Applied Cooperative Game Theory

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Contents

Für Corinna, Ben, Jasper, Samuel

Preface

What is this book about?

This book is on the theory and on the applications of cooperative games. We deal with agents exchanging objects, profit centers within firms, political parties groping for power and many other sorts of "players".

Cooperative game theory focuses on the question of "who gets how much". This question is determined by the two pillars of cooperative game theory. The first pillar is the coalition function (also called characteristic function) that describes the economic (sociologic, political) opportunities open to all possible subgroups of the player set (coalitions). A coalition function may represent a bargaining situation, a market, an election, a cost-division problem and many others.

The second pillar is the solution concept applied to coalition functions. Solutions consist of payoffs attributed to the players. Typically, solutions can be described in one of two ways. Either we provide a formula or an algorithm that tells us how to transfer a coalition function into payoff vectors (formula definition). Or we put down axioms that describe in general terms how much players should get (axiom definition) – axioms in cooperative game theory are general rules of division. For example, Pareto efficiency demands that the worth of the grand coalition (all players taken together) is to be distributed among the players. According to the axiom of symmetry, symmetric (not distinguishable but by name) players should obtain the same payoff.

Ideally, the formula and the axiom definitions coincide. This means that a solution concept can be expressed by a formula or by a set of axiom and that both ways are equivalent – they lead to the very same payoff vectors.

As in any book on cooperative game theory, we, too, talk about matching formulas and axiom definitions. However, we stress applications over theory. This means that we deal with theoretical concepts only if they are helpful for the applications that we have in mind. The knowledgeable reader will excuse us for omitting the von Neumann-Morgenstern sets or the nucleolus. Instead, the Shapley value and derivatives of the Shapley value take center stage.

Which applications do we cover?

We deal with many different institutions that range from markets and elections to coalition governments and hierarchies. In particular, we consider the following applications.

- How does the price obtained on markets depend on the relative scarcity of the traded objects?
- How can we model power and power-over?
- Can we expect unions to be detrimental to employment?
- Will unemployment benefits increase unemployment?
- How can overhead costs be shared?
- How does the number of ministries a party within a government coalition obtains depend on hte number of seats in parliament?
- Which is the optimal percentage of a house price a real estate agent asks for himself?
- How many civil servants an economy can be expected to hold?

Sometimes, cooperative game theory and its axioms are exclusively interpreted in a normative way. While cooperative game theory has a lot to offer for normative analyses, most examples covered in this book are best interpreted in a positive manner.

What about mathematics ... ?

Cooperative game theory need not be too demanding in terms of mathematical sophistication. We explain the mathematical concepts when and where they are needed. Also, since we have an applied focus, we are more interested in interpretation and application than in proofs of axiomatization.

Exercises and solutions

The main text is interspersed with questions and problems wherever they arise. Solutions or hints are given at the end of each chapter. On top, we add a few exercises without solutions.

Thank you!!

I am happy to thank many people who helped me with this book. Several generations of students were treated to (i.e., suffered through) continuously improved versions of this book. Frank Hüttner and Andreas Tutic ... I also thank my coauthors Andre Casajus, Tobias Hiller and ... for the good cooperation with high payoffs to everyone. Some generations of Bachelor and Master students also provided feedback that helped to improve the manuscript.

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Harald Wiese

Overview and Pareto efficiency

Part A

Overview and Pareto efficiency

CHAPTER I

Overview

1. Introduction

In this first chapter, we plan to give the reader a good idea of what to expect in this book. In sections 2 through 4, we briefly introduce the reader to coalition functions and to solution concepts for coalition functions. Section 7 offers some comments on cooperative game theory versus noncooperative game theory. Finally, in section 8, we present the subject matters part by part and chapter by chapter.

2. The players, the coalitions, and the coalition functions

Throughout the book, we deal with a player set $N = \{1, ..., n\}$ and the subsets of N which are also called coalitions. Thus, the coalitions of $N := \{1, 2, 3\}$ include $\{1, 2\}$, $\{2\}$, $\emptyset$ (the empty set – no players at all) and N (all players taken together – the grand coalition).

The general idea of cooperative game theory is that

- coalition functions describe the economic, social or political situation of the agents while
- solution concepts determine the payoffs for all the players from N taking a coalition function as input.

Thus,

$$\begin{array}{rl} & \text{coalition functions} \\ + & \text{solution concepts} \\ \text{yield} & \text{payoffs.} \end{array}$$

In the literature, there are two different sorts of coalition functions, with transferable utility and without transferable utility. We focus on the simpler case of transferable utility in all parts of the book except the last one. In the framework of transferable utility, a coalition function v attributes a real number $v(K)$ to every coalition $K \subseteq N$. Consider, for example, the gloves game v for $N = \{1, 2, 3\}$ where the two players 1 and 2 hold a left glove and player 3 holds a right glove. The idea behind this game is complementarity – pairs of gloves have a worth of 1. Thus, the coalition function for our gloves game is given by

$$\begin{aligned} v(\emptyset) &= 0, \\ v(\{1\}) &= v(\{2\}) = v(\{3\}) = 0, \\ v(\{1,2\}) &= 0, \\ v(\{1,3\}) &= v(\{2,3\}) = 1, \\ v(\{1,2,3\}) &= 1. \end{aligned}$$

Left-glove holders and right-glove holders can stand for the two sides of a market – demand and supply. For example, the left-glove holders buy right gloves in order to form pairs.

3. The Shapley value

In our mind, the Shapley value is the most useful solution concept in cooperative game theory. First of all, it can be applied directly to problems ranging from bargaining over cost division to power. Applying the Shapley value to the above gloves game yields the payoffs

$$Sh_1(v) = \frac{1}{6}, Sh_2(v) = \frac{1}{6}, Sh_3(v) = \frac{2}{3}.$$

We see that the Shapley value

- distributes the worth of the grand coalition $v(N) = 1$ among the three players ($Sh_1(v) + Sh_2(v) + Sh_3(v) = 1$),
- allots the same payoff to players 1 and 2 because they are "symmetric" ($Sh_1(v) = Sh_2(v)$), and
- awards the lion's share to player 3 who possesses the scarce resource of a right glove ($Sh_3(v) = \frac{2}{3} > \frac{1}{6} = Sh_2(v)$).

Thus, the Shapley value tells us how market power is reflected by payoffs. This and many other applications are dealt with in the first part of our book which concentrates on the Shapley value (and some related concepts such as the Banzhaf value).

There are several alternative ways to calculate the Shapley value. Let us denote the payoff to player i by x_i. Assume the players 1, 2 and 3 bargain on how to divide the worth of the grand coalition, $v(N) = 1$, between them, i.e., we have

$$x_1 + x_2 + x_3 = 1. \tag{I.1}$$

Furthermore, let every player use a "where would you be without me" argument. In particular, player 3 could issue the following threat to player 1 (and similarly to player 2): "Without me, there would be only two left gloves and your payoff would be zero rather than x_1, i.e., you, player 1, would lose

$$x_1 - 0$$

without me."

Player 1's counter-threat against player 3 runs as follows: "Without me, you, player 3 would find yourself in an essentially symmetrical situation with player 2 (one right-hand glove versus one left-hand glove) and obtain the payoff $\frac{1}{2}$, i.e., you would lose

$$x_3 - \frac{1}{2}$$

without me."

The Shapley value rests on the premise of equal bargaining power – both arguments carry the same weight. Thus, the two differences are the same:

$$\underbrace{x_1 - 0}_{\substack{\text{loss to player 1} \\ \text{if player 3 withdraws}}} = \underbrace{x_3 - \frac{1}{2}}_{\substack{\text{loss to player 3} \\ \text{if player 1 withdraws}}} . \tag{I.2}$$

Since we have an analogous threat and an analogous counter-threat between players 2 (rather than player 1) and 3, we find

$$\begin{aligned} 1 &= x_1 + x_2 + x_3 \text{ (eq. I.1)} \\ &= \left(x_3 - \frac{1}{2}\right) + \left(x_3 - \frac{1}{2}\right) + x_3 \text{ (eq. I.2)} \end{aligned}$$

and hence

$$(x_1, x_2, x_3) = \left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right). \tag{I.3}$$

The Shapley value is easy to handle. This simplicity gives room for additional structure that may be needed in applications. Thus,

- different players may belong to different groups that work together, bargain as a group etc.
- any two players may or may not be linked together where the links stand for communication or cooperation.

We will briefly introduce

$$\text{Shapley} + \text{structure}$$

in this introductory chapter and treat them in some detail in later chapters.

4. The outside option value

Taking up the gloves game again, assume that the glove traders 1 (left glove) and 3 (right glove) agree to cooperate to form a pair of gloves. We can express this fact by the partition of N

$$\{\{1,3\},\{2\}\}$$

where we address $\{1,3\}$ and $\{2\}$ as that partition's components.

What are the player's payoffs in such a situation? The first idea might be to apply the Shapley value to the individual components. In fact, the resulting value is known as the AD value (where A stands for Aumann and D for Dreze) and given by

$$AD_1(v) = AD_3(v) = \frac{1}{2}, AD_2(v) = 0.$$

More recent developments in cooperative game theory point to the fact that player 3 should obtain more than $\frac{1}{2}$ because he can threaten to join forces with player 2 rather than player 1. Thus, player 2 is an "outside option" for player 3.

How can we find the players' payoffs in that case? First of all, players 1 and 3 will share the value of a glove, i.e., we have

$$x_1 + x_3 = 1 \tag{I.4}$$

and $x_2 = 0$. When 1 and 3 bargain on how to share the payoff of 1, both players may point out that each of them is necessary to form the component $\{1,3\}$. Therefore, the gain from leaving player 2 out should be divided equally where the Shapley value (for the trivial partition $\{\{1,2,3\}\}$ serves as a reference point:

$$\underbrace{x_1 - Sh_1(v)}_{\substack{\text{gain for player 1}\\ \text{from forming component } \{1,3\}}} = \underbrace{x_3 - Sh_3(v)}_{\substack{\text{gain for player 3}\\ \text{from forming component } \{1,3\}}}. \tag{I.5}$$

By

$$\begin{aligned} 1 &= x_1 + x_3 \text{ (eq. I.4)} \\ &= [x_3 - Sh_3(v) + Sh_1(v)] + x_3 \text{ (eq. I.5)} \\ &= 2x_3 - \frac{2}{3} + \frac{1}{6} \text{ (eq. I.3)} \end{aligned}$$

we obtain the outside-option value payoffs due to Casajus (2009)

$$(x_1, x_2, x_3) = \left(\frac{3}{4}, 0, \frac{1}{4}\right).$$

5. The network value

Instead of considering partitions, we may assume a network of links between players. A link between two players means that these two players can communicate or cooperate. The corresponding generalization of the Shapley value is known as the Myerson value.

Departing fromt the gloves game, we assume that players 1 and 3 are the productive or powerful players. This is reflected by the coalition function v given by

$$v(K) = \begin{cases} 1, & K \supseteq \{1,3\} \\ 0, & \text{otherwise} \end{cases}$$

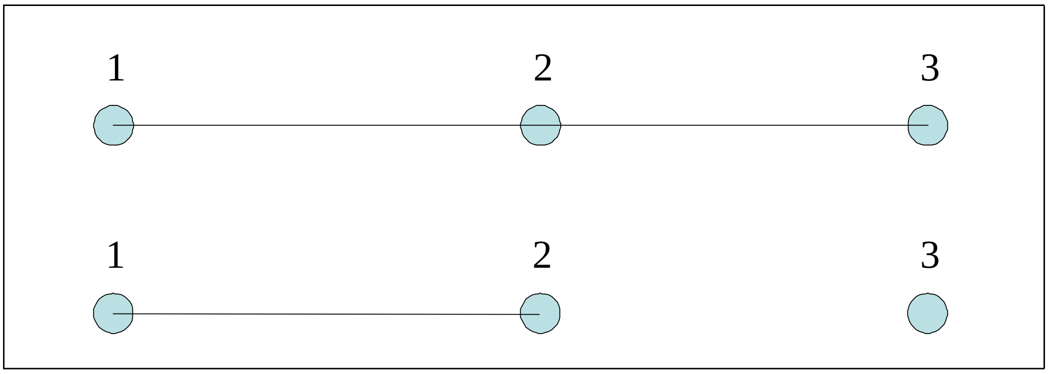

FIGURE 1. A simple network

Coalitions different from $\{1,3\}$ and $\{1,2,3\}$ have the value zero. Without the network, we should expect the Shapley payoffs $\left(\frac{1}{2}, 0, \frac{1}{2}\right)$:

- Player 2 is unimportant (a null player, as we will say later) and obtains nothing.
- The two players 1 and 3 are symmetric and share the worth of 1.

However, we assume restrictions in cooperation or communication. In particular, players 1 and 3 are not directly linked (see the upper part of fig. 1). Player 2's role is to link up the productive players 1 and 3. How should he be rewarded for his linking service?

It is plausible that the payoffs are zero for all players in case of the network linking only players 1 and 2 (lower part of the figure). After all, the two productive players cannot cooperate.

Starting with the upper network and assuming that the link between players 2 and 3 can be formed (or dissolved) by mutual consent only, the removal of this link should harm both players equally:

$$\underbrace{x_2 - 0}_{\substack{\text{loss to player 2}\\ \text{if link is removed}}} = \underbrace{x_3 - 0}_{\substack{\text{loss to player 3}\\ \text{if link is removed}}} . \tag{I.6}$$

Recognizing the symmetry between players 1 and 3 in the upper network (both are productive and both need player 2 to realize their productive potential), we obain

$$\begin{aligned} 1 &= x_1 + x_2 + x_3 \\ &= x_3 + x_2 + x_3 \text{ (symmetry)} \\ &= x_3 + x_3 + x_3 \text{ (eq. I.6)} \end{aligned}$$

and hence

$$(x_1, x_2, x_3) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

6. The permission value

We now turn to directed networks that stand for permission structures. Consider, for example, the left-hand graph in fig. 2. Player 1 is the superior of both player 2 and 3 who have player 4 as their subordinate. We assume that player 4 is the only player with productive capacity, i.e., we work with the coalition function v given by

$$v(K) = \begin{cases} 1, & 4 \in K \\ 0, & \text{otherwise} \end{cases}$$

The Shapley value of that game would give all the payoff to the productive player 4.

However, the permission structure changes the payoffs. A coalition can produce a worth only with its largest autonomous subset. A set K is autonomous if all of K's players' superiors are contained in K – otherwise the permission of somebody outside K is needed. For example, $\{2,4\}$ is not an autonomous set because 2's superior, player 1, is not included.

We want to find the permission payoffs for the left-hand permission structure. However, we look at the right-hand permission structure first. Player 3 is a null player and has no subordinates – he should obtain zero payoff.

The only autonomous subset that contains the productive player 4 and excludes player 3 is the set $\{1,2,4\}$. Therefore, these three players obtain the same payoff which should be $\frac{1}{3}$ be efficiency. Thus, the right-hand graph's permission payoffs can be summarized in the vector $\left(\frac{1}{3}, \frac{1}{3}0, \frac{1}{3}\right)$.

What effect does the deletion of the direct link between players 3 and 4 (3 is not a superior of 4 anymore) have on the players' payoffs? We should expect that player 4 benefits because he does not need player 3's permission anymore. Player 2 should also benefit because he is now the only superior of the productive player 4. The big boss, player 1, may also benefit from giving permission to the permission giver of the productive player.

Indeed, the permission value rests on the premise that players 1, 2 and 4 benefit equally:

$$\underbrace{x_1 - \frac{1}{3}}_{\substack{\text{gain to player 1} \\ \text{if link is removed}}} = \underbrace{x_2 - \frac{1}{3}}_{\substack{\text{gain to player 2} \\ \text{if link is removed}}} = \underbrace{x_4 - \frac{1}{3}}_{\substack{\text{gain to player 4} \\ \text{if link is removed}}} . \quad \text{(I.7)}$$

Therefore, we obtain

$$\begin{aligned} 1 &= x_1 + x_2 + x_3 + x_4 \\ &= x_1 + x_2 + x_2 + x_4 \text{ (symmetry between players 2 and 3)} \\ &= x_2 + x_2 + x_2 + x_2 \text{ (eq. I.7)} \end{aligned}$$

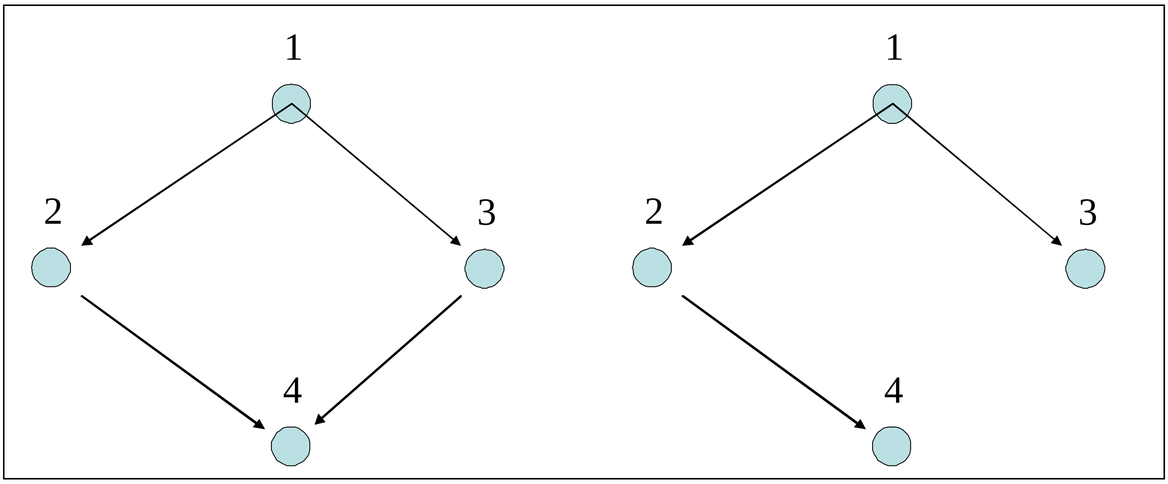

FIGURE 2. A simple hierarchy

and hence

$$(x_1, x_2, x_3, x_4) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right).$$

7. Cooperative and noncooperative game theories

It is sometimes suggested that non-cooperative game theory is more fundamental than cooperative game theory. Indeed, from an economic or sociological point of view, cooperative game theory seems odd in that it does not model people who "act", "know about things", or "have preferences". In cooperative game theory, people just get payoffs. Cooperative game theory is payoff-centered game theory. Noncooperative game theory (which turns around strategies and equilibria) could be termed action-centered or strategy-centered. Of course, non-cooperative game theory's strength does not come without a cost. The modeller is forced to specify in detail (sequences of) actions, knowledge and preferences. More often than not, these details cannot be obtained by the modeller. Cooperative game theory is better at providing a bird's eye view.

