Advanced Microeconomics

Cooperative game theory

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Part D. Bargaining theory and Pareto optimality

1. Pareto optimality in microeconomics
2. **Cooperative game theory**

Nobel price 2012

In 2012, the Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel was awarded to the economists

- 1/2 Alwin Roth (Harvard University and Harvard Business School) and
- 1/2 Lloyd Shapley (University of California, Los Angeles)

for the theory of stable allocations and the practice of market design

Cooperative game theory

overview

1. The coalition function
2. Summing and zeros
3. Solution concepts
4. Pareto efficiency
5. The core
6. The Shapley value: the formula
7. The Shapley value: the axioms
8. Further examples
    - Simple games
    - Three non-simple games
    - Cost-division games

The coalition function I

The coalition function for the gloves game is given by

$$\begin{aligned} v_{L,R} \quad : \quad & 2^N \to \mathbb{R} \\ & K \mapsto v_{L,R}(K) = \min(|K \cap L|, |K \cap R|), \end{aligned}$$

where

- N is the set of players (also called the grand coalition),
- L (set of left-glove holders) and R (set of right-glove holders) form a partition of N,
- $v_{L,R}$ denotes the coalition function for the gloves game,
- 2^N stands for N's power set, i.e., the set of all subsets of N (the domain of $v_{L,R}$),
- K is a coalition, i.e., $K \subseteq N$ or $K \in 2^N$

The coalition function II

Definition (coalition function)

For any finite and nonempty player set $N = \{1, ..., n\}$, a coalition function $v : 2^N \to \mathbb{R}$ fulfills $v(\emptyset) = 0$. The set of all coalition functions (on N) is denoted by $\mathbb{V}$ ($\mathbb{V}_N$). $v(K)$ is called coalition K's worth.

Problem

Assume $N = \{1, 2, 3, 4, 5\}$, $L = \{1, 2\}$ and $R = \{3, 4, 5\}$. Find the worths of the coalitions $K = \{1\}$, $K = \emptyset$, $K = N$ and $K = \{2, 3, 4\}$.

- Interpretation: the gloves game is a market game where the left-glove owners form one market side and the right-glove owners the other.
- Distinguish
  - the worth (of a coalition) from
  - the payoff accruing to players.

Summing and zeros I

Definition

For any finite and nonempty player set $N = \{1, ..., n\}$, a payoff vector

$$x = (x_1, ..., x_n) \in \mathbb{R}^n$$

specifies payoffs for all players $i = 1, ..., n$.

- It is possible to sum coalition functions and it is possible to sum payoff vectors. Summation of vectors is easy – just sum each component individually. For example, determine the sum of the vectors

$$\begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}!$$

- Note the difference between payoff-vector summation and payoff summation $\sum_{i=1}^{n} x_i$.

Summing and zeros II

Vector summation is possible for coalition functions, too. For example, the sum

$$v_{\{1\},\{2,3\}} + v_{\{1,2\},\{3\}}$$

can be seen from

$$\begin{pmatrix} \emptyset : 0 \\ \{1\} : 0 \\ \{2\} : 0 \\ \{3\} : 0 \\ \{1,2\} : 1 \\ \{1,3\} : 1 \\ \{2,3\} : 0 \\ \{1,2,3\} : 1 \end{pmatrix} + \begin{pmatrix} \emptyset : 0 \\ \{1\} : 0 \\ \{2\} : 0 \\ \{3\} : 0 \\ \{1,2\} : 0 \\ \{1,3\} : 1 \\ \{2,3\} : 1 \\ \{1,2,3\} : 1 \end{pmatrix} = \begin{pmatrix} \emptyset : 0 \\ \{1\} : 0 \\ \{2\} : 0 \\ \{3\} : 0 \\ \{1,2\} : 1 \\ \{1,3\} : 2 \\ \{2,3\} : 1 \\ \{1,2,3\} : 2 \end{pmatrix}$$

Summing and zeros III

- Mathematically speaking, $\mathbb{R}^n$ and $\mathbb{V}_N$ can be considered as vector spaces.
- Vector spaces have a zero.
    - The zero from $\mathbb{R}^n$ is
    $$\underset{\in\mathbb{R}^n}{0} = \left(\underset{\in\mathbb{R}}{0}, \ldots, \underset{\in\mathbb{R}}{0}\right)$$
    - In the vector space of coalition functions, $0 \in \mathbb{V}_N$ is the function that attributes the worth zero to every coalition, i.e.,
    $$\underset{\in\mathbb{V}_N}{0}(K) = \underset{\in\mathbb{R}}{0} \text{ for all } K \subseteq N$$

