The mandala theory and four-king chess

Harald Wiese

University of Leipzig

Course overview

Part B. Game theory

- Chapter VI:
  Noncooperative games
- Chapter VII:
  Backward induction
- Chapter VIII:
  Ordeals
- Chapter IX:
  Judicial wagers
- Chapter X:
  Indian Principal-Agent Theory
- **Chapter XI:**
  The mandala theory

Chapter VIII: The mandala theory

1. Introduction
2. Four-king chess with dice
3. Kauṭilya on war and peace
4. Modelling Kauṭilya's maṇḍala
5. Fighting involving friends and enemies
   1. Neighborhood structures and fighting structures
   2. Friends and enemies
   3. Strategies
   4. Payoffs
   5. Equilibria
6. Conclusions

Introduction

- Chess seems to have originated in India, perhaps in the fifth century C.E. Chess boards used to have 8 lines and 8 columns early on and each of the two players commanded 16 pieces.

- The Sanskrit term for that game is *caturaṅga* which means "[boardgame] with four parts". The four parts refer to four different kind of troops in real life or pieces used in Indian chess:

  - elephants
  - chariots
  - horses, and
  - infantry.

- Apart from these four pieces, there was a king.

Introduction

- A variant with four armies each consisting of 8 pieces was also in use. It is also called *caturaṅga* or, tellingly, *catūrājī* ("[boardgame] with four kings").

- This four-king chess seems to have been played by four players, but perhaps also by two players each commanding two armies.

- In some versions, dice are used to tell the players which pieces to move.

- Most later chess historians surmise that four-king chess has developed out of two-king chess.

Introduction

- Throwing dice and engaging in logical reasoning are absolutely heterogenous so that an invention with such an irreconcilable contradiction should be considered a psychological impossibility.

- Chariots stopped to play an important role in Indian warfare and the chess piece "chariot" was eventually replaced by the chess piece "boat". With the old formation not in place any more, *caturaṅga* ("with four parts") was misunderstood in the sense of "four parties" and arising from that misunderstanding four-king chess was constructed.

Introduction

Maṇḍala theory

- The Sanskrit word *maṇḍala* means "circle, wheel".
- In *Kauṭilya*'s *Arthaśāstra* it refers to a ringlike structure of countries.
  - A king should envision his country at the center. This king is then called *vijigīṣu* or "seeker after conquest".
  - *Kauṭilya*'s main theoretical idea:
    - War can only be waged with (direct) neighbors (local warfare).
    - Therefore, neighbors tend to be enemies.
    - Also, since these enemies might be attacked from the other side, the enemies of enemies tend to be friends.

   = *Kauṭilya*n conjecture

Introduction

- Most chess historians agree that chess is a war game, used for didactic purposes.
- Why not consider four-king chess a reflection of a simple *maṇḍala* model with four parties?
- So far, no formal model of neither four-king chess nor *maṇḍala* theory.
- Claim: The *maṇḍala* model and four-king chess show striking similarities. Four-king chess was used in order to teach Indian kings and princes some of the strategic knowledge inherent in the *maṇḍala* model.

Four-king chess with dice

Location on chess board

Tithitattva (probably 16th century CE):

> "Having drawn the eight fields, but having placed in clockwise fashion the red in front, the green army in the south, and, Pārtha, the yellow one in the west and the black army in the north."
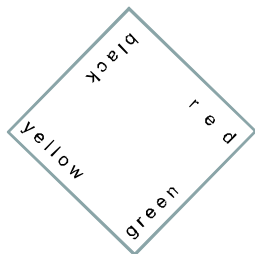
Four-king chess with dice

Using the dice to determine the pieces

Tithitattva:

> "With a five [on the dice] a pawn and a king move, with
> just a four the elephant, but with a three the horse, Pārtha,
> the boat, again, with a two."

Lüders thinks that a *pāśaka* was used whose four sides were indicated
by the numbers 5 through 2.

Tithitattva:

> "When a king, Pārtha, ascends the throne of an ally, then
> the so-called [procedure] siṃhāsana [takes effect] and he
> leads his [the ally's] whole army."

Mānasollāsa:

> "Where the four players [sit], the troops are positioned. The
> white and the red army are to be placed alternatively."

"Good and bad policy pertain to the human realm, while good and bad fortune pertain to the divine realm. Divine and human activity, indeed, makes the world run. The divine consists of what is caused by an invisible agent. Of this, attaining a desirable result is good fortune, while attaining an undesirable result is bad fortune. The human consists of what is caused by a visible agent. Of this, the success of enterprise and security is good policy, while their failure is bad policy. This is within the range of thought, whereas the divine is beyond the range of thought."