On the other hand, cooperative game theory is more demanding in terms of interpretation. It is the modeler's task to imagine a story behind a coalition function or to translate a story into a coalition function. Also, while cooperative game theory yields payoffs, these payoffs often suggest actions.

While the two theories rely on very different methods, they get close for two different reasons. Imagine a cooperative solution concept that produces certain payoffs for the players. One can ask the question whether a noncooperative model exists that leads to the same payoffs. This is the so-called Nash program. Of course, the inverse is also possible. Take a noncooperative model that leads to certain payoffs in equilibrium. Is there a cooperative model that also produces these payoffs?

Second, for some applications, mixtures of noncooperative and cooperative models prove quite useful. The first part of the model is noncooperative and the last cooperative. In this book, we will employ mixed models several times.

8. This book

8.1. Overview. I finally decided on the following order of parts and chapters:

- The present part consists of this introductory chapter and a chapter on Pareto efficiency. In that chapter, we present a wide range of microeconomic models through the lens of the Pareto principle which is one the most welknown cooperative solution concepts.
- Part B is a careful and slow introduction into cooperative game theory. In particular, chapter III uses the gloves game as the leading example to explain the workings of the Shapley value and the core, arguably the two main cooperative concepts. Many other games are presented in chapter IV which also defines general properties of coalition functions. Chapter V is more technical and considers the vector space of coalition functions. The results obtained are used in chapter VI where three different axiomatizations of the Shapley value are presented and discussed. Also, the Banzhaf value gets a short treatment.
- Parts C and D introduce additional structure on the player set. Part C deals with partitional values based on the Shapley value such as the AD value, the union value and the outside-option values.
  - Chapter VII deals with partitions where the players within a component share the component's worth while outside options are taken into account. For example, we consider the power of parties within government coalitions. Here some political parties work together to create power. The outside options concern other parties with which alternative government coalitions could have been formed.
  - Working together to create worth is the reason for forming components in chapter VII. In contrast, forming bargaining groups is the topic treated in chapter VIII. Unions are a prime example.
  - In chapter IX, we present an application that rests on dealing with worth-creating components (firms) and bargaining components (unions) at the same time. We consider the question

of how unions and unemployment benefits influence emoployment.

- Part D concentrates on networks and the Myerson value (chapter X). A first applications concerns the Granovetter thesis (that weak links are more important than strong links) in chapter **??**. We then go on to consider subordination structures in chapter XI which come as permission and use structures. Chapter XII concentrates on hierarchies and on their payoff consequences.
- Part E deviates from the previous parts in two respects.
  - First, players work part-time in a firm (chapter XIII) or as a lazy civil servant (chapter XIV). We need to extend the concept of a coalition function in order to deal with this generalization.
  - Second some players' payoffs are given from the outset. Chapter XIV analyzes the size and setup of the public sector in an economy while chapter XV deals with a real-estate agent who decides on his fees.
- Players in parts B to **??** are atomic (indivisible). Part F is concerned with two models where non-atomic agents or players form continua. In order to keep the book self-contained, we present the Solow growth model (chapter XVI) before introducing the continuous Shapley value into a Solow-type growth model (chapter XVII). Finally, we show that an evolutionary cooperative game theory can be developed and produces interesting results (chapter XVIII).
- Finally, part G turns to non-transferable utility. We examine the allocation of goods within the Edgeworth box (chapter XIX) and also present the Nash bargaining solution (chapter XX).

8.2. Alternative paths through the book. The careful reader goes through the book in the above order. However, different "pick and choose" options present themselves.

- The classical path: parts B and G

  Arguably, every economist worth his salt should know the Shapley value, the core, the Banzhaf solution, the core for an exchange economy and the Nash bargaining solution. If that is all you want, stick to the classical path.
- The structured path: parts B, C and D

  If you are interested in applications involving partitions and networks, you may choose to restrict attention to chapters III and VI within part B before turning to Shapley values where players are structured in some way or other. Chapters VII, VIII, and X present the basic theory with some applications while chapters IX, **??**, and XII put additional flesh on these models.

- The innovative path: parts E and F

  Knowledgeable readers may well get bored with most chapters in this book. May-be, some chapters in the innovative path will grab their attention?

CHAPTER II

Pareto optimality in microeconomics

Although the Pareto principle belongs to cooperative game theory, it sheds an interesting light on many different models in microeconomics. We consider bargaining between consumers, producers, countries in international trade, and bargaining in the context of public goods and externalities. We can also subsume profit maximization and household theory under this heading. It turns out that it suffices to consider three different cases with many subcases:

- equality of marginal rates of substitution
- equality of marginal rates of transformation and
- equality of marginal rate of substitution and marginal rate of transformation

Thus, we consider a wide range of microeconomic topics through the lens of Pareto optimality.

1. Introduction: Pareto improvements

Economists are somewhat restricted when it comes to judgements on the relative advantages of economic situations. The reason is that ordinal utility does not allow for comparison of the utilities of different people.

However, situations can be ranked with the concepts provided by Vilfredo Pareto (Italian sociologue, 1848-1923). Situation 1 is called a Pareto superior to situation 2 if no individual is worse off in the first than in the second while at least one individual is strictly better off. Then, the move from situation 2 to 1 is called a Pareto improvement. Situations are called Pareto efficient, Pareto optimal or just efficient if Pareto improvements are not possible.

EXERCISE II.1. *a) Is the redistribution of wealth a Pareto improvement if it reduces social inequality?*
b) Can a situation be efficient if one individual possesses everything?

This chapter rests on the premise that bargaining leads to an efficient outcome under ideal conditions. As long as Pareto improvements are available, there is no reason (so one could argue) not to "cash in" on them.

2. Identical marginal rates of substitution

2.1. Exchange Edgeworth box and marginal rate of substitution. We consider agents or households that consume bundles of goods. A distribution of such bundles among all households is called an allocation. In a two-agent two-good environment, allocations can be visualized via the Edgeworth box. Exchange Edgeworth boxes also allow to depict preferences by the use of indifference curves.

The analysis of bargaining between consumers in an exchange Edgeworth box is due to Francis Ysidro Edgeworth (1845-1926). Edgeworth's (1881) book bears the beautiful title "Mathematical Psychics". Fig. 1 represents the exchange Edgeworth box for goods 1 and 2 and individuals A and B. The exchange Edgeworth box exhibits two points of origin, one for individual A (bottom-left corner) and another one for individual B (top right).

Every point in the box denotes an allocation: how much of each good belongs to which individual. One possible allocation is the (initial) endowment $\omega = \left(\omega^A, \omega^B\right)$. Individual A possesses an endowment $\omega^A = \left(\omega_1^A, \omega_2^A\right)$, i.e., ω_1^A units of good 1 and ω_2^A units of good 2. Similarly, individual B has an endowment $\omega^B = \left(\omega_1^B, \omega_2^B\right)$.

All allocations $\left(x^A, x^B\right)$ with

- $x^A = \left(x_1^A, x_2^A\right)$ for individual A and
- $x^B = \left(x_1^B, x_2^B\right)$ for individual B

that can be represented in an Edgeworth box with initial endowment ω fulfill

$$\begin{aligned} x_1^A + x_1^B &= \omega_1^A + \omega_1^B \text{ and} \\ x_2^A + x_2^B &= \omega_2^A + \omega_2^B. \end{aligned}$$

Exercise II.2. *Do the two individuals in fig. 1 possess the same quantities of good* 1*, i.e., do we have* $\omega_1^A = \omega_1^B$*?*

Exercise II.3. *Interpret the length and the breadth of the Edgeworth box!*

Seen from the respective points of origin, the Edgeworth box depicts the two individuals' preferences via indifference curves. Refer to fig. 1 when you work on the following exercise.

Exercise II.4. *Which bundles of goods does individual A prefer to his endowment? Which allocations do both individuals prefer to their endowments?*

The two indifference curves in fig. 1, crossing at the endowment point, form the so-called exchange lens which represents those allocations that are Pareto improvements to the endowment point. A Pareto efficient allocation is achieved if no further improvement is possible. Then, no individual can be

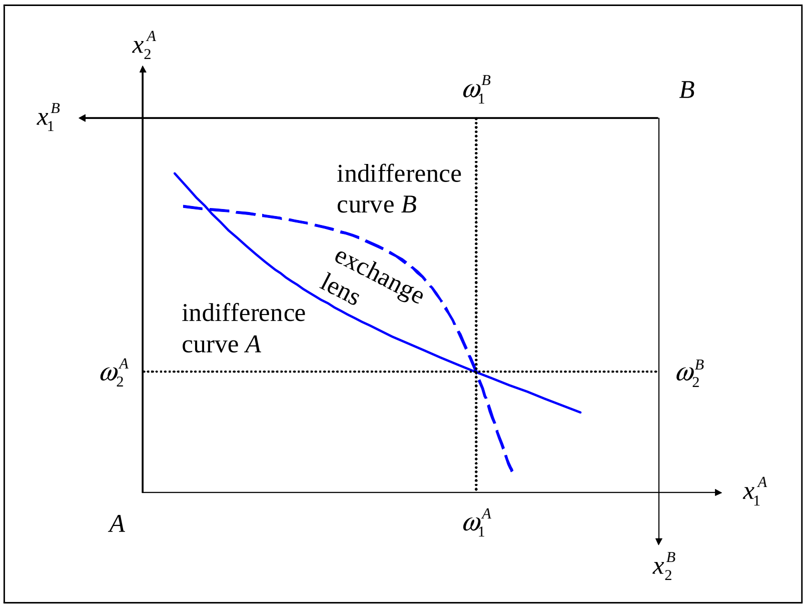

FIGURE 1. The exchange Edgeworth box

made better off without making the other worse off. Oftentimes, we imagine that individuals achieve a Pareto efficient point by a series of exchanges. As long as a Pareto optimum has not been reached, they will try to improve their lots.

Finally, we turn to the equality of the marginal rates of substitution. Graphically, the marginal rate of substitution $MRS = \left|\frac{dx_2}{dx_1}\right|$ is the absolute value of an indifference's slope. If one additional unit of good 1 is consumed while good 2's consumption reduces by MRS units, the consumer stays indifferent. We could also say: MRS measures the willingness to pay for one additional unit of good 1 in terms of good 2.

2.2. Equality of the marginal rates of substitution. Consider, now, an exchange economy with two individuals A and B where the marginal rate of substitution of individual A is smaller than that of individual B:

$$(3 =) \left|\frac{dx_2^A}{dx_1^A}\right| = MRS^A < MRS^B = \left|\frac{dx_2^B}{dx_1^B}\right| (= 5)$$

We can show that this situation allows Pareto improvements. Individual A is prepared to give up a small amount of good 1 in exchange for at least MRS^A units (3, for example) of good 2. If individual B obtains a small amount of good 1, he is prepared to give up MRS^B (5, for example) or less units of good 2. Thus, if A gives one unit of good 1 to B, by $MRS^A < MRS^B$ individual B can offer more of good 2 in exchange than individual A would require for compensation. The two agents might agree on 4 units so that

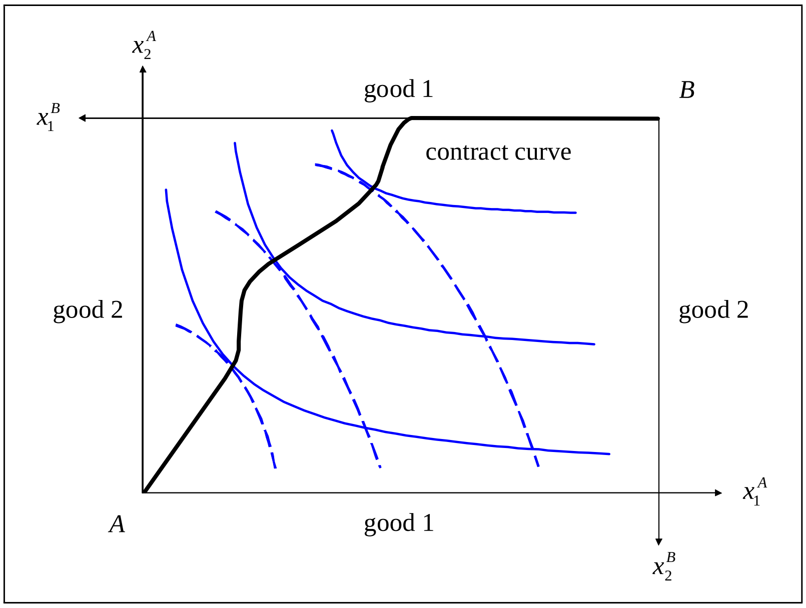

FIGURE 2. The contract curve

both of them would be better off. Thus, the above inequality signals the possibility of mutually beneficial trade.

Differently put, Pareto optimality requires the equality of the marginal rates of substitution for any two agents A and B and any pair of goods 1 and 2. The locus of all Pareto optima in the Edgeworth box is called the contract curve or exchange curve (see fig. 2).

EXERCISE II.5. *Two consumers meet on an exchange market with two goods. Both have the utility function* $U(x_1, x_2) = x_1 x_2$. *Consumer* A*'s endowment is* $(10, 90)$, *consumer* B*'s is* $(90, 10)$.
a) Depict the endowments in the Edgeworth box!
b) Find the contract curve and draw it!
c) Find the best bundle that consumer B *can achieve through exchange!*
d) Draw the Pareto improvement (exchange lens) and the Pareto-efficient Pareto improvements!
e) Sketch the utility frontier!

2.3. Production Edgeworth box. The exchange Edgeworth box looks at two consumers that consume two goods and have preferences indicated by their indifference curves. Similarly, the production Edgeworth box is concerned with two producers that employ two factors of production where the production technology is reflected in isoquants. Consider the example of fig. 3. You see two families of isoquants, one for output A and one for output B (turn the book by 180 degrees). The breadth indicates the amount of factor 1 and the height the amount of factor 2. Every point inside that box shows how the inputs 1 and 2 are allocated to produce the outputs A and B.

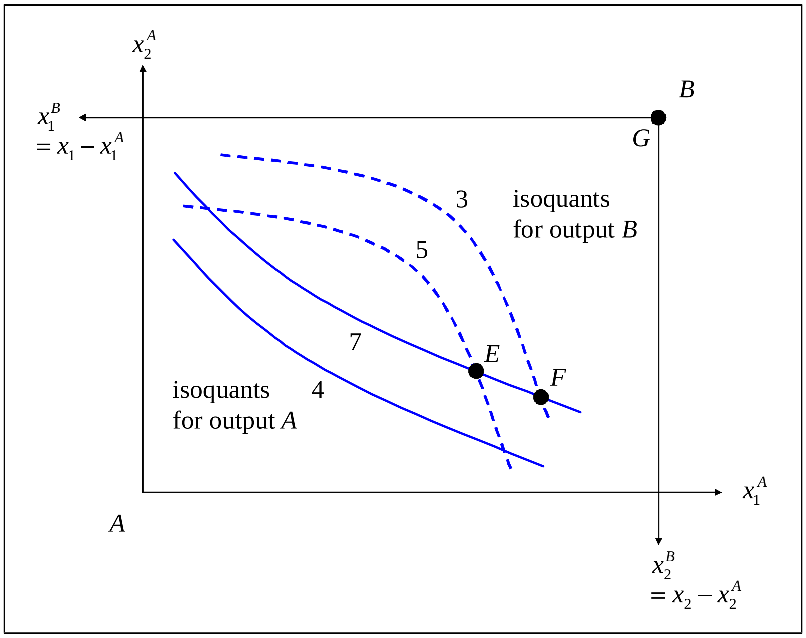

FIGURE 3. A production Edgeworth box

The quantities produced are indicated by the isoquants and the numbers associated with them. Consider, for example, points E and F. They both use the same input tuple (x_1, x_2) (the overall use of both factors), but the output is different, $(7, 5)$ in case of point E and $(7, 3)$ in case of point F.

The marginal rate of technical substitution $MRTS = \left|\frac{dC}{dL}\right|$ is the slope of an isoquant and gives an answer to this question: If we increase the input of labor L by one unit, by how many units can the use of capital C be reduced so that we still produce the same output. The $MRTS$ can be interpreted as the marginal willingness to pay for an additional unit of labor in terms of capital. If two producers 1 and 2 produce goods 1 and 2, respectively, with inputs labor and capital, both can increase their production as long as the marginal rates of technical substitution differ. For example, point E is not efficient.

Thus, Pareto efficiency means

$$\left|\frac{dC_1}{dL_1}\right| = MRTS_1 \stackrel{!}{=} MRTS_2 = \left|\frac{dC_2}{dL_2}\right|$$

so that the marginal willingness to pay for input factors are the same.

2.4. Two markets – one factory. The third subcase under the heading "equality of the marginal willingness to pay" concerns a firm that produces in one factory but supplies two markets 1 and 2. The idea is to consider the marginal revenue $MR = \frac{dR}{dx_i}$ as the monetary marginal willingness to pay for selling one extra unit of good i. How much can a firm pay for the sale of one additional unit?

Thus, the marginal revenue is a marginal rate of substitution $\left|\frac{dR}{dx_i}\right|$. The role of the denominator good is taken over by good 1 or 2, respectively, while the nominator good is "money" (revenue). Now, profit maximization by a firm selling on two markets 1 and 2 implies

$$\left|\frac{dR}{dx_1}\right| = MR_1 \overset{!}{=} MR_2 = \left|\frac{dR}{dx_2}\right|$$

which we can show by contradiction. Assume $MR_1 < MR_2$. The monopolist can transfer one unit from market 1 to market 2. Revenue and profit (we have not changed total output $x_1 + x_2$) increases by $MR_2 - MR_1$.

2.5. Two firms (cartel). The monetary marginal willingness to pay for producing *and* selling one extra unit of good y is a marginal rate of substitution where the denominator good is good 1 or 2 while the nominator good represents "money" (profit). Two cartelists 1 and 2 producing the quantities x_1 and x_2, respectively, maximize their joint profit

$$\Pi_{1,2}(x_1, x_2) = \Pi_1(x_1, x_2) + \Pi_2(x_1, x_2)$$

by obeying the first-order conditions

$$\frac{\partial \Pi_{1,2}}{\partial x_1} \overset{!}{=} 0 \overset{!}{=} \frac{\partial \Pi_{1,2}}{\partial x_2}$$

so that their marginal rates of substitution are the same when profit is understood as joint profit. If $\frac{\partial \Pi_{1,2}}{\partial x_2}$ were higher than $\frac{\partial \Pi_{1,2}}{\partial x_1}$ the cartel could increase profits by shifting the production of one unit from firm 1 to firm 2.