Solution concepts I

Definition (solution function, solution correspondence)

A function σ that attributes, for each coalition function v from $\mathbb{V}$, a payoff to each of v's players,

$$\sigma(v) \in \mathbb{R}^{|N(v)|},$$

is called a solution function. Player i's payoff is denoted by $\sigma_i(v)$. In case of $N(v) = \{1, ..., n\}$, we also write $(\sigma_1(v), ..., \sigma_n(v))$ for $\sigma(v)$ or $(\sigma_i(v))_{i \in N(v)}$. A correspondence that attributes a set of payoff vectors to every coalition function v,

$$\sigma(v) \subseteq \mathbb{R}^{|N(v)|}$$

is called a solution correspondence. Solution functions and solution correspondences are also called solution concepts.

Solution concepts II

- Solution concepts can be described algorithmically. An algorithm is some kind of mathematical procedure (a more or less simple function) that tells how to derive payoffs from the coalition functions. Examples:
  - player 1 obtains $v(N)$ and the other players zero,
  - every player gets 100,
  - every player gets $v(N)/n$,
  - every player i's payoff set is given by $[v(\{i\}), v(N)]$ (which may be the empty set).
- Alternatively, solution concepts can be defined by axioms. For example, axioms might demand that
  - all the players obtain the same payoff,
  - no more than $v(N)$ is to be distributed among the players,
  - player 1 is to get twice the payoff obtained by player 2,
  - every player gets $v(N) - v(N \backslash \{i\})$.
- Ideally, solution concepts can be described both algorithmically and axiomatically.

Pareto efficiency I

- Arguably, Pareto efficiency is the single most often applied solution concept in economics – rivaled only by the Nash equilibrium from noncooperative game theory.
- For any game $v \in \mathbb{V}_N$, Pareto efficiency is defined by

$$\sum_{i \in N} x_i = v(N).$$

- Equivalently, Pareto efficiency means

$$\begin{aligned} \sum_{i \in N} x_i &\leq v(N) \text{ (feasibility) and} \\ \sum_{i \in N} x_i &\geq v(N) \text{ (the grand coalition cannot block } x\text{).} \end{aligned}$$

Pareto efficiency II

In case of $\sum_{i=1}^{n} x_i < v(N)$, the players would leave "money on the table". All players together could block (or contradict) the payoff vector x by proposing (for example) the (feasible!) payoff vector $y = (y_1, ..., y_n)$ defined by

$$y_i = x_i + \frac{1}{n}\left(v(N) - \sum_{i=1}^{n} x_i\right), i \in N.$$

Problem

Find the Pareto-efficient payoff vectors for the gloves game $v_{\{1\},\{2\}}$!

For the gloves game, the solution concept "Pareto efficiency" has two important drawbacks:

- We have very many solutions and the predictive power is weak.
- The payoff for a left-glove owner does not depend on the number of left and right gloves in our simple economy.

The core I

Definition

Definition (blockability and core)

Let $v \in \mathbb{V}_N$ be a coalition function. A payoff vector $x \in \mathbb{R}^n$ is called blockable by a coalition $K \subseteq N$ if

$$\sum_{i \in K} x_i < v(K)$$

holds. The core is the set of all those payoff vectors x fulfilling

$$\begin{aligned} \sum_{i \in N} x_i &\leq v(N) \text{ (feasibility) and} \\ \sum_{i \in K} x_i &\geq v(K) \text{ for all } K \subseteq N \text{ (no blockade by any coalition).} \end{aligned}$$

Do you see that every payoff vector from the core is also Pareto efficient? Just take $K := N$.

The core II

Example

Consider $v_{\{1\},\{2\}}$!

