerson 1980)

Equations (3) and (4) lead to the Shapley values in equations (1) and (2).

THE SHAPLEY VALUE: AXIOMATIC APPROACH II

How is Myerson's axiom of balanced contributions (see eq. (4)) related to the threat of withdrawal?

BĀU 6.1 can be translated into our framework by the inequalities

$$(5) \qquad w(si, h, m, b) > w(sp, si, h, m)$$

or, equivalently,

$$(6) \qquad w(sp, si, \dots, b) - w(si, \dots, b) < w(sp, \dots, m, b) - w(sp, \dots, m)$$

(5) says: The body can get up in the presence of breath even if speech is not present, but not the other way around.

(6) says: The marginal contribution of speech (left side) is smaller than the marginal contribution of the breath (right sight). Or, differently put, the damage of withdrawal that breath can inflict (in terms of worth!) is larger than the corresponding damage that speech or the other vital functions can inflict.

THE SHAPLEY VALUE: AXIOMATIC APPROACH III

These inequalities do not contradict eq. (4) which we rewrite in this manner:

(7) $\quad Sh_b(w \text{ with all players}) - Sh_b(w \text{ with all players except } sp)$

$\quad\quad\quad = Sh_{sp}(w \text{ with all players}) - Sh_{sp}(w \text{ with all players except } b)$

Puzzle? No.

When turning the tribute over to breath within the body (in the grand coalition), speech does not suffer more from breath's leaving the body than breath suffers from speech's exit.

THE SHAPLEY VALUE: AXIOMATIC APPROACH IV

Economic example:

- 1 seller

- 4 potential buyers

Are the buyers more dependent on the seller than the seller is on any particular buyer? After all, we have the inequality

$$(5') \qquad w(\text{seller with three buyers}) > w(\text{four buyers without seller})$$

Why, then, does the seller's threat of withdrawal not carry more weight than any particular buyer's threat of withdrawal?

- The seller obtains a very high price in case of 4 potential buyers and a slightly reduced price in case of 3 potential buyers. So one potential buyer's withdrawal would not do much damage to the seller.

- But this potential buyer's disutility caused by the seller's withdrawal is small also. In the presence of the seller, this buyer will have a small chance (1/4) of getting the item in question and will also have to pay a high price. Therefore, the buyer does not loose much if the seller withdraws and his chance of getting the item is reduced to zero.

CONCLUSION

- 650 BCE   Bṛhadāraṇyaka Upaniṣad

- 1953   Shapley value

- 1962   Emerson (without referring to Shapley or to the Upaniṣads):
  Whenever a person is more dependent on another one (or the second has more power over the first than the other way around), the relation is unbalanced and calls for "balancing operations":
  a high price or a tribute, respectively.

- 1980   Myerson (without referring to Emerson or to the Upaniṣads)