3. Identical marginal rates of transformation

3.1. Marginal rate of transformation. The marginal rate of substitution tells us how much of good 2 an agent is willing to give up if given an extra unit of good 1. In contrast, the marginal rate of transformation MRT informs about the harsh realities of life: how many units of good 2 have to be given up if one extra unit of good 1 is to be consumed or used. Differently put, the marginal rate of substitution is a willingness to pay while the marginal rate of transformation can be seen as a marginal opportunity cost.

The production Edgeworth box introduced above can be used to derive the marginal rate of transformation. If the marginal rates of technical substitutions are equal, we have found an efficient point. The locus of all these points is called the production curve and shown in fig. 4.

A production function associates one specific output with a tuple of inputs. The Edgeworth box shows how to associate a set of two outputs with a tuple of inputs. This set can be read from the isoquants. Referring again to fig. 4, the points $(9, 5)$ and $(11, 3)$ belong to this set. In that manner, a transformation curve (also known as production-possibility frontier) can be derived from a production curve. For an illustration, consider fig. 5.

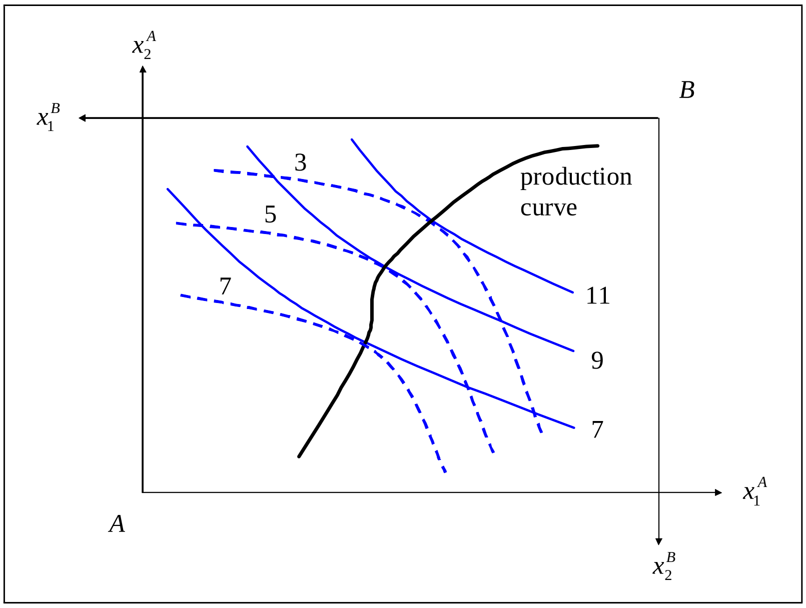

FIGURE 4. The production curve

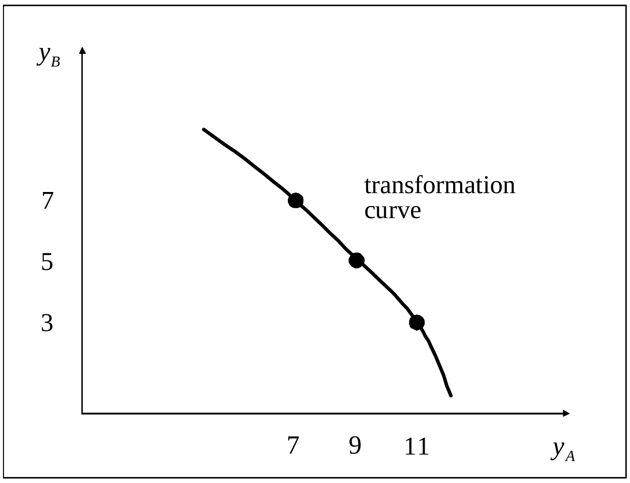

FIGURE 5. The transformation curve

Now, the marginal marginal rate of transformation can be defined as the absolute value of the slope of a transformation curve. With respect to the transformation curve depicted abovewe write $MRT = \left|\frac{dy_B}{dy_A}\right|^{\text{transformation curve}}$.

3.2. Two factories – one market. While the marginal revenue can be understood as the monetary marginal willingness to pay for selling, the marginal cost $MC = \frac{dC}{dy}$ can be seen as the monetary marginal opportunity cost of production. How much money (the second good) must the producer forgo in order to produce an extra unit of y (the first good)? Thus, the margincal cost can be seen as a special case of the marginal rate of transformation.

Similar to section 2.4, we argue that $MC_1 < MC_2$ leaves room for an improvement: A transfer of one unit of production from the (marginally!) more expensive factory 2 to the cheaper factory 1, decreases cost, and increases profit, by $MC_2 - MC_1$. Therefore, a firm supplying a market from two factories (or a cartel in case of homogeneous goods), obeys the equality

$$MC_1 \stackrel{!}{=} MC_2.$$

The cartel also makes clear that Pareto improvements and Pareto optimality have to be defined relative to a specific group of agents. While the cartel solution (maximizing the sum of profits) can be optimal for the producers, it is not for the economy as a whole because the sum of producers' and consumers' (!) rent may well be below the welfare optimum.

3.3. Bargaining between countries (international trade). David Ricardo (1772–1823) has shown that international trade is profitable as long as the rates of transformation between any two countries are different. Let us consider the classic example of England and Portugal producing wine (W) and cloth (Cl). Suppose that the marginal rates of transformation differ:

$$4 = MRT^P = \left|\frac{dW}{dCl}\right|^P > \left|\frac{dW}{dCl}\right|^E = MRT^E = 2.$$

In that case, international trade is Pareto-improving. Indeed, let England produce another unit of cloth Cl that it exports to Portugal. England's production of wine reduces by $MRT^E = 2$ gallons. Portugal, that imports one unit of cloth, reduces the cloth production and can produce additional $MRT^P = 4$ units of wine. Therefore, if England obtains 3 gallons of wine in exchange for the one unit of cloth it gives to Portugal, both countries are better off.

Ricardo's theorem is known under the heading of "comparative cost advantage". It seems that "differing marginal rates of transformation" might be a better name. However, you take my word that the marginal rate of transformation equals the ratio of the marginal costs (when factor prices are given),

$$MRT = \left|\frac{dW}{dCl}\right| = \frac{MC_{Cl}}{MC_W},$$

so that we have Ricardo's result in the form it is usually presented: As long as the comparative costs (more precise: the ratio of marginal costs) between two goods differ, international trade is worthwhile for both countries.

Thus, Pareto optimality requires the equality of the marginal opportunity costs between any two goods produced in any two countries. The economists before Ricardo clearly saw that absolute cost advantages make international trade profitable. If England can produce cloth cheaper than Portugal while Portugal can produce wine cheaper than England, we have

$$\begin{aligned} MC_{Cl}^{E} &< MC_{Cl}^{P} \text{ and} \\ MC_{W}^{E} &> MC_{W}^{P} \end{aligned}$$

so that England should produce more cloth and Portugal should produce more wine. Ricardo observed that for the implied division of labor to be profitable, it is sufficient that the ratio of the marginal costs differ:

$$\frac{MC_{Cl}^{E}}{MC_{W}^{E}} < \frac{MC_{Cl}^{P}}{MC_{W}^{P}}.$$

Do you see that this inequality follows from the two inequalities above, but not vice versa?

4. Equality between marginal rate of substitution and marginal rate of transformation

4.1. Base case. Imagine two goods consumed at a marginal rate of substitution MRS and produced at a marginal rate of transformation MRT. We now show that optimality also implies $MRS = MRT$. Assume, to the contrary, that the marginal rate of substitution (for a consumer) is lower than the marginal rate of transformation (for a producer):

$$MRS = \left| \frac{dx_2}{dx_1} \right|^{\text{indifference curve}} < \left| \frac{dx_2}{dx_1} \right|^{\text{transformation curve}} = MRT.$$

If the producer reduces the production of good 1 by one unit, he can increase the production of good 2 by MRT units. The consumer has to renounce the one unit of good 1, and he needs at least MRS units of good 2 to make up for this. By $MRT > MRS$ the additional production of good 2 (come about by producing one unit less of good 1) more than suffices to compensate the consumer. Thus, the inequality of marginal rate of substitution and marginal rate of transformation points to a Pareto-inefficient situation.

4.2. Perfect competition. We want to apply the formula

$$MRS \stackrel{!}{=} MRT$$

to the case of perfect competition. For the output space, we have the profit-maximizing condition

$$p \stackrel{!}{=} MC.$$

We have derived "price equals marginal cost" by forming the derivative of profit $\pi(y) = py - c(y)$ with respect to y and setting this derivative equal to zero.

We can link the two formulae by letting good 2 be money with price 1.

- Then, the marginal rate of substitution tells us the consumer's monetary marginal willingness to pay for one additional unit of good 1. Cum grano salis, the price can be taken to measure this willingness to pay for the marginal consumer (the last consumer prepared to buy the good).
- The marginal rate of transformation is the amount of money one has to forgo for producing one additional unit of good 1, i.e., the marginal cost.

Therefore, we obtain

$$\text{price} = \text{marginal willingness to pay } \stackrel{!}{=} \text{marginal cost.}$$

In a similar fashion, we can argue for inputs. Let x be the amount of an input and $y = f(x)$ the amount of an output. The marginal value product $MVP = p\frac{dy}{dx}$ is the product of output price p and marginal product $\frac{dy}{dx}$. It can be understood as the monetary marginal willingness to pay for the factor use. The factor price w can be perceived as the monetary marginal opportunity cost of employing the factor. Thus, we obtain

$$\text{marginal value product } \stackrel{!}{=} \text{factor price}$$

which is the optimization condition for a price taker on both the input and the output market. Just consider the profit function $\pi(x) = pf(x) - wx$, form the derivative ... .

4.3. Cournot monopoly. A trivial violation of Pareto optimality ensues if a single agent acts in a non-optimal fashion. Just consider consumer and producer as a single person. For the Cournot monopolist, the $MRS \stackrel{!}{=} MRT$ formula can be rephrased as the equality between

- the monetary marginal willingness to pay for selling – this is the marginal revenue $MR = \frac{dR}{dy}$ (see above p. 19) – and
- the monetary marginal opportunity cost of production, the marginal cost $MC = \frac{dC}{dy}$ (p. 21).

4.4. Household optimum. A second violation of efficiency concerns the consuming household. It "produces" goods by using his income to buy them, $m = p_1x_1 + p_2x_2$ in case of two goods.

Exercise II.6. *The prices of two goods* 1 *and* 2 *are* $p_1 = 6$ *and* $p_2 = 2$, *respectively. If the household consumes one additional unit of good* 1, *how many units of good* 2 *does he have to renounce?*

The exercise helps us understand that the marginal rate of transformation is the price ratio,

$$MRT = \frac{p_1}{p_2},$$

that we also know under the heading of "marginal opportunity cost". (Alternatively, consider the transformation function $x_2 = f(x_1) = \frac{m}{p_2} - \frac{p_1}{p_2} x_1$.). Seen this way, $MRS \stackrel{!}{=} MRT$ is nothing but the famous condition for household optimality.

4.5. External effects and the Coase theorem.

4.5.1. *External effects and bargaining.* The famous Coase theorem can also be interpreted as an instance of $MRS \stackrel{!}{=} MRT$. We present this example in some detail.

External effects are said to be present if consumption or production activities are influenced positively or negatively while no compensation is paid for this influence. Environmental issues are often discussed in terms of negative externalities. Also, the increase of production exerts a negative influence on other firms that try to sell subsitutes. Reciprocal effects exist between beekeepers and apple planters.

Consider a situation where A pollutes the environment doing harm to B. In a very famous and influential paper, Coase (1960) argues that economists have seen environmental and similar problems in a misguided way.

First of all, externalities are a "reciprocal problem". By this Coase means that restraining A from polluting harms A (and benefits B). According to Coase, the question to be decided is whether the harm done to B (suffering the polluting) is greater or smaller than the harm done to A (by stopping A's polluting activities).

Second, many problems resulting from externalities stem from missing property rights. Agent A may not be in a position to sell or buy the right to pollute from B simply because property exists for cars and real estate but not for air, water or quietness. Coase suggests that the agents A and B bargaing about the externality. If, for example, A has the right to pollute (i.e., is not liable for the damage cause by him), B can give him some money so that A reduce his harmful (to B) activity. If B has the right not to suffer any pollution (i.e., A is liable), A could approach B and offer some money in order to pursue some of the activity benefitting him. Coase assumes (as we have done in this chapter) that the two parties bargain about the externality so as to obtain a Pareto-efficient outcome.

The Nobel prize winner (of 1991) presents a startling thesis: the externality (the pollution etc.) is independent on the initial distribution of property rights. This thesis is also known as the invariance hypothesis.

4.5.2. *Straying cattle.* Coase (1960) discusses the example of a cattle raiser and a crop farmer who possess adjoining land. The cattle regularly destroys part of the farmer's crop. In particular, consider the following table:

number of steers	marginal profit	marginal crop loss
1	4	1
2	3	2
3	2	3
4	1	4

The cattle raiser's marginal profit from steers is a decreasing function of the number of steers while the marginal crop loss increases. Let us begin with the case where the cattle raiser is liable. He can pay the farmer up to 4 (thousand Euros) for allowing him to have one cattle destroy crop. Since the farmer's compensating variation is 1, the two can easily agree on a price of 2 or 3.

The farmer and cattle raiser will also agree to have a second steer roam the fields, for a price of $2\frac{1}{2}$. However, there are no gains from trade to be had for the third steer. The willingness to pay of 2 is below the compensation money of 3.

If the cattle raiser is not liable, the farmer has to pay for reducing the number of steers from 4 to 3. A Pareto improvement can be have for any price between 1 and 4. Also, the farmer will convince the cattle raiser to take the third steer, but not the second one, off the field.

Thus, Coase seems to have a good point – irrespective of the property rights (the liability question), the number of steers and the amount of crop damaged is the same.

The reason for the validity (so far) of the Coase theorem is the fact that forgone profits are losses and forgone losses are profits. Therefore, the numbers used in the comparisons are the same.

It is about time to tell the reader why we talk about the Coase theorem in the $MRS \stackrel{!}{=} MRT$ section. From the cartel example, we are familiar with the idea of finding a Pareto optimum by looking at joint profits. We interpret the cattle raiser's marginal profit as the (hypothetical) joint firm's willingness to pay for another steer and the marginal crop loss incurred by the farmer as the joint firm's marginal opportunity cost for that extra steer.

We close this section by throwing in two caveats:

- If consumers are involved, the distribution of property rights has income effects. Then, Coase's theorem does not hold any more (see Varian 2010, chapter 31).
- More important is the objection raised by Wegehenkel (1980). The distribution of property rights determine who pays whom. Thus, if the property rights were to change from non-liability to liability, cattle raising becomes a less profitable business while growing crops is more worthwhile as before. In the medium run, agents will move

to the profitable occupations with effects on the crop losses (the sign is not clear a priori).

4.6. Public goods. Public goods are defined by non-rivalry in consumption. While an apple can be eaten only once, the consumption of a public good by one individual does not reduce the consumption possibilities by others. Often-cited examples include street lamps or national defence.

Consider two individuals A and B who consume a private good x (quantities x^A and x^B, respectively) and a public good G. The optimality condition is

$$\begin{aligned} & MRS^A + MRS^B \\ = \; & \left| \frac{dx^A}{dG} \right|^{\text{indifference curve}} + \left| \frac{dx^B}{dG} \right|^{\text{indifference curve}} \\ \overset{!}{=} \; & \left| \frac{d\left(x^A + x^B\right)}{dG} \right|^{\text{transformation curve}} = MRT. \end{aligned}$$

Assume that this condition is not fulfilled. For example, let the marginal rate of transformation be smaller than the sum of the marginal rates of substitution. Then, it is a good idea to produce one additional unit of the public good. The two consumers need to forgo MRT units of the private good. However, they are prepared to give up $MRS^A + MRS^B$ units of the private good in exchange for one additional unit of the public good. Thus, they can give up more than they need to. Assuming monotonicity, the two consumers are better off than before and the starting point (inequality) does not characterize a Pareto optimum.

Once more, we can assume that good x is the numéraire good (money with price 1). Then, the optimality condition simplifies and Pareto efficiency requires that the sum of the marginal willingness' to pay equals the marginal cost of the public good.

EXERCISE II.7. *In a small town, there live 200 people* $i = 1, ..., 200$ *with identical preferences. Person* i*'s utility function is* $U_i(x_i, G) = x_i + \sqrt{G}$*, where* x_i *is the quantity of the private good and* G *the quantity of the public good. The prices are* $p_x = 1$ *and* $p_G = 10$*, respectively. Find the Pareto-optimal quantity of the public good.*

Thus, by the non-rivalry inconsumption, we do not quite get a subrule of $MRS \overset{!}{=} MRT$ but something similar.

5. Topics and literature

The main topics in this chapter are

- Pareto efficiency
- Pareto improvement

- exchange Edgeworth box
- contract curve
- exchange lense
- core
- international trade
- external effects
- quantity cartel
- public goods
- first-degree price discrimination

We recommend the textbook by

6. Solutions

Exercise II.1

a) A redistribution that reduces inequality will harm the rich. Therefore, such a redistribution is not a Pareto improvement.
b) Yes. It is not possible to improve the lot of the have-nots without harming the individual who possesses everything.

Exercise II.2

No, obviously ω_1^A is much larger than ω_1^B.

Exercise II.3

The length of the exchange Edgeworth box represents the units of good 1 to be divided between the two individuals, i.e., the sum of their endowment of good 1. Similarly, the breadth of the Edgeworth box is $\omega_2^A + \omega_2^B$.

Exercise II.4

Individual A prefers all those bundels x_A that lie to the right and above the indifference curve that crosses his endowment point. The allocations preferred by both individuals are those in the hatched part of fig. 1.

Exercise II.5

a) See fig. 6,
b) $x_1^A = x_2^A$,
c) $(70, 70)$.
d) The exchange lens is dotted in fig. 6. The Pareto efficient Pareto improvements are represented by the contract curve within this lens.
e) The utility frontier is downward sloping and given by $U_B(U_A) = \left(100 - \sqrt{U_A}\right)^2$.

Exercise II.6

If the household consumers one additional unit of good 1, he has to pay Euro 6. Therefore, he has to renounce 3 units of good 2 that also cost Euro 6 = Euro 2 times 3.