- Pareto efficiency requires $x_1 + x_2 = 1$.
- Furthermore, x may not be blocked by one-man coalitions:

$$\begin{aligned} x_1 &\geq v_{\{1\},\{2\}}\left(\{1\}\right) = 0 \text{ and} \\ x_2 &\geq v_{\{1\},\{2\}}\left(\{2\}\right) = 0. \end{aligned}$$

- Hence, the core is the set of payoff vectors $x = (x_1, x_2)$ obeying

$$x_1 + x_2 = 1, x_1 \geq 0, x_2 \geq 0.$$

- Are we not forgetting about $K := \emptyset$? No:

$$\sum_{i \in \emptyset} x_i = 0 \geq 0 = v\left(\emptyset\right).$$

The core III:

Exercises

- Determine the core for the gloves game $v_{L,R}$ with $L = \{1, 2\}$ and $R = \{3\}$.
- Why does the Pareto-efficient payoff vector

$$y = \left(\frac{1}{10}, \frac{1}{10}, \frac{8}{10}\right)$$

not lie in the core?

The Shapley formula I: general idea

- In contrast to Pareto efficiency and the core, the Shapley value is a point-valued solution concept, i.e., a solution function.
- Shapley's (1953) article is famous for pioneering the twofold approach of algorithm and axioms.
- The idea behind the Shapley value is that every player obtains
  - an average of
  - his marginal contributions.

The Shapley formula II: marginal contribution

The marginal contribution of player i with respect to coalition K is

"the worth with him" minus "the worth without him".

Definition (marginal contribution)

Player i's marginal contribution with respect to a coalition K is denoted by $MC_i^K(v)$ and given by

$$MC_i^K(v) := v(K \cup \{i\}) - v(K \setminus \{i\}).$$

Problem

Determine the marginal contributions for $v_{\{1,2,3\},\{4,5\}}$ *and*

- $i = 1, K = \{1, 3, 4\}$,
- $i = 1, K = \{3, 4\}$,
- $i = 4, K = \{1, 3, 4\}$,
- $i = 4, K = \{1, 3\}$.

The Shapley formula III: rank orders

- Imagine three players outside the door who enter this room, one after the other.
- We have the rank orders:

$$(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1).$$

- Why 6 rank orders?
  - For a single player 1, we have just one rank order (1).
  - The second player 2 can be placed before or after player 1 so that we obtain the $1 \cdot 2$ rank orders

$$\begin{array}{c}(1,2), \\ (2,1).\end{array}$$

  - For each of these two, the third player 2 can be placed before the two players, in between or after them:

$$\begin{array}{c}(3,1,2),(1,3,2),(1,2,3), \\ (3,2,1),(2,3,1),(2,1,3).\end{array}$$

  - Therefore, we have $2 \cdot 3 = 6$ rank orders.

The Shapley formula IV: rank orders and marginal contributions

Definition (rank order)

Bijective functions $\rho : N \rightarrow N$ are called rank orders or permutations on N. The set of all permutations on N is denoted by RO_N. The set of all players "up to and including player i under rank order ρ" is denoted by $K_i(\rho)$ and given by

$$\rho(j) = i \text{ and } K_i(\rho) = \{\rho(1), .., \rho(j)\}.$$

Player i's marginal contribution with respect to rank order ρ:

$$MC_i^{\rho}(v) := MC_i^{K_i(\rho)}(v) = v(K_i(\rho)) - v(K_i(\rho) \setminus \{i\}).$$

The Shapley formula V: an example

For rank order $(3, 1, 2)$, one finds the marginal contributions

$$v(\{3\}) - v(\emptyset), v(\{1,3\}) - v(\{3\}) \text{ and } v(\{1,2,3\}) - v(\{1,3\}).$$

They add up to $v(N) - v(\emptyset) = v(N)$.

Problem

Consider $N = \{1,2,3\}$, $L = \{1,2\}$ *and* $R = \{3\}$ *and determine player* 1*'s marginal contribution for each rank order.*

Problem

Find player 2*'s marginal contributions for the rank orders* $(1,3,2)$ *and* $(3,1,2)$*!*

The Shapley formula VI: algorithm

- We first determine all the possible rank orders.
- We then find the marginal contributions for every rank order.
- For every player, we add his marginal contributions.
- Finally, we divide the sum by the number of rank orders.