"Between these two, when one questions: 'Is this an advantage or not?' 'Is this a disadvantage or not?' 'Is this advantage actually a disadvantage?' 'Is this disadvantage actually an advantage?'–it is uncertainty. There is uncertainty as to whether it is an advantage or not to rouse up an ally of the foe. There is uncertainty as to whether it is a disadvantage or not to entice troops of the foe with money and honors."

"The seeker after conquest is a king who is endowed with the exemplary qualities both of the self and of material constituents, and who is the abode of good policy. Forming a circle all around him and with immediately contiguous territories is the constituent comprising his enemies. In like manner, with territories once removed from his, is the constituent comprising his allies."

Modelling Kauṭilya's maṇḍala

- *saṃdhi* ("peace pact")
- *vigraha* ("initiating hostilities")
- *āsana* ("remaining stationary")
- *yāna* ("marching into battle")
- *saṃśraya* ("seeking refuge")
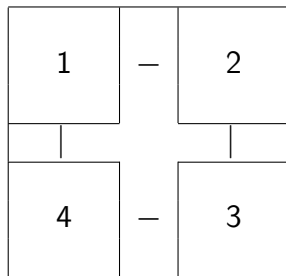- *dvaidhībhāva* ("double stratagem")

Central strategies for our model:

- "remaining stationary" (= "not attacking") and
- "marching into battle" (= "attacking")

Fighting involving friends and enemies

- $I = \{1, ..., n\}$: a set of n countries.
- $I^{(2)} := \{i - j : i, j \in I, i \neq j\}$: the set of all links between countries
- $N \subseteq I^{(2)}$: neighborhood structure
- $N(i) = \{i - j : j \in I, i - j \in N\}$: links of country i
- $F \subseteq N$: fighting structure

Fighting involving friends and enemies

- Payoff function

$$p_i : \mathfrak{F} \to \mathbb{R}$$

Definition

For three countries i, j, and k assume

      (a) $i - j \notin F$,

      (b) $j - k \in F$,

      (c) $i - k \in N \backslash F$.

Country i is called a friend of j against k at F if

$$p_i \left(F \cup \{ i - k \} \right) > p_i \left(F \right)$$

holds.

Fighting involving friends and enemies

Friends and enemies II

Definition

Assume countries i, j with $i - j \in N$. Country i is called an enemy of country j if one of two conditions hold:

- either i fights against j ($i - j \in F$),
- or, if i does not fight j, she would like to do so:

$$i - j \notin F \Rightarrow p_i \left(F \cup \{i - j\} \right) > p_i \left(F \right).$$

In above figure,

- country 1 is never a friend of country 2 and
- country 1 is never an enemy of country 3.

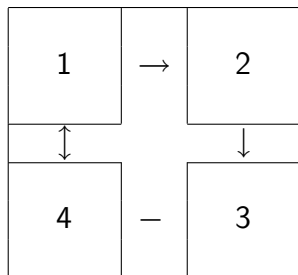# Fighting involving friends and enemies
Strategies I

- Let i be a country. For $i - j \in N(i)$ (i.e., each of i's neighbors):
  - $i \rightarrow j$ (i attacks j) or
  - $i \nrightarrow j$ (i does not attack j).
- Let $s = (s_1, ..., s_n)$ be a tuple of strategies.
  The fighting structure $F(s)$ contains $i - j$ if $i \rightarrow j$ or $j \rightarrow i$ hold.

In above figure, country 1 has four strategies:

- country 1 attacks both neighboring countries: $(1 \rightarrow 2, 1 \rightarrow 4)$
- country 1 attacks country 2, but not country 4: $(1 \rightarrow 2, 1 \nrightarrow 4)$
- country 1 attacks does not attack country 2, but attacks country 4: $(1 \nrightarrow 2, 1 \rightarrow 4)$
- country 1 does not attack any country: $(1 \nrightarrow 2, 1 \nrightarrow 4)$

Fighting involving friends and enemies

Strategies II

represents the strategy combination $s = (s_1, ..., s_4)$ with

$$s_1 = (1 \rightarrow 2, 1 \rightarrow 4), s_2 = (2 \nrightarrow 1, 2 \rightarrow 3),$$
$$s_3 = (3 \nrightarrow 2, 3 \nrightarrow 4), s_4 = (4 \rightarrow 1, 4 \nrightarrow 3)$$

and the induced fighting structure

$$F(s) = \{1 - 2, 2 - 3, 4 - 1\}$$

Fighting involving friends and enemies

Payoffs and equilibria

Definition

We define a utility function $u_i : S \rightarrow \mathbb{R}$ on S by $u_i(s) = p_i(F(s))$.