Exercise II.7

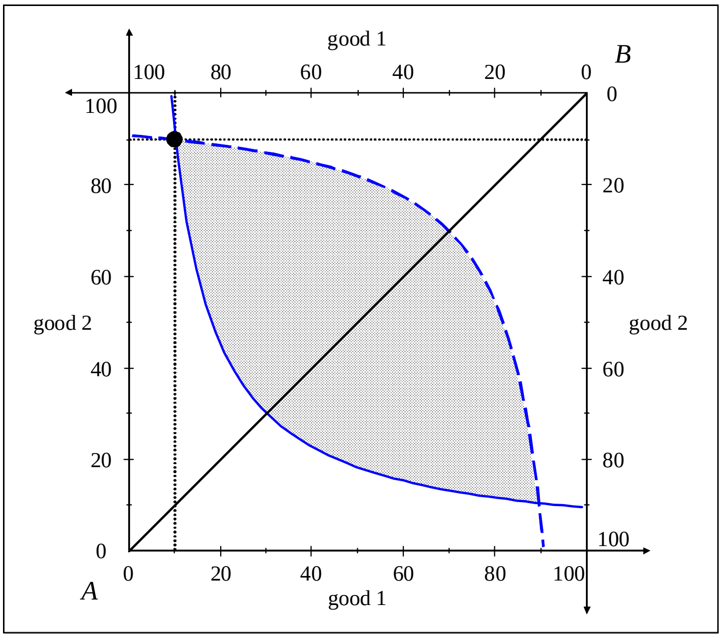

FIGURE 6. The answer to parts a) and d)

The marginal rate of transformation $\left|\frac{d\left(\sum_{i=1}^{200} x_i\right)}{dG}\right|$ equals $\frac{p_G}{p_x} = \frac{10}{1} = 10$. The marginal rate of substitution for inhabitant i is

$$\left|\frac{dx^i}{dG}\right|^{\text{indifference curve}} = \frac{MU_G}{MU_{x^i}} = \frac{\frac{1}{2\sqrt{G}}}{1} = \frac{1}{2\sqrt{G}}.$$

Applying the optimality condition yields

$$200 \cdot \frac{1}{2\sqrt{G}} \stackrel{!}{=} 10$$

and hence $G = 100$.

7. Further exercises without solutions

Agent A has preferences on (x_1, x_2), that can be represented by $u^A(x_1^A, x_2^A) = x_1^A$. Agent B has preferences, which are represented by the utility function $u^B(x_1^B, x_2^B) = x_2^B$. Agent A starts with $\omega_1^A = \omega_2^A = 5$, and B has the initial endowment $\omega_1^B = 4, \omega_2^B = 6$.

(a) Draw the Edgeworth box, including
   - ω,
   - an indifference curve for each agent through ω!

(b) Is $(x_1^A, x_2^A, x_1^B, x_2^B) = (6, 0, 3, 11)$ a Pareto-improvement compared to the initial allocation?

(c) Find the contract curve!

The Shapley value and the core

Part B

The Shapley value and the core

The second part of our course explains some important basic concepts. Chapter III introduces Pareto efficiency, the Shapley value and the core for a simple game, the gloves game. We present many examples of cooperative games in chapter IV. Games can be understood as vectors – this is the point of view we mention in the following chapter and discuss in detail in chapter V. We then deal with the axiomatization of the Shapley value in chapter VI. In that chapter, the Banzhaft index also gets a brief treatment. Partitions and networks have no role to play in this part of the book.

CHAPTER III

The gloves game

1. Introduction

This chapter lies the groundwork in cooperative game theory. First of all, section 2 familiarizes the reader with the player set N (the set of all players), subsets of N (that we also call coalitions) and the set of coalitions for a player set N.

We then use the specific example of gloves games to introduce the concept of a coalition function in section 3. As in most part of the book, we focus on transferable utility where v attaches a real number to every coalition. Thus, $v(K)$ is the worth or the "utility sum" created by the members from K. The basic idea is to distribute $v(K)$ or $v(N)$ among the members from K or N, respectively. Thus, the utility is "transferable".

Transferability is a serious assumption and does not work well in every model. Transferable utility is justfied if utility can be measured in terms of money and if the agents are risk neutral. We will need non-transferable utility for the analysis of exchange within an Edgeworth box (part G, chapter XIX).

Section 4 is devoted to a technical point. We define zero payoff vectors (everybody gets nothing) and zero coalition functions (every coalition creates nothing). We then turn to the main topic of cooperative game theory: solution concepts. We present a general definition in section 5 before presenting four specific examples:

(1) Most solution concepts presented in this book obey Pareto efficiency – we introduce this central concept in section 6. An efficient payoff vector is feasible (the players can afford it) and cannot be blocked by the player set N (it is not possible to improve upon that vector).

(2) A well-known subset of efficient payoff vectors is called the core (presented in section 7). A payoff vector from the core cannot be blocked by the player set N nor by any subset of N. The core for coalition functions has first been defined by Gillies (1959). Shubik (1981, S. 299) mentions that Lloyd Shapley proposes this concept as early as 1953 in unpublished lecture notes. In contrast to Pareto efficiency and the core, the rank-order values and the Shapley value

are point-valued solution concepts – for every coalition function, they spit out exactly one payoff vector.

(3) In order to prepare the reader for the Shapley value, we introduce the ρ-value in section 8. Its idea is to order the players (for example, player 2 first, then player 3 and player 1 last) and attribute to each player his marginal contribution – by how much does the worth of the coalition increase because this particular player joined.

(4) Shapley's (1953a) article is famous for pioneering the twofold approach of algorithm and axioms. The algorithmic definition of the Shapley value (which is a mean of the ρ-values for all different orders ρ) can be found in section 9 while section 10 introduces the axiomatic definition. The equivalence of these two approaches will be shown much later, in chapter VI.

2. Coalitions

All players together are assembled in the player set N. More often than not, we have $N = \{1, ..., n\}$ with $n \in \mathbb{N}$. Any subset K of N, $K \subseteq N$, is called a coalition. Two coalitions stand out:

- N itself is called the grand coalition.
- The empty set, denoted by $\emptyset$, is a subset of every player set N – it stands for no players at all.

Sometimes, we want to address the number of players in a coalition. There is a special symbol for that operation, $||$. Thus $|K|$ denotes the number of players in K which is also called K's cardinality.

EXERCISE III.1. *Determine* $|\emptyset|$ *and* $|N|$.

Consider the player set $N = \{1, 2, 3\}$. How many coalitions can we find? Here they are:

$$\begin{aligned}
&\emptyset, \\
&\{1\}, \{2\}, \{3\}, \\
&\{1,2\}, \{1,3\}, \{2,3\}, \\
&N
\end{aligned}$$

A three-player set has eight subsets. The set of $\{1, 2, 3\}$'s subsets is denoted by $2^{\{1,2,3\}}$. Thus, we find $\left|2^{\{1,2,3\}}\right| = 2^{|\{1,2,3\}|}$. (Look at it again and express this formula in words!) In fact, this is a general rule:

$$\left|2^N\right| = 2^{|N|}$$

where 2^N denotes the set of subsets of N. The above formula is a good reason for denoting the set of N's subsets by 2^N. There is another, very good reason. Consider a subset K of N. Every player i from N belongs to K ($i \in K$) or not ($i \in K$). Therefore, a coalition is characterized by giving one of two states ("in" or "out") for every player from N. Differently put, a coalition is a function

$$N \to \{\text{in, out}\}.$$

The set of these functions are also written as $\{\text{in, out}\}^N$ or simpler as 2^N. The set of all subsets of N (or any other set) is sometimes called N's power set.

EXERCISE III.2. *Which of the following propositions make sense? Any coalition K and any grand coalition N fulfill*

- $K \in N$ *and* $K \in 2^N$,
- $K \subseteq N$ *and* $K \subseteq 2^N$,
- $K \in N$ *and* $K \subseteq 2^N$ *and/or*
- $K \subseteq N$ *and* $K \in 2^N$?

We often need the set-theoretic concept of a complement:

DEFINITION III.1 (complement). *The set $N \backslash K := \{i \in N : i \notin K\}$ is called K's complement (with respect to N).*

":=" indicates that $N \backslash K$ (on the :-side) is defined to be equal to $\{i \in N : i \notin K\}$ (on the =-side).

EXERCISE III.3. *Consider $K = \{1, 3\}$. Determine K's complement with respect to $N = \{1, 2, 3\}$ and with respect to $N = \{1, 2, 3, 4\}$!*

3. The coalition function

In this chapter, we concentrate on a particular game, the gloves game. Some players have a left glove and others a right glove. Single gloves have a worth of zero while pairs have a worth of 1 (Euro). The coalition function for the gloves game is given by

$$\begin{aligned} v_{L,R} \quad : \quad & 2^N \to \mathbb{R} \\ & K \mapsto v_{L,R}(K) = \min(|K \cap L|, |K \cap R|), \end{aligned}$$

where

- L the set of players holding a left glove and R the set of right-glove owners together with $L \cap R = \emptyset$ and $L \cup R = N$,
- $v_{L,R}$ denotes the coalition function for the gloves game,
- 2^N is N's power set (the domain of $v_{L,R}$),
- $\mathbb{R}$ is the set of real numbers (the range of $v_{L,R}$),
- $|K \cap L|$ stands for the number of left gloves the players in coalition K possess, and
- $\min(x, y)$ is the smallest of the two numbers x and y.

Thus, the coalition function $v_{L,R}$ attributes the number of pairs in possession of some coalition K to that coalition.

Definition III.2 (player sets and coalition functions). *Player sets and coalition functions are specified by the following definitions:*

- $v : 2^N \to \mathbb{R}$ *is called a coalition function if* v *fulfills* $v(\emptyset) = 0$. $v(K)$ *is called coalition* K*'s worth.*
- *For any given coalition function* v, *its player set can be addressed by* $N(v)$ *or, more simply,* N.
- *We denote the set of all games on* N *by* $\mathbb{V}_N$ *and the set of all games (for any player set* N*) by* $\mathbb{V}$.

Exercise III.4. *Assume* $N = \{1,2,3,4,5\}$, $L = \{1,2\}$ *and* $R = \{3,4,5\}$. *Find the worths of the coalitions* $K = \{1\}$, $K = \emptyset$, $K = N$ *and* $K = \{2,3,4\}$.

The above exercise makes clear that $v_{L,R}$ is, indeed, a coalition function. The requirement of $v(\emptyset) = 0$ makes perfect sense: a group of zero agents cannot achieve anything.

We can interpret the gloves game as a market game where the left-glove owners form one market side and the right-glove owners the other. We need to distinguish the worth (of a coalition) from the payoff acrruing to players.

4. Summing and zeros

Payoffs for players are summarized in payoff vectors:

Definition III.3. *For any finite and nonempty player set* $N = \{1, ..., n\}$, *a payoff vector*

$$x = (x_1, ..., x_n) \in \mathbb{R}^n$$

specifies payoffs for all players $i = 1, ..., n$.

It is possible to sum coalition functions and it is possible to sum payoff vectors. Summation of vectors is easy – just sum each component individually:

Exercise III.5. *Determine the sum of the vectors*

$$\begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}!$$

Note the difference between payoff-vector summation

$$x + y = \begin{pmatrix} x_1 \\ x_2 \\ \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \\ x_n + y_n \end{pmatrix}$$

and payoff summation

$$\sum_{i=1}^{n} x_i.$$

Vector summation is possible for coalition functions, too. For example, we obtain the sum $v_{\{1\},\{2,3\}}+v_{\{1,2\},\{3\}}$ by summing the worths $v_{\{1\},\{2,3\}}(K)+v_{\{1,2\},\{3\}}(K)$ for every coalition K, from the empty set $\emptyset$ down to the grand coalition $\{1,2,3\}$:

$$\begin{pmatrix} \emptyset : 0 \\ \{1\} : 0 \\ \{2\} : 0 \\ \{3\} : 0 \\ \{1,2\} : 1 \\ \{1,3\} : 1 \\ \{2,3\} : 0 \\ \{1,2,3\} : 1 \end{pmatrix} + \begin{pmatrix} \emptyset : 0 \\ \{1\} : 0 \\ \{2\} : 0 \\ \{3\} : 0 \\ \{1,2\} : 0 \\ \{1,3\} : 1 \\ \{2,3\} : 1 \\ \{1,2,3\} : 1 \end{pmatrix} = \begin{pmatrix} \emptyset : 0 \\ \{1\} : 0 \\ \{2\} : 0 \\ \{3\} : 0 \\ \{1,2\} : 1 \\ \{1,3\} : 2 \\ \{2,3\} : 1 \\ \{1,2,3\} : 2 \end{pmatrix}$$

Of course, we need to agree upon a specific order of coalitions.

Mathematically speaking, $\mathbb{R}^n$ and $\mathbb{V}_N$ can be considered as vector spaces. Vector spaces have a zero. The zero from $\mathbb{R}^n$ is

$$\underset{\in\mathbb{R}^n}{0} = \left(\underset{\in\mathbb{R}}{0}, ..., \underset{\in\mathbb{R}}{0}\right)$$

where the zero on the left-hand side is the zero vector while the zeros on the right-hand side are just the zero payoffs for all the individual players. In the vector space of coalition functions, $0 \in \mathbb{V}_N$ is the function that attributes the worth zero to every coalition, i.e.,

$$\underset{\in\mathbb{V}_N}{0}(K) = \underset{\in\mathbb{R}}{0} \text{ for all } K \subseteq N$$

We will opresent some vector-space theory in chapter V.

5. Solution concepts

For the time being, cooperative game theory consists of coalition functions and solution concepts. The task of solution concepts is to define and defend payoffs as a function of coalition functions. That is, we take a coalition function, apply a solution concept and obtain payoffs for all the players.

Solution concepts may be point-valued (solution function) or set-valued (solution correspondence). In each case, the domain is the set of all games $\mathbb{V}$ for any finite player sets N. A solution function associates each game with exactly one payoff vector while a correspondence allows for several or no payoff vectors.

Definition III.4 (solution function, solution correspondence). *A function σ that attributes, for each coalition function v from $\mathbb{V}$, a payoff to each of v's players,*

$$\sigma(v) \in \mathbb{R}^{|N(v)|},$$

is called a solution function (on $\mathbb{V}$*)*[1]. *Player* i*'s payoff is denoted by* $\sigma_i(v)$. *In case of* $N(v) = \{1, ..., n\}$, *we also write* $(\sigma_1(v), ..., \sigma_n(v))$ *for* $\sigma(v)$ *or* $(\sigma_i(v))_{i \in N(v)}$.

A correspondence that attributes a set of payoff vectors to every coalition function v,

$$\sigma(v) \subseteq \mathbb{R}^{|N(v)|}$$

is called a solution correspondence (on $\mathbb{V}$*).*

Solution functions and solution correspondences are also called solution concepts (on $\mathbb{V}$*).*

Ideally, solution concepts are described both algorithmically and axiomatically. An algorithm is some kind of mathematical procedure (a more less simple function) that tells how to derive payoffs from the coalition functions. Consider, for example, these four solutions concepts in algorithmic form:

- player 1 obtains $v(N)$ and the other players zero,
- every player gets 100,
- every player gets $v(N)/n$,
- every player i's payoff set is given by $[v(\{i\}), v(N)]$ (which may be the empty set).

Alternatively, solution concepts can be defined by axioms. For example, axioms might demand that

- all the players obtain the same payoff,
- no more than $v(N)$ is to be distributed among the players,
- player 1 is to get twice the payoff obtained by player 2,
- the names of players have no role to play,
- every player gets $v(N) - v(N \backslash \{i\})$.

Axioms pin down the players' payoffs, more or less. Axioms may also make contradictory demands. We present the most familiar axioms in the following sections.

6. Pareto efficiency

Arguably, Pareto efficiency is the single most often applied solution concept in economics – rivaled only by Nash equilibrium from noncooperative game theory. For the gloves game, Pareto efficiency is defined by

$$\sum_{i \in N} x_i = v_{L,R}(N).$$

[1]More formallay, a solution function on G is given by

$$\sigma : G \to \cup_{k \in \mathbb{N}} \mathbb{R}^k, \sigma(v) \in \mathbb{R}^{|N(v)|}.$$

Thus, the sum of all payoffs is equal to the number of glove pairs. It is instructive to write this equality as two inequalities:

$$\begin{aligned}\sum_{i\in N} x_i &\leq v_{L,R}(N) \text{ (feasibility) and}\\ \sum_{i\in N} x_i &\geq v_{L,R}(N) \text{ (the grand coalition cannot block } x\text{).}\end{aligned}$$

According to the first inequality, the players cannot distribute more than they (all together) can "produce". This is the requirement of feasibility.

Imagine that the second inequality were violated. Then, we have $\sum_{i=1}^n x_i < v_{L,R}(N)$ and the players would leave "money on the table". All players together could block (or contradict) the payoff vector x. This means they can propose another payoff vector that is both feasible and better for them. Indeed, the payoff vector $y = (y_1, ..., y_n)$ defined by

$$y_i = x_i + \frac{1}{n}\left(v_{L,R}(N) - \sum_{i=1}^n x_i\right), i \in N,$$

does the trick. y improves upon x.

Exercise III.6. *Show that the payoff vector y is feasible.*

Normally, Pareto efficiency is defined by "it is impossible to improve the lot of one player without making other players worse off". If a sum of money is distributed among the player, we can also define Pareto efficiency by "it is impossible to improve the lot of all players". The additional sum of money that makes one player better off (first definition) can be spread among all the players (second definition).

Definition III.5 (feasibility and efficiency). *Let $v \in \mathbb{V}_N$ be a coalition function and let $x \in \mathbb{R}^n$ be a payoff vector. x is called*

- *blockable by N in case of*
$$\sum_{i=1}^n x_i < v(N),$$
- *feasible in case of*
$$\sum_{i\in N} x_i \leq v(N)$$
- *and efficient or Pareto efficient in case of*
$$\sum_{i\in N} x_i = v(N).$$

Thus, an efficient payoff vector is feasible and cannot be blocked by the grand coalition N. Obviously, Pareto efficiency is a solution correspondence, not a solution function.

EXERCISE III.7. *Find the Pareto-efficient payoff vectors for the gloves game* $v_{\{1\},\{2\}}$!

For the gloves game, the solution concept "Pareto efficiency" has two important drawbacks:

- We have very many solutions and the predictive power is weak. In particular, a left-hand glove can have any price, positive or negative.
- The payoffs for a left-glove owner does not depend on the number of left and right gloves in our simple economy. Thus, the relative scarcity of gloves is not reflected by this solution concept.

We now turn to a solution concept that generalizes the idea of blocking from the grand coalition to all coalitions.

7. The core

Pareto efficiency demands that the grand coalition should not be in a position to make all players better off. Extending this idea to all coalitions, the core consists of those feasible (!) payoff vectors that cannot be improved upon by any coalition with its own means. Formally, we have

DEFINITION III.6 (blockability and core). *Let* $v \in \mathbb{V}_N$ *be a coalition function. A payoff vector* $x \in \mathbb{R}^n$ *is called blockable by a coalition* $K \subseteq N$ *if*

$$\sum_{i \in K} x_i < v(K)$$

holds. The core is the set of all those payoff vectors x *fulfilling*

$$\begin{aligned} \sum_{i \in N} x_i &\leq v(N) \text{ (feasibility) and} \\ \sum_{i \in K} x_i &\geq v(K) \text{ for all } K \subseteq N \text{ (no blockade by any coalition).} \end{aligned}$$

Do you see that every payoff vector from the core is also Pareto efficient? Just take $K := N$.