Definition (Shapley value)

The Shapley value is the solution function Sh given by

$$Sh_i(v) = \frac{1}{n!} \sum_{\rho \in RO_N} MC_i^{\rho}(v).$$

The Shapley formula VII: the gloves game

- Consider the gloves game $v_{\{1,2\},\{3\}}$. We find

$$Sh_1\left(v_{\{1,2\},\{3\}}\right) = \frac{1}{6}.$$

- The Shapley values of the other two players can be obtained by the same procedure.
- However, there is a more elegant possibility.
  - The Shapley values of players 1 and 2 are identical because they hold a left glove each and are symmetric (in a sense to be defined shortly).
  - Thus, we have $Sh_2\left(v_{\{1,2\},\{3\}}\right) = \frac{1}{6}$.
  - Also, the Shapley value satisfies Pareto efficiency:

$$\sum_{i=1}^{3} Sh_i\left(v_{\{1,2\},\{3\}}\right) = v\left(\{1,2,3\}\right) = 1$$

  - Thus, we find

$$Sh\left(v_{\{1,2\},\{3\}}\right) = \left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right).$$

The Shapley formula VIII: results for the gloves game

The following table reports the Shapley values for an owner of a right glove in a market with r right-glove owners and l left-glove owners:

		number l of left-glove owners				
		0	1	2	3	4
number r	1	0	0,500	**0,667**	0,750	0,800
of	2	0	0,167	0,500	0,650	0,733
right-glove	3	0	0,083	0,233	0,500	0,638
owners	4	0	0,050	0,133	0,271	0,500

This table clearly shows how the payoff increases with the number of players on the other market side. The payoff $Sh_3\left(v_{\{1,2\},\{3\}}\right) = \frac{2}{3}$ is highlighted.

The Shapley axioms I: overview

The Shapley value fulfills four axioms:

- the efficiency axiom: the worth of the grand coalition is to be distributed among all the players,
- the symmetry axiom: players in similar situations obtain the same payoff,
- the null-player axiom: a player with zero marginal contribution to every coalition, obtains zero payoff, and
- additivity axiom: if players are subject to two coalition functions, it does not matter whether we apply the Shapley value to the sum of these two coalition functions or apply the Shapley value to each coalition function separately and sum the payoffs.

The Shapley axioms II: equality and efficiency

Let us consider a player set $N = \{1, ..., n\}$ and a solution function σ. It may or may not obey one or several of these axioms:

Definition (equality axiom)

A solution function σ is said to obey the equality axiom if

$$\sigma_i(v) = \sigma_j(v)$$

holds for all players $i, j \in N$.

Definition (efficiency axiom)

A solution function σ is said to obey the efficiency axiom or the Pareto axiom if

$$\sum_{i \in N} \sigma_i(v) = v(N)$$

holds.

The Shapley axioms III: symmetry

Definition (symmetry)

Two players i and j are called symmetric (with respect to $v \in \mathbb{V}_N$) if we have

$$v(K \cup \{i\}) = v(K \cup \{j\})$$

for every coalition K that contains neither i nor j.

Problem

Are left-glove holders symmetric?

Definition (symmetry axiom)

A solution function σ is said to obey the symmetry axiom if we have

$$\sigma_i(v) = \sigma_j(v)$$

for any coalition function $v \in \mathbb{V}$ and for any two symmetric players i and j.

The Shapley axioms IV: null player

Definition (null player)

A player $i \in N$ is called a null player (with respect to $v \in \mathbb{V}_N$) if

$$v(K \cup \{i\}) = v(K)$$

holds for every coalition K.

Can a left-glove holder be a null player? Shouldn't a null player obtain nothing?

Definition (null-player axiom)

A solution function σ is said to obey the null-player axiom if we have

$$\sigma_i(v) = 0$$

for any null player i.

The Shapley axioms V: additivity

Definition (additivity axiom)

A solution function σ is said to obey the additivity axiom if we have

$$\sigma(v+w) = \sigma(v) + \sigma(w)$$

for any player set N and any two coalition functions $v, w \in \mathbb{V}_N$.

- On the left-hand side, we add the coalition functions first and then apply the solution function.
- On the right-hand side we apply the solution function to the coalition functions individually and then add the payoff vectors.

Problem

Can you deduce $\sigma(0) = 0$ *from the additivity axiom? Hint: use* $v = w := 0$.

The Shapley axioms VI: equivalence

Theorem (Shapley theorem)

The Shapley formula is the unique solution function that fulfills the symmetry axiom, the efficiency axiom, the null-player axiom and the additivity axiom.