Definition

A strategy combination s^* obeying

$$u_i\left(s_i^*, s_{-i}^*\right) \geq u_i\left(s_i, s_{-i}^*\right)$$

for all countrys $i \in I$ and all strategies $s_i \in S_i$ is called an equilibrium.

Definition

The strategy combination given by $i \rightarrow j$ for all $i, j \in I$ with $i - j \in N$ is called the trivial equilibrium.

Fighting involving friends and enemies

How about

$$p_i : \mathfrak{F} \to \mathbb{R}, i \in I$$

Assumptions:

- The size of each country is 1 (basic payoff and fighting power).
- If a country wins against another country, the winning country takes over the territory of the losing one.
- If several winning countries are involved, the losing country's territory is split evenly between the winning ones. Note *Kauṭilya*'s recommendation to "urge a neighboring ruler to march into battle after concluding a pact, saying, "You should march in this direction, and I will march in that direction. The spoils shall be equal".

Fighting involving friends and enemies

Payoffs II

- If a country is involved in two or more fights, its fighting power is split evenly.
- The fighting powers of several attackers are added. The relative fighting power determines the winner. If the fighting power is the same, the outcome is a "draw".
- Fighting is costly. For each fight, the fighting power is reduced by $\delta > 0$. We assume that the cost of fighting is relatively small in comparison to a country's basic payff and fighting power of 1 and let $\delta < \frac{1}{2}$.
- Basically, payoff equals fighting power. But each country prefers weaker neighbors to stronger ones. $\varepsilon > 0$ stands for the advantage of being stronger than neighbors while $-\varepsilon < 0$ represents the disadvantage of being weaker.

Fighting involving friends and enemies

$$p_1(F)$$

$$= \begin{cases} 1, & F = \varnothing & \text{(i)} \\ 1 + \varepsilon, & F = \{2-3\} & \text{(ii)} \\ 1 - \varepsilon, & F = \{2-3, 3-4\} & \text{(iii)} \\ 1 - \delta, & F = \{1-2, 3-4\} \text{ or } F = N & \text{(iv)} \\ 1 - \delta - \varepsilon, & F = \{1-2\} & \text{(v)} \\ \frac{3}{2} - \delta + \varepsilon, & F = \{1-2, 2-3\} & \text{(vi)} \\ 0, & F = \{1-2, 1-4\} & \text{(vii)} \\ 0, & F = \{1-2, 1-4, 2-3\} & \text{(viii)} \\ 2 - \delta, & F = \{1-2, 2-3, 3-4\} & \text{(ix)} \end{cases}$$

Fighting involving friends and enemies
Payoffs IV

Here, (i) through (ix) can be seen from

(i)	no fighting
(ii)	two other countries fight
(iii)	3 loses against 2 and 4
(iv)	all neighbors fight
(v)	1 fights against 2 or 4
(vi)	1 joins 3 to win against 2 or 4
(vii)	1 loses against 2 and 4
(viii)	1 and 2 lose, or 1 and 4 lose
(ix)	1 and 4 win, or 1 and 2 win

Fighting involving friends and enemies

Identifying friends and enemies I

Theorem

Assuming, $\varepsilon > \delta < \frac{1}{2}$:

A Country 1 is a friend of country 3 against 2 (against 4) at $\{2-3\}$ (at $\{3-4\}$) by $\delta < \frac{1}{2}$,

B Country 1 is an enemy of country 2 (of country 4) at $\{2-3\}$ (at $\{3-4\}$) by $\delta < \frac{1}{2}$,

C Country 1 is a friend of country 3 against 2 (against 4) at $\{2-3, 3-4\}$ by $\delta < 1 + \varepsilon$ (1 turns against either 2, or 4),

...

Fighting involving friends and enemies

Identifying friends and enemies II

Theorem

Assuming, $\varepsilon > \delta < \frac{1}{2}$:

> ...

> D Country 1 is an enemy of 2 (of 4) at $\{2-3, 3-4\}$ by $\delta < 1 + \varepsilon$ (1 turns against either 2, or 4),

> E Country 1 is not a friend of country 3 against 2 at $\{1-4, 2-3, 3-4\}$ (or against 4 at $\{1-2, 2-3, 3-4\}$),

> F Country 1 is not an enemy of country 2 (or country 4) at $F = \varnothing$.

- Consider the fighting structure $\{2 - 3\}$. Then, 1 is a friend of 3 against 2 (see A) and an enemy of 2 (see B). In that situation, the enemy of country 1's enemy is a friend.

- But *Kauṭilya*'s conjecture does not hold in E:
  - 4 is 1's enemy by $1 - 4$,
  - 3 is 4's enemy by $3 - 4$, but
  - 3 is not 1's friend because 1 is not prepared to attack 2.