The core is a stricter concept than Pareto efficiency. It demands that no coalition (not just the grand coalition) can block any of its payoff vectors. Let us consider the gloves game for $L = \{1\}$ and $R = \{2\}$. By Pareto efficiency, we can restrict attention to those payoff vectors $x = (x_1, x_2)$ that fulfill $x_1 + x_2 = 1$. Furthermore, x may not be blocked by one-man coalitions:

$$\begin{aligned} x_1 &\geq v_{L,R}(\{1\}) = 0 \text{ and} \\ x_2 &\geq v_{L,R}(\{2\}) = 0. \end{aligned}$$

Hence, the core is the set of payoff vectors $x = (x_1, x_2)$ obeying

$$x_1 + x_2 = 1, x_1 \geq 0, x_2 \geq 0.$$

Are we not forgetting about $K = \emptyset$? Let us check

$$\sum_{i \in \emptyset} x_i \geq v_{L,R}(\emptyset).$$

Since there is not i from $\emptyset$ (otherwise $\emptyset$ would not be the empty set), the sum $\sum_{i \in \emptyset} x_i$ has no summands and is equal to zero. Since all coalition functions have worth zero for the empty set, we find $\sum_{i \in \emptyset} x_i = 0 = v_{L,R}(\emptyset)$ for the gloves game and also for any coalition function.

EXERCISE III.8. *Determine the core for the gloves game $v_{L,R}$ with $L = \{1, 2\}$ and $R = \{3\}$.*

In case of $|L| = 2 > 1 = |R|$ right gloves are scarcer than left gloves. In such a situation, the owner of a right glove should be better off than the owner of a left glove. The core reflects the relative scarcity in a drastic way. Consider the Pareto-efficient payoff vector

$$y = \left(\frac{1}{10}, \frac{1}{10}, \frac{8}{10}\right).$$

It can be blocked by coalition $\{1, 3\}$. Its worth is $v(\{1, 3\}) = 1$ which can be distributed among its members in a manner that both are better off. Thus, y does not lie in the core.

Note that the core is a set-valued solution concept. It can contain one payoff vector (see the above exercise) or very many payoff vectors (in case of $L = \{1\}$ and $R = \{2\}$). Later on, we will see coalition functions with an empty core: every feasible payoff vector is blockable by at least one coalition.

8. The rank-order value

8.1. Rank orders. The rank-order value (this section) and the Shapley value (the two following sections) are point-valued solution concepts. We begin with the rank-order values because the Shapley value builds on these values.

Consider the player set $N = \{1, 2, 3\}$ and assume that these three players stand outside our lecture hall and enter, one after the other. Player 1 may be first, player 3 second and player 2 last – this is the rank order $(1, 3, 2)$. All in all, we find these rank orders:

$$\begin{aligned} &(1, 2, 3), (1, 3, 2), \\ &(2, 1, 3), (2, 3, 1), \\ &(3, 1, 2), (3, 2, 1). \end{aligned}$$

It is not difficult to see, why, for three players, there are 6 different rank orders. For a single player 1, we have just one rank order (1). The second player 2 can be placed before or after player 1 so that we obtain the $1 \cdot 2$ rank orders

$$\begin{aligned}&(1,2)\,,\\&(2,1)\,.\end{aligned}$$

For each of these two, the third player 3 can be placed before the two players, in between or after them:

$$\begin{aligned}&(3,1,2)\,,(1,3,2)\,,(1,2,3)\,,\\&(3,2,1)\,,(2,3,1)\,,(2,1,3)\,.\end{aligned}$$

Therefore, we have $1 \cdot 2 \cdot 3 = 6$ rank orders. Generalizing, , for n players, we have $1 \cdot 2 \cdot ... \cdot n$ rank orders. We can also use the abbreviation

$$n! := 1 \cdot 2 \cdot ... \cdot n$$

which is to be read "n factorial".

Exercise III.9. *Determine the number of rank oders for* 5 *and for* 6 *players!*

Definition III.7 (rank order). *Let $N = \{1, ..., n\}$ be a player set. Bijective function $\rho : N \to N$ are called rank orders or permutations on N. The set of all permutations on N is denoted by RO_N. The set of all players "up to and including player i under rank order ρ" is denoted by $K_i(\rho)$ and given by*

$$\rho(j) = i \text{ and } K_i(\rho) = \{\rho(1), .., \rho(j)\}\,.$$

Thus, $K_i(\rho)$ is the set of players who enter our lecture hall in the rank order ρ just after player i has entered.

Exercise III.10. *Determine $K_2(\rho)$ for*

- $\rho = (2,1,3)$ *and*
- $\rho = (3,1,2)$*!*

8.2. Marginal contributions with respect to rank orders. The rank-order values give every players his marginal contribution. The marginal contribution of player i with respect to coalition K is

"the value with him" minus "the value without him".

Thus, the marginal contributions reflect a player's productivity:

Definition III.8 (marginal contribution with respect to coalitions). *Let $i \in N$ be a player from N and let $v \in \mathbb{V}_N$ be a coalition function on N. Player i's marginal contribution with respect to a coalition K is denoted by $MC_i^K(v)$ and given by*

$$MC_i^K(v) := v(K \cup \{i\}) - v(K \backslash \{i\})\,.$$

The marginal contribution of a player depends on the coalition function and the coalition. It does not matter whether i is a member of K or not, i.e., we have $MC_i^{K\cup\{i\}}(v) = MC_i^{K\backslash\{i\}}(v)$.

EXERCISE III.11. *Determine the marginal contributions for* $v_{\{1,2,3\},\{4,5\}}$ *and*

- $i = 1, K = \{1,3,4\}$,
- $i = 1, K = \{3,4\}$,
- $i = 4, K = \{1,3,4\}$,
- $i = 4, K = \{1,3\}$.

We now shift from the marginal contribution with respect to some coalition K to the marginal contribution with respect to some rank order ρ. For rank order $(3,1,2)$, one finds the marginal contributions

$$\begin{aligned}
&v(\{3\}) - v(\emptyset) \ \text{ (player 3 enters first),}\\
&v(\{1,3\}) - v(\{3\}) \ \text{ (player 1 joins player 3), and}\\
&v(\{1,2,3\}) - v(\{1,3\}) \ \text{ (player 2 enters last).}
\end{aligned}$$

DEFINITION III.9 (marginal contribution with respect to rank orders). *Player* i*'s marginal contribution with respect to rank order* ρ *is denoted by* $MC_i^\rho(v)$ *and given by*

$$MC_i^\rho(v) := MC_i^{K_i(\rho)}(v) = v(K_i(\rho)) - v(K_i(\rho)\setminus\{i\}).$$

EXERCISE III.12. *Find player* 2*'s rank-order values for the rank orders* $(1,3,2)$ *and* $(3,1,2)$*!*

Do you see that the players' marginal contributions add up to $v(\{1,2,3\}) - v(\emptyset) = v(N)$? When you sum the three marginal contributions, the worths $v(\{3\})$ and $v(\{1,3\})$ cancel! In fact, this holds in general:

LEMMA III.1 (Adding-up lemma for rank-order values). *For any player set* N*, any rank order* ρ *on* N *and any player* $i \in N$*, we have*

$$\sum_{j\in K_i(\rho)} MC_i^\rho(v) = v(K_i(\rho))$$

9. The Shapley value: the formula

The Shapley formula rests on a simple idea. Every player obtains

- an average of
- his rank-order values,
- where each rank order is equally likely.

EXERCISE III.13. *Consider* $N = \{1,2,3\}$, $L = \{1,2\}$ *and* $R = \{3\}$ *and determine player* 1*'s marginal contribution for each rank order.*

We employ the following algorithm:

- We first determine all the possible rank orders.
- We then find the marginal contributions for every rank order (the rank-order values).
- For every player, we add his marginal contributions.
- Finally, we divide the sum by the number of rank orders.

DEFINITION III.10 (Shapley value). *The Shapley value is the solution function Sh given by*

$$Sh_i(v) = \frac{1}{n!} \sum_{\rho \in RO_N} MC_i^\rho(v)$$

According to the previous exercise, we have

$$Sh_1\left(v_{\{1,2\},\{3\}}\right) = \frac{1}{6}.$$

The Shapley values of the other two players can be obtained by the same procedure. However, there is a more elegant possibility. The Shapley values of players 1 and 2 are identical because they hold a left glove each and are symmetric (in a sense to be defined shortly). Thus, we have $Sh_2\left(v_{\{1,2\},\{3\}}\right) = \frac{1}{6}$. Also, the Shapley value satisfies Pareto efficiency which means that the sum of the payoffs equals the worth of the grand coalition:

$$\sum_{i=1}^{3} Sh_i\left(v_{\{1,2\},\{3\}}\right) = v(\{1,2,3\}) = 1$$

Thus, we find

$$Sh\left(v_{\{1,2\},\{3\}}\right) = \left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right).$$

10. The Shapley value: the axioms

The Shapley value fulfills four axioms:

- the efficiency axiom: the worth of the grand coalition is to be distributed among all the players,
- the symmetry axiom: players in similar situations obtain the same payoff,
- the null-player axiom: a player with zero marginal contribution to every coalition obtains zero payoff, and
- additivity axiom: if players are subject to two coalition functions, it does not matter whether we apply the Shapley value to the sum of these two coalition functions or apply the Shapley value to each coalition function separately and sum the payoffs.

A solution function σ may or may not obey the four axioms mentioned above.

DEFINITION III.11 (efficiency axiom). *A solution function σ is said to obey the efficiency axiom or the Pareto axiom if*

$$\sum_{i\in N}\sigma_i(v) = v(N)$$

holds for all coalition functions $v \in \mathbb{V}$.

In the gloves game, two left-glove owners are called symmetric.

DEFINITION III.12 (symmetry). *Two players i and j are called symmetric (with respect to $v \in \mathbb{V}$) if we have*

$$v(K \cup \{i\}) = v(K \cup \{j\})$$

for every coalition K that does not contain i or j.

EXERCISE III.14. *Show that any two left-glove holders are symmetric in a gloves game $v_{L,R}$.*

EXERCISE III.15. *Show $MC_i^K = MC_j^K$ for two symmetric players i and j fulfilling $i \notin K$ and $j \notin K$.*

It may seem obvious that symmetric players obtain the same payoff:

DEFINITION III.13 (symmetry axiom). *A solution function σ is said to obey the symmetry axiom if we have*

$$\sigma_i(v) = \sigma_j(v)$$

for any game $v \in \mathbb{V}$ and any two symmetric players i and j.

In any gloves game obeying $L \neq \emptyset \neq R$, every player has a non-zero marginal contribution sometimes.

DEFINITION III.14 (null player). *A player $i \in N$ is called a null player (with respect to v) if*

$$v(K \cup \{i\}) = v(K)$$

holds for every coalition K.

Shouldn't a null player obtain nothing?

DEFINITION III.15 (null-player axiom). *A solution function σ is said to obey the null-player axiom if we have*

$$\sigma_i(v) = 0$$

for any game $v \in \mathbb{V}$ and for any null player $i \in N$.

EXERCISE III.16. *Under which condition is a player from L a null player in a gloves game $v_{L,R}$?*

The last axiom that we consider at present is the additivity axiom. It rests on the possibility to add both payoff vectors and coalition functions (see section 4).

DEFINITION III.16 (additivity axiom). *A solution function σ is said to obey the additivity axiom if we have*

$$\sigma(v+w) = \sigma(v) + \sigma(w)$$

for any two coalition functions $v, w \in \mathbb{V}$ with $N(v) = N(w)$.

Do you see the difference? On the left-hand side, we add the coalition functions first and then apply the solution function. On the right-hand side we apply the solution function to the coalition functions individually and then add the payoff vectors.

EXERCISE III.17. *Can you deduce $\sigma(0) = 0$ from the additivity axiom? Hint: use $v = w := 0$.*

Now we note a stunning result:

THEOREM III.1 (Shapley axiomatization). *The Shapley formula is the unique solution function that fulfills the symmetry axiom, the efficiency axiom, the null-player axiom and the additivity axiom.*

The theorem means that the Shapley formula fulfills the four axioms. Consider now a solution function that fulfills the four axioms. According to the theorem, the Shapley formula is the only solution function to do so.

Differently put, the Shapley formula and the four axioms are equivalent – they specify the same payoffs. Cooperative game theorists say that she Shapley formula is "axiomatized" by the set of the four axioms. The chapter after next will show you how to prove this wonderful result.

EXERCISE III.18. *Determine the Shapley value for the gloves game for $L = \{1\}$ and $R = \{2, 3, 4\}$! Hint: You do not need to write down all $4!$ rank orders. Try to find the probability that player 1 does not complete a pair.*

11. Topics and literature

The main topics in this chapter are

- coalition
- coalition function
- gloves game
- core
- efficiency
- feasibility
- marginal contribution
- axioms
- symmetry
- null player
- Shapley value

We introduce the following mathematical concepts and theorems:

- t
-

We recommend

12. Solutions

Exercise III.1

We have $|\emptyset| = 0$ and $|N| = n$.

Exercise III.2

The first three propositions are nonsensical, the last one is correct.

Exercise III.3

We have $\{1,2,3\} \backslash K = \{2\}$ and $\{1,2,3,4\} \backslash K = \{2,4\}$.

Exercise III.4

The values are

$$\begin{aligned} v_{L,R}(\{1\}) &= \min(1,0) = 0, \\ v_{L,R}(\emptyset) &= \min(0,0) = 0, \\ v_{L,R}(N) &= \min(2,3) = 2 \text{ and} \\ v_{L,R}(\{2,3,4\}) &= \min(2,1) = 1. \end{aligned}$$

Exercise III.5

We obtain the sum of vectors

$$\begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+2 \\ 3+5 \\ 6+1 \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \\ 7 \end{pmatrix}$$

Exercise III.6

Feasibility follows from

$$\begin{aligned}\sum_{i=1}^{n} y_i &= \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{1}{n}\left(v_{L,R}(N) - \sum_{j=1}^{n} x_j\right) \\ &= \sum_{i=1}^{n} x_i + \frac{1}{n}\left(\sum_{i=1}^{n} v_{L,R}(N) - \sum_{i=1}^{n}\sum_{j=1}^{n} x_j\right) \\ &= \sum_{i=1}^{n} x_i + \frac{1}{n}\left(n v_{L,R}(N) - n\sum_{j=1}^{n} x_j\right) \\ &= v_{L,R}(N).\end{aligned}$$

Exercise III.7

The set of Pareto-efficient payoff vectors (x_1, x_2) are described by $x_1 + x_2 = 1$. In particular, we may well have $x_1 < 0$.

Exercise III.8

The core obeys the conditions

$$\begin{aligned} x_1 + x_2 + x_3 &= v_{L,R}(N) = 1, \\ x_i &\geq 0, i = 1, 2, 3, \\ x_1 + x_2 &\geq 0, \\ x_1 + x_3 &\geq 1 \text{ and} \\ x_2 + x_3 &\geq 1. \end{aligned}$$

Substituting $x_1 + x_3 \geq 1$ into the efficiency condition yields

$$x_2 = 1 - (x_1 + x_3) \leq 1 - 1 = 0.$$

Hence (because of $x_2 \geq 0$), we have $x_2 = 0$. For reasons of symmetry, we also have $x_1 = 0$. Applying efficiency once again, we obtain $x_3 = 1 - (x_1 + x_2) = 1$. Thus, the only candidate for the core is $x = (0, 0, 1)$. Indeed, this payoff vector fulfills all the conditions noted above. Therefore,

$$(0, 0, 1)$$

is the only element in the core.

Exercise III.11

The marginal contributions are

$$\begin{aligned}
MC_1^{\{1,3,4\}}\left(v_{\{1,2,3\},\{4,5\}}\right) &= v(\{1,3,4\}\cup\{1\}) - v(\{1,3,4\}\setminus\{1\}) \\
&= v(\{1,3,4\}) - v(\{3,4\}) \\
&= 1-1=0, \\
MC_1^{\{3,4\}}\left(v_{\{1,2,3\},\{4,5\}}\right) &= v(\{3,4\}\cup\{1\}) - v(\{3,4\}\setminus\{1\}) \\
&= v(\{1,3,4\}) - v(\{3,4\}) \\
&= 1-1=0, \\
MC_4^{\{1,3,4\}}\left(v_{\{1,2,3\},\{4,5\}}\right) &= v(\{1,3,4\}\cup\{4\}) - v(\{1,3,4\}\setminus\{4\}) \\
&= v(\{1,3,4\}) - v(\{1,3\}) \\
&= 1-0=1, \\
MC_4^{\{1,3\}}\left(v_{\{1,2,3\},\{4,5\}}\right) &= v(\{1,3\}\cup\{4\}) - v(\{1,3\}\setminus\{4\}) \\
&= v(\{1,3,4\}) - v(\{1,3\}) \\
&= 1-0=1.
\end{aligned}$$

Exercise III.12

The marginal contributions and hence the rank-order values are the same: $v(\{1,2,3\}) - v(\{1,3\})$.

Exercise III.9

We find $5! = 1\cdot 2\cdot 3\cdot 4\cdot 5 = 120$ rank orders of 5 players and $6! = 5!\cdot 6 = 120\cdot 6 = 720$ rank orders for 6 players.

Exercise III.10

We find $K_2((2,1,3)) = \{2\}$ and $K_2((3,1,2)) = \{1,2,3\}$.