- The Shapley formula fulfills the four axioms.
- The Shapley formula is the only solution function to do so.
- Differently put, the Shapley formula and the four axioms are equivalent – they specify the same payoffs.
- Cooperative game theorists say that the Shapley formula is "axiomatized" by the set of the four axioms.

Problem

Determine the Shapley value for the gloves game for $L = \{1\}$ *and* $R = \{2, 3, 4\}$! *Hint: You do not need to write down all* $4!$ *rank orders. Try to find the probability that player* 1 *does not complete a pair.*

Simple games

Definition (monotonic game)

A coalition function $v \in \mathbb{V}_N$ is called monotonic if $\emptyset \subseteq S \subseteq S'$ implies $v(S) \leq v(S')$

Thus, monotonicity means that the worth of a coalition cannot decrease if other players join. Simple games are a special subclass of monotonic games:

Definition (simple game)

A coalition function $v \in \mathbb{V}_N$ is called simple if

- we have $v(K) = 0$ (losing coalition) or $v(K) = 1$ (winning coalition) for every coalition $K \subseteq N$,
- the grand coalition's worth is 1, and
- v is monotonic.

Definition (veto player, dictator)

Let v be a simple game. A player $i \in N$ is called a veto player if

$$v\left(N \backslash \left\{i\right\}\right) = 0$$

holds. i is called a dictator if

$$v\left(S\right) = \begin{cases} 1, & i \in S \\ 0, & \text{otherwise} \end{cases}$$

holds for all $S \subseteq N$.

Unanimity games

Definition (unanimity game)

For any $T \neq \emptyset$,

$$u_T(K) = \begin{cases} 1, & K \supseteq T \\ 0, & \text{otherwise} \end{cases}$$

defines a unanimity game.

- The players from T are the productive or powerful members of society.
  - Every player from T is a veto player and no player from $N \backslash T$ is a veto player.
  - In a sense, the players from T exert common dictatorship.
- For example, each player $i \in T$ possesses part of a treasure map.

Problem

Find the core and the Shapley value for $N = \{1, 2, 3, 4\}$ *and* $u_{\{1,2\}}$.

Weighted voting games

Definition (weighted voting game)

A voting game v is specified by a quota q and voting weights g_i, $i \in N$, and defined by

$$v(K) = \begin{cases} 1, & \sum_{i \in K} g_i \geq q \\ 0, & \sum_{i \in K} g_i < q \end{cases}$$

In that case, the voting game is also denoted by $[q; g_1, ..., g_n]$.

UN Security Council

- 5 permanent members: China, France, Russian Federation, the United Kingdom and the United States
- 10 non-permanent members
- For substantive matters, the voting rule can be described by

$$[39; 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$$

where the weights 7 accrue to the five permanent and the weights 1 to the non-permanent members.

Problem

- *Show that every permanent member is a veto player.*
- *Show also that the five permanent members need the additional support of four non-permanent ones.*

UN Security Council II

- For the fifteen members of the Security Council, we have

$$15! = 1.307.674.368.000$$

  rank orders.
- The Shapley values are

$$\begin{aligned} &0,19627 \text{ for each permanent member} \\ &0,00186 \text{ for each non-permanent member.} \end{aligned}$$

Buying a car I

- Andreas (A) has a used car he wants to sell, Frank (F) and Tobias (T) are potential buyers with willingness to pay of 700 and 500, respectively.
- Coalition function:

$$
\begin{aligned}
v(A) &= v(F) = v(T) = 0, \\
v(A, F) &= 700, \\
v(A, T) &= 500, \\
v(F, T) &= 0 \text{ and} \\
v(A, F, T) &= 700.
\end{aligned}
$$

Buying a car II

The core is the set of those payoff vectors (x_A, x_F, x_T) that fulfill

$$x_A + x_F + x_T = 700$$

and

$$\begin{aligned} x_A &\geq 0, x_F \geq 0, x_T \geq 0, \\ x_A + x_F &\geq 700, \\ x_A + x_T &\geq 500 \text{ and} \\ x_F + x_T &\geq 0. \end{aligned}$$

Buying a car III

- Tobias obtains

$$\begin{aligned} x_T &= 700 - (x_A + x_F) \text{ (efficiency)} \\ &\leq 700 - 700 \text{ (by } x_A + x_F \geq 700) \\ &= 0 \end{aligned}$$

- and hence zero, $x_T = 0$.
- By $x_A + x_T \geq 500$, the seller Andreas can obtain at least 500.
- The core is the set of vectors (x_A, x_F, x_T) obeying

$$\begin{aligned} 500 &\leq x_A \leq 700, \\ x_F &= 700 - x_A \text{ and} \\ x_T &= 0. \end{aligned}$$

- Therefore, the car sells for a price between 500 and 700.