How about C versus D at fighting structure $F = \{2 - 3, 3 - 4\}$?

Theorem

We have 20 equilibria:

    a) *the trivial equilibrium s^* leading to $F(s^*) = N$,*

    b) *the no-attack equilibrium s^* given by $F(s^*) = \emptyset$,*

    c) *the equilibrium s^* with mutual attacks given by*

$$
\begin{aligned}
s_1^* &= (1 \to 2,\ 1 \nrightarrow 4), \\
s_2^* &= (2 \to 1,\ 2 \nrightarrow 3), \\
s_3^* &= (3 \nrightarrow 2,\ 3 \to 4), \\
s_4^* &= (4 \nrightarrow 1,\ 4 \to 3)
\end{aligned}
$$

   *together with the analogous equilibrium s^* with two fighting pairs $1 - 4$ and $2 - 3$,*

Theorem

...

d) the $2 \times 2 = 4$ asymmetric equilibria s^* given by
$F(s^*) = \{1-2, 2-3, 3-4\}$ and $2 \rightarrow 3$ and $3 \rightarrow 2$,
where $1-2$ (and also $3-4$) may come about by both
countries attacking or country 1 attacking country 2
(country 4 attacking country 3),
together with the analogous $3 \times 4 = 12$ equilibria with
no fighting between 1 and 2, 2 and 3, or 3 and 4,
respectively.

Fighting involving friends and enemies

Equilibria III
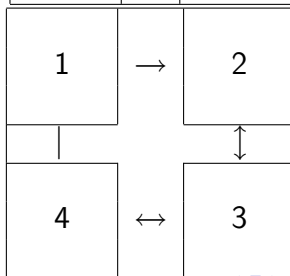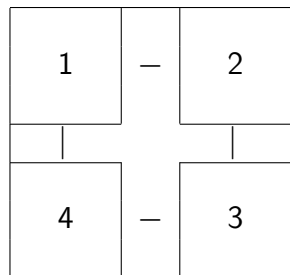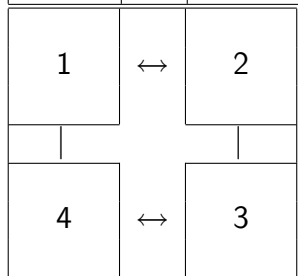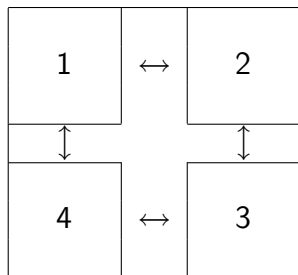
Conclusions

1. The oldest *caturaṅga* was of the two-king variety.
2. It served the didactic purpose of practicing war strategies.
3. Dice were not used, intelligence alone determined the outcome.
4. In due time, onlookers wished to join and hence four-king chess was invented.
5. This chess for four players was a popularized variant, did not (necessarily) belong to the court anymore and did not (necessarily) serve the didactic purpose of teaching strategy in peace and war.

Conclusions
Using dice

Syed comments: "When [four-king chess] was not about teaching the art of war any more, the cognitive ability could be combined with the contingency of dice."

But:

- The use of dice stands for "the divine realm, good and bad fortune, the invisible, what is beyond the range of thought, uncertainty" (*Kautilya*/Olivelle).
- Is a pure game of strategy somehow intellectually and morally superior above chess with dice? (van der Linde)
- Dice outcomes could be related to spellpower or to dexterity, rather than to mere luck. One might conjecture that a king or any other army ruler might try to get the particular parts of the army in motion, but that he may fail sometimes. The dice (that regulate whose pieces to move) might stand for this problem?

Conclusions

- Syed: A didactic model, a sort of "Kriegsübung im Sandkasten", which also carried the name *caturaṅga*, formed a connecting link between real-world warfare and two-king chess.
- Board games in ancient civilizations are typically otherworldly oriented (Schädler).
- Thieme's dictum: four-king chess is "von [...] wirklichkeitsfremder Konstruktion".
- Objection (ours): Four-king chess is as "unrealistic" as formal models in economics and elsewhere tend to be. It was deemed to be realistic enough so that princes could be taught the tricks of coping with friendly and unfriendly countries. In a similar manner, the paper on backward in induction in Indian fables claims that Indian princes were meant to learn backward induction by way of animal tales.

Conclusions
Critique of our model

1. *Kauṭilya*'s *maṇḍala* theory is clearly dynamic in nature. We try to capture the gist of his theory by way of a static model, where ε reflects the dynamic aspects. A truly dynamic model is still missing.

2. Middle and neutral kings play an important role in *Kauṭilya*'s thinking, but not in our model.