Exercise III.13

We find the marginal contributions

$$\begin{aligned}
v(\{1\}) - v(\emptyset) &= 0-0=0, \text{ rank order } (1,2,3) \\
v(\{1\}) - v(\emptyset) &= 0-0=0, \text{ rank order } (1,3,2) \\
v(\{1,2\}) - v(\{2\}) &= 0-0=0, \text{ rank order } (2,1,3) \\
v(\{1,2,3\}) - v(\{2,3\}) &= 1-1=0, \text{ rank order } (2,3,1) \\
v(\{1,3\}) - v(\{3\}) &= 1-0=1, \text{ rank order } (3,1,2) \\
v(\{1,2,3\}) - v(\{2,3\}) &= 1-1=0, \text{ rank order } (3,2,1).
\end{aligned}$$

Exercise III.14

Let i and j be players from L and let K be a coalition that contains neither i nor j. Then $K\cup\{i\}$ contains the same number of left and the same number of right gloves as $K\cup\{j\}$. Therefore,

$$\begin{aligned}
v_{L,R}(K\cup\{i\}) &= \min\left(\left|(K\cup\{i\})\cap L\right|, \left|(K\cup\{i\})\cap R\right|\right) \\
&= \min\left(\left|(K\cup\{j\})\cap L\right|, \left|(K\cup\{j\})\cap R\right|\right) \\
&= v_{L,R}(K\cup\{j\}).
\end{aligned}$$

Exercise III.15

The equality follows from

$$\begin{aligned} MC_i^K &= v(K\cup\{i\}) - v(K\backslash\{i\}) \\ &= v(K\cup\{i\}) - v(K) \\ &= v(K\cup\{j\}) - v(K) \\ &= v(K\cup\{j\}) - v(K\backslash\{j\}) \\ &= MC_j^K. \end{aligned}$$

Exercise III.16

A player i from L is a null player iff $R=\emptyset$ holds. $R=\emptyset$ implies

$$\begin{aligned} v_{L,\emptyset}(K) &= \min(|K\cap L|, |K\cap\emptyset|) \\ &= \min(|K\cap L|, 0) \\ &= 0 \end{aligned}$$

for every coalition K. $R\neq\emptyset$ means that i has a marginal contribution of 1 when he comes second after a right-glove holder.

Exercise III.18

The left-glove holder 1 completes a pair (the only one) whenever he does not come first. The probability for coming first is $\frac{1}{4}$ for player 1 (and any other player). Thus, player 1 obtains $\left(1-\frac{1}{4}\right)\cdot 1$. The other players share the rest. Therefore, symmetry and efficiency lead to

$$\begin{aligned} \varphi_1\left(v_{\{1\},\{2,3,4\}}\right) &= \frac{3}{4}, \\ \varphi_2\left(v_{\{1\},\{2,3,4\}}\right) &= \varphi_3\left(v_{\{1\},\{2,3,4\}}\right) = \varphi_4\left(v_{\{1\},\{2,3,4\}}\right) = \frac{1}{12}. \end{aligned}$$

13. Further exercises without solutions

CHAPTER IV

Many games

0.1. Introduction. In the previous chapter, we focus on a specific class of games, the gloves games. In this chapter, we aim to familiarize the reader with many other interesting games.

Simple games are simple – all the coalitions have worth 0 or 1. We address worth-0 coalitions as loosing coalitions and worth-1 coalitions as winning coalitions. Simple games can be used to model these interesting situations:

- Political parties form a winning coalition if they command more than fifty percent of a parliament's seats. In Germany, one particular winning coalition of political parties forms the government coalition in order to elect the chancellor.
- The United Nation's Security Council has peculiar voting rules according to which each permanent member (China, France, ...) has veto power.
- Some players may be powerful or productive if they combine while all the other players are "useless". For example, each productive player possesses part of a treasure map. The treasure can be found only if all the different parts of the map are put together. This type of game is called a unanimity game.

We also introduce non-simple games:

- For example, a car is sold by one player to one of two prospective buyers. The willingness' to pay by both buyers should influence the seller's payoff.
- Many organizations have the problem of dividing overhead cost to several units. Examples are doctors with a common secretary or commonly used facilities, firms organized as a collection of profit-centers, universities with computing facilities used by several departments or faculties. We show that the core and also the Shapley value can provide solutions to this problem. This sections rests on Young (1994a) and chapter 5 from Young (1994b).
- We consider endowment games which are generalizations of gloves games. Players may possess any number of gloves are any other goods.

Finally, this chapter presents general properties of coalition functions such as monotonicity or superadditivity.

1. Simple games

1.1. Definition. We first define monotonic games and then simple games.

DEFINITION IV.1 (monotonic game). *A coalition function $v \in \mathbb{V}_N$ is called monotonic if $\emptyset \subseteq S \subseteq S'$ implies $v(S) \leq v(S')$.*

Thus, monotonicity means that the worth of a coalition cannot decrease if other players join. Differently put, if S' is a superset of S (or S a subset of S'), we cannot have $v(S) = 1$ and $v(S') = 0$.

Simple games are a special subclass of monotonic games:

DEFINITION IV.2 (simple game). *A coalition function $v \in \mathbb{V}_N$ is called simple if*

- *we have $v(K) = 0$ or $v(K) = 1$ for every coalition $K \subseteq N$,*
- *the grand coalition's worth is 1 and.*
- *v is monotonic.*

Coalitions with $v(K) = 1$ are called winning coalitions and coalitions with $v(K) = 0$ are called loosing coalitions. A winning coalition K is a minimal winning coalition if every strict subset of K is not a winning coalition.

Simple games can be characterized by the pivotal coalitions of all the players:

DEFINITION IV.3 (pivotal coalition). *For a simple game v, $K \subseteq N$ is a pivotal coalition for $i \in N$ if $v(K) = 0$ and $v(K \cup \{i\}) = 1$. The number of i's pivotal coalitions is denoted by $\eta_i(v)$,*

$$\eta_i(v) := |\{K \subseteq N : v(K) = 0 \text{ and } v(K \cup \{i\}) = 1\}|.$$

We have $\eta(v) := (\eta_1(v), ..., \eta_n(v))$ and $\bar{\eta}(v) := \sum_{i \in N} \eta_i(v)$. We sometimes omit v and write η_i $(\eta, \bar{\eta})$ rather than $\eta_i(v)$ $(\eta(v), \bar{\eta}(v))$.

By $\left|2^{N \backslash \{i\}}\right| = 2^{n-1}$, no player can have more pivotal coalitions than 2^{n-1}.

EXERCISE IV.1. *How do you call a player $i \in N$ who has no pivotal coalitions?*

1.2. Veto players and dictators. According to the previous exercise, all interesting simple games have $v(N) = 1$. Sometimes, some players are of central importance:

DEFINITION IV.4 (veto player, dictator). *Let v be a simple game. A player $\in N$ is called a veto player if*

$$v(N \backslash \{i\}) = 0$$

holds. i is called a dictator if

$$v(S)=\begin{cases}1, & i\in S\\ 0, & sonst\end{cases}$$

holds for all $S\subseteq N$.

Thus, without a veto player, the worth of a coalition is 0 while a dictator can produce the worth 1 just by himself.

Exercise IV.2. *Can there be a coalition K such that $v(K\backslash\{i\})=1$ for a veto player i or a dictator i?*

Exercise IV.3. *Is every veto player a dictator or every dictator a veto player?*

Exercise IV.4. *How do you call a player $i\in N$ with $\eta_i=2^{n-1}$?*

1.3. Simple games and voting mechanisms. Oftentimes, simple games can be used to model voting mechanisms. As a matter of consistency, complements of winning coalitions have to be loosing coalitions. Otherwise, a coalition K could vote for something and $N\backslash K$ would vote against it, both of them successfully.

Definition IV.5 (contradictory, decidable). *A simple game $v\in\mathbb{V}_N$ is called non-contradictory if $v(K)=1$ implies $v(N\backslash K)=0$.*

A simple game $v\in\mathbb{V}_N$ is called decidable if $v(K)=0$ implies $v(N\backslash K)=1$.

Thus, a contradictory voting game can lead to opposing decisions – for example, some candidate A is voted president (with the support of some coalition K) and then some other candidate B (with the support of $N\backslash K$) is also voted president. A non-decidable voting game can prevent any decision. Neither A nor B can gain enough support because coalition K blocks candidate B while $N\backslash K$ blocks candidate A.

Exercise IV.5. *Show that a simple game with a veto player cannot be contradictory. A simple game with two veto players cannot be decidable.*

1.4. Unanimity games. Unanimity games are famous games in cooperative game theory. We will use them to prove the Shapley theorem.

Definition IV.6 (unanimity game). *For any $T\neq\emptyset$,*

$$u_T(K)=\begin{cases}1, & K\supseteq T\\ 0, & otherwise\end{cases}$$

defines a unanimity game.

The T-players exert a kind of common dictatorship.

EXERCISE IV.6. *Find the null players in the unanimity game u_T.*

EXERCISE IV.7. *Find the core and the Shapley value for $N = \{1, 2, 3, 4\}$ and $u_{\{1,2\}}$.*

1.5. Apex-Spiel. The apex game has one important player $i \in N$ who is nearly a veto player and nearly a dictator.

DEFINITION IV.7 (apex game). *For $i \in N$ with $n \geq 2$, the apex game h_i is defined by*

$$h_i(K) = \begin{cases} 1, & i \in K \text{ and } K \backslash \{i\} \neq \emptyset \\ 1, & K = N \backslash \{i\} \\ 0, & \text{otherwise} \end{cases}$$

Player i is called the main, or apex, player of that game.

Thus, there are two types of winning coalitions in the apex game:

- i together with at least one other player or
- all the other players taken together.

Generally, we work with apex games for $n \geq 4$.

EXERCISE IV.8. *Consider h_1 for $n = 2$ and $n = 3$. How do these games look like?*

EXERCISE IV.9. *Is the apex player a veto player or a dictator?*

EXERCISE IV.10. *Show that the apex game is decidable and not contradictory.*

Let us now think find the Shapley value for the apex game. Consider all the rank orders. The apex player $i \in N$ obtains the marginal contribution 1 unless

- he is the first player in a rank order (then his marginal contribution is $v(\{i\}) - v(\emptyset) = 0 - 0 = 0$) or
- he is the last player (with marginal contribution $v(N) - v(N \backslash \{i\}) = 1 - 1 = 0$).

Since every position of the apex player in a rank order has the same probability, the following exercise is easy:

EXERCISE IV.11. *Find the Shapley value for the apex game h_1!*

1.6. Weighted voting games.

1.6.1. *Definition.* Weighted voting games form an important subclass of the simple games. We specify weights for every player and a quota. If the sum of weights for a coalition is equal to or above the quota, that coalition is a winning one.

DEFINITION IV.8 (weighted voting game). *A voting game v is specified by a quota q and voting weights g_i, $i \in N$, and defined by*

$$v(K) = \begin{cases} 1, & \sum_{i\in K} g_i \geq q \\ 0, & \sum_{i\in K} g_i < q \end{cases}$$

In that case, the voting game is also denoted by $[q; g_1, ..., g_n]$.

For example,

$$\left[\frac{1}{2}; \frac{1}{n}, ..., \frac{1}{n}\right]$$

is the majority rule, according to which fifty percent of the votes are necessary for a winning coalition. Do you see that $n = 4$ implies that the coalition $\{1, 2\}$ is a winning coalition and also the coalition of the other players, $\{3, 4\}$? Thus, this voting game is contradictory.

The apex game h_1 for n players can be considered a weighted voting game given by

$$\left[n - 1; n - \frac{3}{2}, 1, ..., 1\right].$$

EXERCISE IV.12. *Consider the unanimity game u_T given by $t < n$ and $T = \{1, ..., t\}$. Can you express it as a weighted voting game?*

1.6.2. *UN Security Council.* Let us consider the United Nations' Security Council. According to http://www.un.org/sc/members.asp:, it has 5 permanent members and 10 non-permanent ones. The permanent members are China, France, Russian Federation, the United Kingdom and the United States. In 2009, the non-permanent members were Austria, Burkina Faso, Costa Rica, Croatia, Japan, Libyan Arab Jamahiriya, Mexico, Turkey, Uganda and Viet Nam.

We read:

> Each Council member has one vote. ... Decisions on substantive matters require nine votes, including the concurring votes of all five permanent members. This is the rule of "great Power unanimity", often referred to as the "veto" power.
>
> Under the Charter, all Members of the United Nations agree to accept and carry out the decisions of the Security Council. While other organs of the United Nations make recommendations to Governments, the Council alone has the power to take decisions which Member States are obligated under the Charter to carry out.

Obviously, the UN Security Council has a lot of power and so its voting mechanism deserves analysis. The above rule for "substantive matters" can be translated into the weighted voting game

$$[39; 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$$

where the weights 7 accrue to the five permanent and the weights 1 to the non-permanent members.

Exercise IV.13. *Using the above voting game, show that every permanent member is a veto player. Show also that the five permanent members need the additional support of four non-permanent ones.*

Exercise IV.14. *Is the Security Council's voting rule non-contradictory and decidable?*

It is not easy to calculate the Shapley value for the Security Council. After all, we have

$$15! = 1.307.674.368.000$$

rank orders for the 15 players. Anyway, the Shapley values are

$$\begin{array}{l} 0,19627 \text{ for each permanent member} \\ 0,00186 \text{ für each non-permanent member.} \end{array}$$

2. Three non-simple games

2.1. Buying a car. Following Morris (1994, S. 162), we consider three agents envolved in a car deal. Andreas (A) has a used car he wants to sell, Frank (F) and Tobias (T) are potential buyers with willingness to buy of 700 and 500, respectively. This leads to the coalition function v given by

$$\begin{aligned} v(A) &= v(F) = v(T) = 0, \\ v(A,F) &= 700, \\ v(A,T) &= 500, \\ v(F,T) &= 0 \text{ and} \\ v(A,F,T) &= 700. \end{aligned}$$

One-man coalitions have the worth zero. For Andreas, the car is useless (he believes in cycling rather than driving). Frank and Tobias cannot obtain the car unless Andreas cooperates. In case of a deal, the worth is equal to the (maximal) willingness to pay.

We use the core to find predictions for the car price. The core is the set of those payoff vectors (x_A, x_F, x_T) that fulfill

$$x_A + x_F + x_T = 700$$

and

$$\begin{aligned} x_A &\geq 0, x_F \geq 0, x_T \geq 0, \\ x_A + x_F &\geq 700, \\ x_A + x_T &\geq 500 \text{ and} \\ x_F + x_T &\geq 0. \end{aligned}$$

Tobias obtains

$$\begin{aligned} x_T &= 700 - (x_A + x_F) \text{ (efficiency)} \\ &\leq 700 - 700 \text{ (by } x_A + x_F \geq 700) \\ &= 0 \end{aligned}$$

and hence zero, $x_T = 0$. By $x_A + x_T \geq 500$, the seller Andreas can obtain at least 500.

Summarizing (and checking all the conditions above), we see that the core is the set of vectors (x_A, x_F, x_T) obeying

$$\begin{aligned} 500 &\leq x_A \leq 700, \\ x_F &= 700 - x_A \text{ and} \\ x_T &= 0. \end{aligned}$$

Therefore, the car sells for a price between 500 and 700.

2.2. The Maschler game. Aumann & Myerson (1988) present the Maschler game which is the three-player game given by

$$v(K) = \begin{cases} 0, & |K| = 1 \\ 60, & |K| = 2 \\ 72, & |K| = 3 \end{cases}$$

Obviously, the three players are symmetric. It is easy to see that all players of symmetric games are symmetric.

Definition IV.9 (symmetric game). *A coalition function v is called symmetric if there is a function $f : N \to \mathbb{R}$ such that*

$$v(K) = f(|K|), \ K \subseteq N.$$

Exercise IV.15. *Find the Shapley value for the Maschler game!*

According to the Shapley value, the players 1 and 2 obtain less than their common worth. Therefore, they can block the payoff vector suggested by the Shapley value. Indeed, for any efficient payoff vector, we can find a two-man coalition that can be made better off. Differently put: the core is empty.

This can be seen easily. We are looking for vectors (x_1, x_2, x_3) that fulfill both

$$x_1 + x_2 + x_3 = 72$$

and

$$\begin{aligned} x_1 &\geq 0, x_2 \geq 0, x_3 \geq 0, \\ x_1 + x_2 &\geq 60, \\ x_1 + x_3 &\geq 60 \text{ and} \\ x_2 + x_3 &\geq 60. \end{aligned}$$

Summing the last three inequalities yields

$$2x_1 + 2x_2 + 2x_3 \geq 3 \cdot 60 = 180$$

and hence a contradiction to efficiency.

2.3. The gloves game, once again. In chapter III, we have calculated the core for the gloves game $L = \{1, 2\}$ and $R = \{3\}$. The core clearly shows the bargaining power of the right-glove owner. We will now consider the core for a case where the scarcity of right gloves seems minimal:

$$\begin{aligned} L &= \{1, 2, ..., 100\} \\ R &= \{101, ..., 199\}. \end{aligned}$$

If a payoff vector

$$(x_1, ..., x_{100}, x_{101}, ..., x_{199})$$

is to be long to the core, we have

$$\sum_{i=1}^{199} x_i = 99$$

by the efficiency axiom. We now pick any left-glove holder $j \in \{1, 2, ..., 100\}$. We find

$$v(L \backslash \{j\} \cup R) = 99$$

and hence

$$\begin{aligned} x_j &= 99 - \sum_{\substack{i=1, \\ i \neq j}}^{199} x_i \text{ (efficiency)} \\ &\leq 99 - 99 \text{ (blockade by coalition } L \backslash \{j\} \cup R) \\ &= 0. \end{aligned}$$

Therefore, we have $x_j = 0$ for every $j \in L$.

Every right-glove owner can claim at least 1 because he can point to coalitions where he is joined by at least one left-glove owner. Therefore, every right-glove owner obtains the payoff 1 and every left-glove owner the payoff zero. Inspite of the minimal scarcity, the right-glove owners get everything.

If two left-glove owners burned their glove, the other left-glove owners would get a payoff increase from 0 to 1. (Why?)

Exercise IV.16. *Consider a generalized gloves game where*

- *player* 1 *has one left glove,*
- *player* 2 *has two left gloves and*
- *players* 3 *and* 4 *have one right glove each.*

Calculate the core. How does the core change if player 2 *burns one of his two gloves?*

The burn-a-glove strategy may make sense if payoffs depend on the scarcity in an extreme fashion as they do for the core.

3. Cost division games

We model cost-division games (for doctors sharing a secretarial office or faculties sharing computing facilities) by way of cost functions and cost-savings functions.

DEFINITION IV.10 (cost-division game). *For a player set N, let $c : 2^N \to \mathbb{R}_+$ be a coalition function that is called a cost function. On the basis of c, the cost-savings game is defined by $v : 2^N \to \mathbb{R}$ and*

$$v(K) = \sum_{i \in K} c(\{i\}) - c(K), K \subseteq N.$$

The idea behind this definition is that cost savings can be realized if players pool their resources so that $\sum_{i \in K} c(\{i\})$ is greater than $c(K)$ and $v(K)$ is positive.