The Maschler game

Coalition function:

$$v(K) = \begin{cases} 0, & |K| = 1 \\ 60, & |K| = 2 \\ 72, & |K| = 3 \end{cases}$$

Core:

- Efficiency:

$$x_1 + x_2 + x_3 = 72$$

- and non-blockability:

$$\begin{aligned} x_1 &\geq 0, x_2 \geq 0, x_3 \geq 0, \\ x_1 + x_2 &\geq 60, x_1 + x_3 \geq 60 \text{ and } x_2 + x_3 \geq 60. \end{aligned}$$

- Summing the last three inequalities yields

$$2x_1 + 2x_2 + 2x_3 \geq 3 \cdot 60 = 180$$

  and hence a contradiction to efficiency.
- The core is empty!

The gloves game, once again I

- Gloves game with minimal scarcity:

$$\begin{aligned} L &= \{1, 2, ..., 100\} \\ R &= \{101, ..., 199\}. \end{aligned}$$

- Are the right-hand glove owners much better off?
- If

$$x = (x_1, ..., x_{100,} x_{101}, ..., x_{199}) \in core(v_{L,R})$$

then, by efficiency,

$$\sum_{i=1}^{199} x_i = 99.$$

The gloves game, once again II

- We now pick any left-glove holder $j \in \{1, 2, ..., 100\}$. We find

$$v\left(L\backslash\left\{j\right\}\cup R\right) = 99$$

and hence

$$\begin{aligned} x_j &= 99 - \sum_{\substack{i=1,\\ i\neq j}}^{199} x_i \text{ (efficiency)} \\ &\leq 99 - 99 \text{ (blockade by coalition } L\backslash\left\{j\right\}\cup R\text{)} \\ &= 0. \end{aligned}$$

- Therefore, we have $x_j = 0$ for every $j \in L$.
- Every right-glove owner can claim at least 1 because he can point to coalitions where he is joined by at least one left-glove owner.
- Therefore, every right-glove owner obtains the payoff 1 and every left-glove owner the payoff zero.

Cost division games I

- doctors with a common secretary or commonly used facilities
- firms organized as a collection of profit-centers
- universities with computing facilities used by several departments or faculties

Definition (cost-division game)

For a player set N, let $c : 2^N \to \mathbb{R}_+$ be a coalition function that is called a cost function. On the basis of c, the cost-savings game is defined by $v : 2^N \to \mathbb{R}$ and

$$v(K) = \sum_{i \in K} c(\{i\}) - c(K), K \subseteq N.$$

The idea behind this definition is that cost savings can be realized if players pool their resources so that $\sum_{i \in K} c(\{i\})$ is greater than $c(K)$ and $v(K)$ is positive.

Cost division games II

- Two towns A and B plan a water-distribution system.
- Cost:
  - Town A could build such a system for itself at a cost of 11 million Euro and
  - town B would need 7 million Euro for a system tailor-made to its needs.
  - The cost for a common water-distribution system is 15 million Euro.
- The cost function is given by

$$\begin{aligned} c(\{A\}) &= 11, c(\{B\}) = 7 \text{ and} \\ c(\{A,B\}) &= 15. \end{aligned}$$

- The associated cost-savings game is $v : 2^{\{A,B\}} \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} v(\{A\}) &= 0, v(\{B\}) = 0 \text{ and} \\ v(\{A,B\}) &= 7 + 11 - 15 = 3. \end{aligned}$$

Cost division games III

- v's core is obviously given by

$$\left\{ (x_A, x_B) \in \mathbb{R}^2_+ : x_A + x_B = 3 \right\}.$$

- The cost savings of $3 = 11 + 7 - 15$ can be allotted to the towns such that no town is worse off compared to going it alone. Thus, the set of undominated cost allocations is

$$\left\{ (c_A, c_B) \in \mathbb{R}^2 : c_A + c_B = 15, c_A \leq 11, c_B \leq 7 \right\}.$$

Problem

Calculate the Shapley values for c and v! Comment!