We consider a specific example. Two towns A and B plan a water-distribution system.Town A could build such a system for itself at a cost of 11 million Euro and twon B would need 7 million Euro for a system tailor-made to its needs. The cost for a common water-distribution system is 15 million Euro. The cost function is given by

$$\begin{aligned} c(\{A\}) &= 11, c(\{B\}) = 7 \text{ and} \\ c(\{A, B\}) &= 15. \end{aligned}$$

The associated cost-savings game is $v : 2^{\{A,B\}} \to \mathbb{R}$ defined by

$$\begin{aligned} v(\{A\}) &= 0, c(\{B\}) = 0 \text{ and} \\ v(\{A, B\}) &= 7 + 11 - 15 = 3. \end{aligned}$$

v's core is obviously given by

$$\left\{(x_A, x_B) \in \mathbb{R}^2_+ : x_1 + x_2 = 3\right\}.$$

The cost savings of $3 = 11 + 7 - 15$ can be allotted to the towns such that no town is worse off compared to going alone. Thus, the set of undominated cost allocations is

$$\left\{(c_A, c_B) \in \mathbb{R}^2 : c_A + c_B = 15, c_A \le 11, c_B \le 7\right\}.$$

4. Endowment games

Gloves games are a specific class of endowment games. In these games, players own an endowment (in the gloves game: a right or a left glove). We first define the endowment economy and then, on that basis, the endowment game.

Definition IV.11 (endowment economy). *An endowment economy is a tuple*

$$\mathcal{E} = \left(N, G, \left(\omega^i\right)_{i \in N}, agg\right)$$

consisting of

- *the set of agents* $N = \{1, 2, ..., n\}$,
- *the finite set of goods* $G = \{1, ..., \ell\}$,
- *for every agent* $i \in N$, *an endowment* $\omega^i = \left(\omega_1^i, ..., \omega_\ell^i\right) \in \mathbb{R}_+^\ell$ *where*

$$\omega := \sum_{i \in N} \omega^i = \left(\sum_{i \in N} \omega_1^i, ..., \sum_{i \in N} \omega_\ell^i\right)$$

  is the economy's total endowment, and
- *an aggregation functions* $agg : \mathbb{R}^\ell \to \mathbb{R}$.

Two remarks are in order:

- Do you see the connection between ω and the exchange Edgeworth box introduced in chapter II on pp. 16?
- The aggregation function aggregates the different goods' amounts into a specific real number in the same way as the min-operator does in the gloves game.

Definition IV.12 (endowment game). *Consider an endowment economy* $\mathcal{E}$. *An endowment game* $v^{\mathcal{E}} : 2^N \to \mathbb{R}$ *is defined by*

$$v^{\mathcal{E}}(K) := agg\left(\sum_{i \in K} \omega_1^i, ..., \sum_{i \in K} \omega_\ell^i\right).$$

We sometimes write v^ω *rather than* $v^{\mathcal{E}}$.

Within the class of endowment games, we can define the sum of two coalition functions on N in the usual manner – just sum the worths of every coalition. For example, we have

$$\begin{aligned} & \left(v_{\{1,2\},\{3\}} + v_{\{1\},\{2,3\}}\right)(\{2\}) \\ = \; & v_{\{1,2\},\{3\}}(\{2\}) + v_{\{1\},\{2,3\}}(\{2\}) \\ = \; & 0 + 0 = 0 \end{aligned}$$

However, taking the specific nature of endowment games into account, it is also plausible to sum endowments and take it from there. In that case, we find that player 2 has a left glove (in $v_{\{1,2\},\{3\}}$) and a right glove (in $v_{\{1\},\{2,3\}}$) and hence the worth 1. We capture this idea by the following definition:

Definition IV.13 (summing of endowment games). *Consider two endowment economies* $\mathcal{E}$ *and* $\mathcal{F}$ *which have the same player set* N, *the same set of goods* G *and the same aggregation function* agg. *In that case,* $\mathcal{E}$ *and* $\mathcal{F}$ *are called structurally identical. The (possibly different) endowments are denoted* $\omega_{\mathcal{E}}$ *and* $\omega_{\mathcal{F}}$, *respectively, and the derived endowment games by* $v_{\mathcal{E}}$

and $v_{\mathcal{F}}$. *The endowment-based sum of these games is denoted by* $v_{\mathcal{E}} \oplus v_{\mathcal{F}}$ *and defined by*

$$\begin{aligned} \omega_g^i &= (\omega_{\mathcal{E}})_g^i + (\omega_{\mathcal{F}})_g^i \,,\ i \in N, g \in G \text{ and} \\ (v_{\mathcal{E}} \oplus v_{\mathcal{F}})(K) &:= agg\left(\sum_{i \in K} \omega_1^i, ..., \sum_{i \in K} \omega_\ell^i\right). \end{aligned}$$

Note that the sum of two gloves games need not be a gloves game, but a generalized gloves game where players can have any number of left or right gloves.

Endowment-based summing is of economic interest. For example, we can consider two autarkic economies that open up for trade and define the gains from trade:

Definition IV.14 (summing of endowment games). *For a player set* N*, consider two endowment economies* $\mathcal{E}$ *and* $\mathcal{F}$ *. The gains from trade are defined by*

$$GfT(\mathcal{E}, \mathcal{F}) = (v_{\mathcal{E}} \oplus v_{\mathcal{F}})(N) - [v_{\mathcal{E}}(N) + v_{\mathcal{F}}(N)].$$

Thus the usual sum of coalition function ignores all substantial linkages that might exist between them.

Exercise IV.17. *Show that the gains from trade are zero for any gloves game* $v_{\mathcal{E}} := v_{\{L\},\{R\}}$ *and* $v_{\mathcal{F}} := v_{\mathcal{E}}$.

A specific class of endowment games has been proposed by Owen (1975): production games. In these games, players' endowments represent factors of production rather than consumption goods. The idea is that the players pool their factors of production and sell the output. We define the aggregation function $agg : \mathbb{R}^\ell \to \mathbb{R}$ by

$$agg(\omega_1, ..., \omega_\ell) := p \cdot f(\omega_1, ..., \omega_\ell)$$

where f is a production function and p the price vector. If m goods are produced, p is a price vector with m entries and $\cdot$ stands for the scalar product. Thus, the endowment game's worths stand for

- the revenue
- generated by the output
- produced with the factors of production
- a coalition is endowed with.

5. Properties of coalition functions

5.1. Zero players and symmetric players.

Definition IV.15 (zero player). *A player $i \in N$ is a zero player for a coalition function $v \in \mathbb{V}_N$ if*

$$v(K \cup \{i\}) = v(K \setminus \{i\})$$

holds for every coalition $K \subseteq N$.

Definition IV.16 (inessential player). *A player $i \in N$ is an inessential player for a coalition function $v \in \mathbb{V}_N$ if*

$$v(K \cup \{i\}) - v(K \setminus \{i\}) = v(\{i\})$$

holds for every coalition $K \subseteq N$.

5.2. Inessentiality and additivity. We begin with boring coalition functions.

Definition IV.17 (triviality). *A coalition function $v \in \mathbb{V}_N$ is called trivial if*

$$v(K) = 0$$

holds for every coalition $K \subseteq N$.

Thus, a trivial coalition function $v \in \mathbb{V}_N$ is the zero coalition function $v = 0$.

Definition IV.18 (inessentiality). *A coalition function $v \in \mathbb{V}_N$ is called inessential if*

$$v(K) = \sum_{i \in K} v(\{i\})$$

holds for all $K \subseteq N$.

Definition IV.19. *A coalition function is called additive if $v(R \cup S) = v(R) + v(S)$ holds for all coalitions R and $S \subseteq N$ obeying $R \cap S = \emptyset$.*

Lemma IV.1. *A coalition function v is inessential if and only if every player $i \in N$ is an inessential player for v and if and only if v is additive.*

5.3. Monotonicity and superadditivity. Nearly all the coalition functions we work with in this book are monotonic (see definition IV.1 on p. 52) and superadditive. Monotonicity and superadditivity are closely related:

- Monotonicity means that adding players never decreases the worth.
- Superadditivity can be tanslated as "cooperation pays".

Definition IV.20 (superadditivity). *A coalition function $v \in \mathbb{V}_N$ is called superadditive if for any two coalitions R and S*

$$R \cap S = \emptyset$$

implies

$$v(R) + v(S) \leq v(R \cup S).$$

$v(R \cup S) - (v(R) + v(S)) \geq 0$ is called the gain from cooperation.

Glove games are monotonic because the number of glove pairs cannot decrease if additional players (and hence additional gloves) are added. They are also superadditive because the number of glove pairs cannot decrease when two disjoint coalitions pool their gloves.

EXERCISE IV.18. *Is the coalition function v, given by $N = \{1, 2, 3\}$ and*

$$\begin{aligned} v(\{1,2,3\}) &= 5, \\ v(\{1,2\}) &= v(\{1,3\}) = v(\{2,3\}) = 4, \\ v(\{1\}) &= v(\{2\}) = v(\{3\}) = 0 \end{aligned}$$

superadditive?

EXERCISE IV.19. *How about superadditivity of unanimity games, of the Maschler game or of a contradictory simple game?*

While monotonicity and superadditivity seem very similar properties, monotonicity does not imply superadditivity as you can see from $N = \{1, 2\}$ and $v(\{1\}) = v(\{2\}) = 3$ and $v(\{1, 2\}) = 4$.

EXERCISE IV.20. *Show that every monotonic game v is non-negative, i.e., fulfills $v(K) \geq 0$ for alle $K \subseteq N$.*

EXERCISE IV.21. *Show that superadditivity and non-negativity imply monotonicity.*

5.4. Convexity. Superadditivity means: cooperation pays. Convexity implies superadditivity, but is stronger. Convexity is interesting because the Shapley value can be shown to lie in the core of any convex game.

DEFINITION IV.21 (convexity). *A coalition function $v \in \mathbb{V}_N$ is called convex if for any two coalitions S and S' with $S \subseteq S'$ and for all players $i \in N \backslash S'$, we have*

$$v(S \cup \{i\}) - v(S) \leq v(S' \cup \{i\}) - v(S').$$

v is called strictly convex if the inequality is strict.

Thus, the marginal contribution is large for large coalitions. May-be, you find fig. 1 helpful.

Let us consider the example of by $N = \{1, 2, 3, 4\}$ and the coalition function v given by

$$v(S) = |S| - 1, S \neq \emptyset.$$

Note that the marginal contribution is zero for any player who joins the empty set,

$$v(\emptyset \cup \{i\}) - v(\emptyset) = [|\{i\}| - 1] - 0 = 0,$$

while the marginal contribution with respect to any nonempty coalition is 1. Thus, this coalition function is convex.

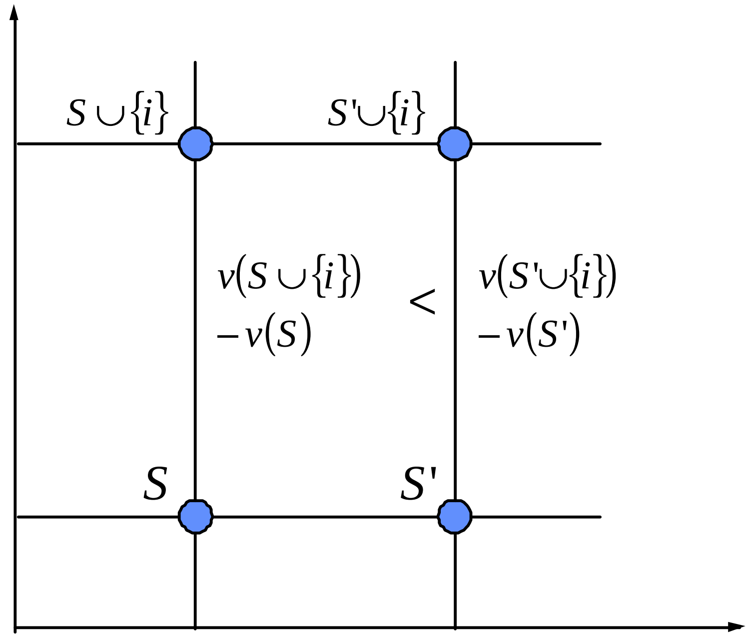

FIGURE 1. Strict convexity

EXERCISE IV.22. *Is the unanimity game u_T convex? Distinguish between $i \in T$ and $i \notin T$. Is u_T strictly convex?*

Why are convex coalition functions called convex? The reader remembers that function $f : \mathbb{R} \to \mathbb{R}$ that are defined by $f(x) = x^2$ or $f(x) = e^x$ are called convex. If they are twice differentiable, the second derivatives (2 and e^x in our examples) are positive.

To see that convex coalition functions behave similarly, we consider the special case of symmetric coalition functions. In fig. 2, you see that the differences increase as they do for x^2.

Sometimes, an alternative characterization of convexity is helpful:

THEOREM IV.1 (criterion for convexity). *A coalition function v is convex if and only if for all coalitions R and S, we have*

$$v(R \cup S) + v(R \cap S) \geq v(R) + v(S).$$

v is strictly convex if and only if

$$v(R \cup S) + v(R \cap S) > v(R) + v(S)$$

holds for all coalitions R and S with $R \backslash S \neq \emptyset$ and $S \backslash R \neq \emptyset$.

We do not present a proof for this criterion. The reader can find a proof in the textbook on lattice theory by Topkis (1998).

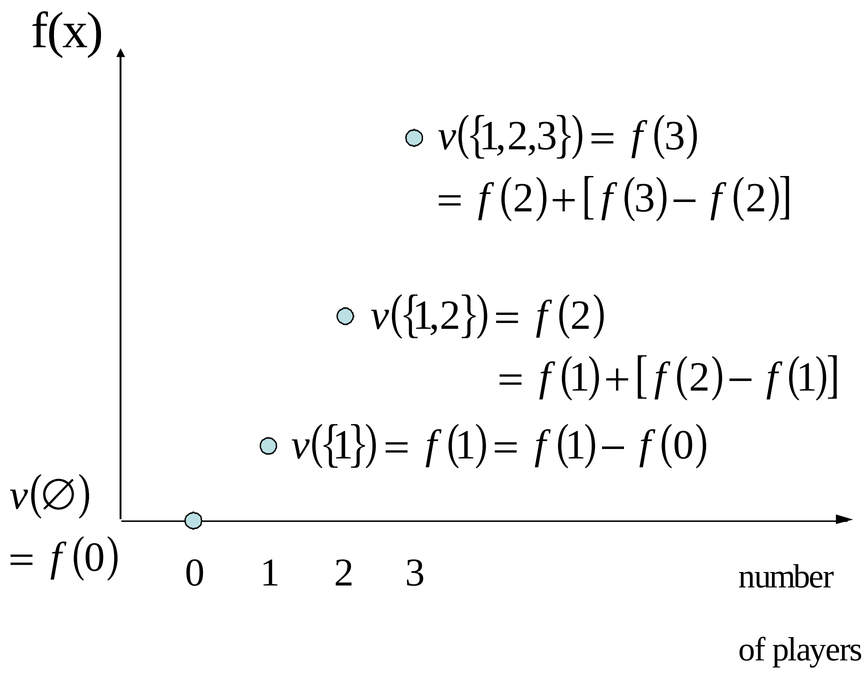

FIGURE 2. Convexity for symmetric coalition functions

We now turn to the relationship between superadditivity and convexity.

EXERCISE IV.23. *Is the Maschler game convex? Is it superadditive?*

Thus, a superadditive coalition function need not be convex. However, the inverse is true.

EXERCISE IV.24. *Using the above criterion for convexity, show that every convex coalition function is superadditive.*

5.5. The Shapley value and the core. The Shapley value need not be in the core even if the core is nonempty. This assertion follows from the following exercise that is taken from Moulin (1995, S. 425).

EXERCISE IV.25. *Consider the coalition function given by* $N = \{1, 2, 3\}$ *and*

$$v(K) = \begin{cases} 0, & |K| = 1 \\ \frac{1}{2}, & K = \{1,3\} \text{ or } K = \{2,3\} \\ \frac{8}{10}, & K = \{1,2\} \\ 1, & K = \{1,2,3\} \end{cases}$$

Show that $\left(\frac{4}{10}, \frac{4}{10}, \frac{2}{10}\right)$ *belongs to the core but that the Shapley value does not.*

However, the Shapley value can be shown to lie in the core for convex coalition functions:

THEOREM IV.2. *If a coalition function v is convex, the Shapley value $Sh(v)$ lies in the core.*

6. Topics and literature

The main topics in this chapter are

- simple game
- winning coalition
- veto player
- dictator
- null player
- unanimity game
- apex game
- weighted voting game
- buying-a-car game
- Maschler-Spiel
- endowment game
- superadditivity
- convexity
- monotonicity

We introduce the following mathematical concepts and theorems:

- linear independence
- span
- basis
- coefficients

We recommend .

7. Solutions

Exercise IV.1

$\eta_i = 0$ means that player i's marginal contribution is zero with respect to every coalition and hence player i is a null player.

Exercise IV.2

Can there be a coalition K such that $v(K\backslash\{i\}) = 1$ for a veto player i or a dictator i?

If i is a veto player, we have $v(K\backslash\{i\}) \leq v(N\backslash\{i\}) = 0$ for every coalition $K \subseteq N$ and hence $v(K\backslash\{i\}) = 0$. Thus, a veto player $i \in N$ cannot fulfill $v(K\backslash\{i\}) = 1$. A dictator i cannot fulfill $v(K\backslash\{i\}) = 1$ because the worth of a coalition is 1 if and only if the dictator belongs to the coalition.

Exercise IV.3

A dictator is always a veto player – without him the coalition cannot win. However, a veto player need not be a dictator. Just consider the simple game v on the player set $N = \{1,2\}$ defined by $v(\{1\}) = v(\{2\}) = 0$, $v(\{1,2\}) = 1$. Players 1 and 2 are two veto players but not dictators.

Exercise IV.4

$\eta_i = 2^{n-1}$ implies that every subset K of $N\backslash\{i\}$ is a loosing coalition while $K \cup \{i\}$ is winning. Player i is a dictator and a veto player.

Exercise IV.5

Let v be a simple game with a veto player $i \in N$. Then $v(K) = 1$ implies $i \in K$. By $i \notin N\backslash K$, we obtain $v(N\backslash K) = 0$ – the desired result.

Let v be a simple game with two veto players i and j, $i \neq j$. Then $v(\{i\}) = 0$ (by $j \notin \{i\}$) and $v(K\backslash\{i\}) = 0$ (by $i \notin K\backslash\{i\}$) hold.

Exercise IV.6

For the unanimity game u_T, the null players are the players from $N\backslash T$.

Exercise IV.7

The core is

$$\left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4_+ : x_1 + x_2 = 1\right\}$$

and the Shapley value is given by

$$Sh\left(u_{\{1,2\}}\right) = \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right).$$

Exercise IV.8

For $n = 2$, we have

$$\begin{aligned} h_1(K) &= \begin{cases} 0, & K = \{1\} \text{ or } K = \emptyset \\ 1, & \text{otherwise} \end{cases} \\ &= u_{\{2\}}. \end{aligned}$$

$n = 3$ yields the symmetric game

$$h_1(K) = \begin{cases} 1, & |K| \geq 2 \\ 0, & \text{otherwise} \end{cases}$$

(Symmetry means that the worths depend on the number of the players, only.)

Exercise IV.9

No, the apex player is not a veto player. If all the other player unite against the apex player, they win:

$$h_i(N\backslash\{i\}) = 1.$$

For the same reason, the apex player is not a dictator, either.

Exercise IV.10

We first show that h_i is not contradictory. Assume $h_i(K) = 1$ for any coalition $K \subseteq N$. Then, one of two cases holds. Either we have $K = N\backslash\{i\}$. This implies $h_i(N\backslash K) = h_i(\{i\}) = 0$. Or we have $i \in K$ and $|K| \geq 2$. Then, $h_i(N\backslash K) = 0$. Thus, h_i is noct contradictory.

We now show that h_i is decidable. Take any $K \subseteq N$ with $h_i(K) = 0$. This implies $K = \{i\}$ or $K \subsetneqq N \backslash \{i\}$. In both cases, the complements are winning coalitions: $N \backslash K = N \backslash \{i\}$ or $N \backslash K \supsetneqq \{i\}$.

Exercise IV.11

Since the apex player obtains the marginal contributions for positions 2 through $n-1$, his Shapley payoff is

$$\frac{n-2}{n} \cdot 1.$$

Due to efficiency, the other (symmetric!) players share the rest so that each of them obtains

$$\frac{1}{n-1}\left(1 - \frac{n-2}{n}\right) = \frac{2}{n(n-1)}.$$

Thus, we have

$$Sh(h_1) = \left(\frac{n-2}{n}, \frac{2}{n(n-1)}, ..., \frac{2}{n(n-1)}\right).$$

Exercise IV.12

One possible solution is

$$\left[1; \frac{1}{t}, ..., \frac{1}{t}, 0, ..., 0\right]$$

where $\frac{1}{t}$ is the weight for the powerful T-players while 0 is the weight for the unproductive $N \backslash T$-players.

Exercise IV.13

Every permanent member is a veto player by $4 \cdot 7 + 10 \cdot 1 = 38 < 39$. Because of $5 \cdot 7 + 4 \cdot 1 = 39$, four non-permanent members are necessary for passing a resolution.

Exercise IV.14

The voting rule is not contradictory and not decidable. This is just a corollary of exercise IV.5 (p. IV.5).

Exercise IV.15

By efficiency and symmetry, we have

$$Sh(v) = (24, 24, 24).$$

Exercise IV.16

The core has to fulfill

$$x_1 + x_2 + x_3 + x_4 = 2$$

and also the inequalities

$$\begin{aligned} x_i &\geq 0, i = 1, ..., 4, \\ x_1 + x_3 &\geq 1, \\ x_1 + x_4 &\geq 1, \\ x_2 + x_4 &\geq 1 \text{ and} \\ x_2 + x_3 + x_4 &\geq 2. \end{aligned}$$

We then find

$$x_1 = 2 - (x_2 + x_3 + x_4) \leq 0$$

and hence

$$\begin{aligned} x_1 &= 0 \text{ (because of } x_1 \geq 0), \\ x_3 &\geq 1 \text{ and } x_4 \geq 1. \end{aligned}$$

Using efficiency once more supplies $x_2 = 0$ and

$$(0, 0, 1, 1)$$

is the only candidate for a core. Indeed, this is the core. Just check all the inequalities above and also those omitted. Player 2's payoff is 0 in this situation. If he burns his second glove, we find (non-generalized) gloves game $v_{\{1,2\},\{3,4\}}$ where player 2 may achieve any core payoff between 0 and 1.

Exercise IV.17

The number of gloves pairs in $v_{\mathcal{E}} \oplus v_{\mathcal{E}}$ is twice the number of glove pairs in $v_{\mathcal{E}}$.

Exercise IV.18

For any $i, j \in \{1, 2, 3\}$, $i \neq j$, we have $v(\{i\}) + v(\{j\}) = 0 + 0 < 4 = v(\{i, j\})$ and $v(\{i\}) + v(N \backslash \{i\}) = 0 + 4 < 5$. Hence, v is superadditive.

Exercise IV.19

Every unanimity game is superadditive. Assume a unanimity game u_T that is not superadditive. Then, we would have to disjunct coalitions R and S with $v(R) + v(S) > v(R \cup S)$. The whole set of productive players T cannot be contained in both R and S. If it is contained in R (or in S), it is also contained in $R \cup S$. Then, we have $v(R) + v(S) = 1 = v(R \cup S)$ and the desired contradiction. If T is not contained in R and not contained in S, we have $v(R) + v(S) = 0$ and the inequality cannot be true, either.

The Maschler game is also superadditive. We need to consider the two inequalities

$$0 + 0 \leq 60 \text{ and } 0 + 60 \leq 72.$$

A simple game is contradictory if we have a coalition K such that $v(K) = v(N \backslash K) = 1$. By $v(K) + v(N \backslash K) = 2 > 1 = v(N)$, superadditivity is violated.

Exercise IV.20

For all coalitions $K \subseteq N$, we have $K \supseteq \emptyset$ and, by monotonicity $v(K) \geq v(\emptyset) = 0$.

Exercise IV.21

Consider two coalitions $S, S' \subseteq N$ with $S \subseteq S'$ gegeben. Monotonicity follows from

$$\begin{aligned} v(S') &= v\left(S \cup \left(S' \backslash S\right)\right) \\ &\geq v(S) + v\left(S' \backslash S\right) \text{ (superadditivity)} \\ &\geq v(S) \text{ (non-negativity).} \end{aligned}$$

Exercise IV.22

Yes, u_T is convex. For $i \in T$ and $S \subseteq S' \subseteq N$ with $i \notin S'$, we obtain

$$\begin{aligned} u_T(S \cup \{i\}) - u_T(S) &= u_T(S \cup \{i\}) - 0 \ (S \not\supseteq T) \\ &\leq u_T\left(S' \cup \{i\}\right) - 0 \ (u_T \text{ is monotonic}) \\ &= u_T\left(S' \cup \{i\}\right) - u_T\left(S'\right) \ (S' \not\supseteq T). \end{aligned}$$

If, however, i is not included in T, both $v(S \cup \{i\}) - v(S)$ and $v(S' \cup \{i\}) - v(S')$ are equal to zero. This shows that u_T is convex, but not strictly convex.

Exercise IV.23

The Maschler game is superadditive (see exercise **IV.19, p. 63),** but not convex. For $S = \{1\}$, $S' = \{1, 2\}$ and $i = 3$, we have

$$\begin{aligned} v(S \cup \{i\}) - v(S) &= v(\{1,3\}) - v(\{1\}) = 60 \\ &> 12 = v(\{1,2,3\}) - v(\{1,2\}) \\ &= v\left(S' \cup \{i\}\right) - v\left(S'\right). \end{aligned}$$

Exercise IV.24

Let R and S be disjunct coalitions. If v is convex, we obtain

$$\begin{aligned} v(R \cup S) &= v(R \cup S) + v(\emptyset) \\ &= v(R \cup S) + v(R \cap S) \\ &\geq v(R) + v(S). \end{aligned}$$

Thus, v is superadditive.

Exercise IV.25

Player 3's Shapley value is

$$Sh_3(v) = \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{2}{10} = \frac{7}{30}.$$

Symmetry and efficiency yield

$$Sh_1(v) = Sh_2(v) = \frac{1}{2} \cdot \left(1 - \frac{7}{30}\right) = \frac{23}{60}.$$

Since we have

$$Sh_1(v) + Sh_2(v) = 2 \cdot \frac{23}{60} = \frac{23}{30} < \frac{24}{30} = \frac{8}{10} = v(\{1,2\}),$$

the Shapley value does not belong to the core. You can check that $\left(\frac{4}{10}, \frac{4}{10}, \frac{2}{10}\right)$ fulfills all the necessary inequalities.

8. Further exercises without solutions

Show that the Shapley value for the cost function and the Shapley value for the cost-savings function amount to the same result.

CHAPTER V

Dividends

1. Introduction

This chapter is rather technical in nature. We discuss the vector space of coalition functions. It is a well-known result from linear algebra that every vector space has a basis.

It turns out that the unanimity games form a basis of the vector space of coalition functions on a player set N. This means that every coalition function can be "expressed" by unanimity games.

2. Definition and interpretation

Harsanyi (1963) defines devidends:

DEFINITION V.1 (Harsanyi dividend). *Let $v \in \mathbb{V}_N$ be a coalition function. The dividend (also called Harsanyi dividend) is a coalition function d^v on N defined by*

$$d^v(S) = \sum_{K \subseteq S} (-1)^{|S|-|K|} v(K).$$

THEOREM V.1 (Harsanyi dividend). *For any coalition function $v \in \mathbb{V}_N$, its Harsanyi dividends are defined by the induction formula*

$$\begin{aligned} d^v(S) &= v(S) \text{ for } |S| = 1, \\ d^v(S) &= v(S) - \sum_{K \subset S} d^v(K) \text{ for } |S| > 1 \end{aligned}$$

Why are the values of the coalition function d^v called dividends? Consider a player i who is a member of 2^{n-1} coalitions $S \subseteq N$. Player i "owns" coalition S together with the other players from S where his ownership fraction is $\frac{1}{|S|}$. Let us, now, assume that each coalition S brings forth a dividend $d^v(S)$. Then, player i should obtain the sum of average dividends

$$\sum_{i \in S \subseteq N} \frac{d^v(S)}{|S|}.$$

It can be shown that this sum equals the Shapley value $Sh_i(v)$. Thus, the term dividend makes sense if we assume that players get the Shapley value.

3. Coalition functions as vectors

As noted in chapter III, $\mathbb{V}_N$ can be considered the vector space of coalition functions on N. Since we have 2^n subsets of N, $2^n - 1$ (the worth of $\emptyset$ is always zero!) entries suffice to describe any game $v \in \mathbb{V}_N$. For example, $u_{\{1,2\}} \in G_{\{1,2,3\}}$ can be identified with the vector from $\mathbb{R}^7$

$$\left(\underbrace{0}_{\{1\}}, \underbrace{0}_{\{2\}}, \underbrace{0}_{\{3\}}, \underbrace{1}_{\{1,2\}}, \underbrace{0}_{\{1,3\}}, \underbrace{0}_{\{2,3\}}, \underbrace{1}_{\{1,2,3\}}\right).$$

EXERCISE V.1. *Write down the vector that describes the Maschler game*

$$v(K) = \begin{cases} 0, & |K| = 1 \\ 60, & |K| = 2 \\ 72, & |K| = 3 \end{cases}$$

You know how to sum vectors. We can also multiply a vector by a real number (scalar multiplication). Both operations proceed entry by entry:

EXERCISE V.2. *Consider* $v = (1, 3, 3)$, $w = (2, 7, 8)$ *and* $\alpha = \frac{1}{2}$ *and determine* $v + w$ *and* αw.

4. Spanning and linear independence

$\mathbb{R}^m$, $m \geq 1$, is a prominent class of vector spaces some of which obey $m = 2^n - 1$. We need some vector-space theory:

DEFINITION V.2 (linear combination, spanning). *A vector* $w \in \mathbb{R}^m$ *is called a linear combination of vectors* $v_1, ..., v_k \in \mathbb{R}^m$ *if there exist scalars (also called coefficients)* $\alpha_1, ..., \alpha_k \in \mathbb{R}$ *such that*

$$w = \sum_{\ell=1}^{k} \alpha_\ell v_\ell$$

holds. The set of vectors $\{v_1, ..., v_k\}$ *is said to span* $\mathbb{R}^m$ *if every vector from* $\mathbb{R}^m$ *is a linear combinations of the vectors* $v_1, ..., v_k$.

Consider, for example, $\mathbb{R}^2$ and the set of vectors

$$\{(1,2), (0,1), (1,1)\}.$$

Any vector (x_1, x_2) is a linear combination of these vectors. Just consider

$$\begin{aligned} & 2x_1(1,2) - (3x_1 - x_2)(0,1) - x_1(1,1) \\ = \; & (2x_1 - x_1, 4x_1 - (3x_1 - x_2) - x_1) \\ = \; & (x_1, x_2). \end{aligned}$$

EXERCISE V.3. *Show that* $(0,1)$ *is a linear combination of the other two vectors,* $(1,2)$ *and* $(1,1)$*!*

Using the result of the above exercise, we have

$$\begin{aligned}
& (x_1, x_2) \\
= \; & 2x_1 (1,2) - (3x_1 - x_2)(0,1) - x_1 (1,1) \\
= \; & 2x_1 (1,2) - (3x_1 - x_2)[(1,2) - (1,1)] - x_1 (1,1) \\
= \; & [2x_1 - (3x_1 - x_2)](1,2) - [x_1 + (3x_1 - x_2)](1,1)
\end{aligned}$$

so that any vector from $\mathbb{R}^2$ is a linear combination of just $(1,2)$ and $(1,1)$.

If we want to span $\mathbb{R}^2$ (or any $\mathbb{R}^m$), we try to find a minimal way to do so. Any vector in a spanning set that is a linear combination of other vectors in that set, can be eliminated.

Definition V.3 (linear independence). *A set of vectors* $\{v_1, ..., v_k\}$ *is called linearly independent if no vector from that set is a linear combination of other vectors from that set.*

Exercise V.4. *Are the vectors* $(1,3,3), (2,1,1)$ *and* $(8,9,9)$ *linearly independent?*

Merging these two definitions gives rise to one of the most important concept for vector spaces.

Definition V.4 (basis). *A set of vectors* $\{v_1, ..., v_k\}$ *is called a basis for* $\mathbb{R}^m$ *if it spans* $\mathbb{R}^m$ *and is linearly independent.*

An obvious basis for $\mathbb{R}^m$ consists of the m unit vectors

$$\begin{aligned}
& (1, 0, ..., 0), \\
& (0, 1, 0, ...,), \\
& ..., \\
& (0, ..., 0, 1).
\end{aligned}$$

Let us check whether they really do form a basis. Any $x = (x_1, ..., x_m)$ is a linear combination of these vectors by

$$\begin{aligned}
& x_1 (1, 0, ..., 0) + x_2 (0, 1, 0, ...,) + ... + x_m (0, ..., 0, 1) \\
= \; & (x_1, 0, ..., 0) + (0, x_2, 0, ...,) + ... + (0, ..., 0, x_m) \\
= \; & (x_1, ..., x_m).
\end{aligned}$$

This proves that the unit vectors do indeed span $\mathbb{R}^m$.

In order to show linear independence, consider any linear combination of $m-1$ unit vectors, for example

$$\alpha_1 (1, 0, ..., 0) + \alpha_2 (0, 1, 0, ...,) + ... + \alpha_{m-1} (0, ..., 0, 1, 0)$$

which is equal to $(\alpha_1, ..., \alpha_{m-1}, 0)$ and unequal to $(0, ..., 0, 1)$ for any coefficients $\alpha_1, ..., \alpha_{m-1}$.

LEMMA V.1 (basis of unit vectors). *The m unit vectors* $(1, 0, ..., 0), ..., (0, ..., 0, 1) \in \mathbb{R}^m$ *form a basis of the vector space* $\mathbb{R}^m$.

According to the above definition, a basis is a set of

(1) linearly independent vectors
(2) that span $\mathbb{R}^m$.

However, we do not need to check both conditions:

THEOREM V.2 (basis criterion). *Every basis of the vector space $\mathbb{R}^m$ has m elements. Any set of m elements of the vector space $\mathbb{R}^m$ that span $\mathbb{R}^m$ form a basis. Any set of m elements of the vector space $\mathbb{R}^m$ that are linearly independent form a basis.*

The reader might have noticed that the coefficients needed to express x as a linear combinations of unit vectors are uniquely determined. This is true for any basis:

THEOREM V.3 (uniquely determined coefficients). *Let $\{v_1, ..., v_m\}$ be a basis of $\mathbb{R}^m$ and let x be any vector such that*

$$x = \sum_{i=1}^{m} \alpha_i v_i = \sum_{i=1}^{m} \beta_i v_i.$$

Then $\alpha_i = \beta_i$ for all $i = 1, .., m$.

5. The basis of unanimity games

We have shown in the previous section that the unit games (that attribute the worth of one to exactly one nonempty coalition) form a basis of $\mathbb{V}_N$. They are the $2^n - 1$ coalition functions v_T, $T \neq \emptyset$, given by

$$v_T(S) = \begin{cases} 1, & S = T \\ 0, & S \neq T \end{cases}$$

An alternative and prominent basis of $\mathbb{V}_N$ is given by the unanimity games:

LEMMA V.2 (unanimity games form basis). *The $2^n - 1$ unanimity games u_T, $T \neq \emptyset$, form a basis of the vector space $\mathbb{V}_N$.*

According to theorem V.2, it is sufficient to show that the unanimity games are linearly independent. We use a proof by contradiction and assume that there is a unanimity game u_T that is a linear combination of the others:

$$u_T = \sum_{\ell=1}^{k.} \beta_\ell u_{T_\ell}$$

where

- the coalitions $T, T_1, ..., T_k$ are all pairwise different,
- $k \leq 2^n - 2$ holds and
- $\beta_\ell \neq 0$ holds for all $\ell = 1, ..., k$.

Let us assume $|T| \leq |T_\ell|$ for all $\ell = 1, .., k$. We can always rearrange the equation and rename the coalitions so that this condition is fulfilled. Using the coalition T as an argument, we now obtain

$$\begin{aligned} 1 &= u_T(T) \\ &= \sum_{\ell=1}^{k} \beta_\ell u_{T_\ell}(T) \\ &= \sum_{\ell=1}^{k} \beta_\ell \cdot 0 \\ &= 0 \end{aligned}$$

and hence the desired contradiction.

EXERCISE V.5. *In the above proof, do you see why $u_{T_\ell}(T) = 0$ holds for all $\ell = 1, ..., k$?*

Now, let us reconsider lemma V.2 and theorem V.3. They say that for any $v \in \mathbb{V}_N$ there exist uniquely determined coefficients $\lambda^v(T)$ such that

$$v = \sum_{T \in 2^N \setminus \{\emptyset\}} \lambda^v(T) u_T$$

holds. This equation can also be expressed by

$$v(S) = \sum_{T \in 2^N \setminus \{\emptyset\}} \lambda^v(T) u_T(S), S \subseteq N. \tag{V.1}$$

Indeed, the coefficients can be shown to be the Harsanyi dividends:

$$\lambda^v(T) := d^v(T).$$

We will not provide a proof for this intriguing fact. Instead, we borrow an example from Slikker & Nouweland (2001, p. 7)). Consider $N := \{1, 2, 3\}$ and the coalition function v given by

$$v(S) = \begin{cases} 0, & |S| = 1 \\ 60, & S = \{1, 2\} \\ 48, & S = \{1, 3\} \\ 30, & S = \{2, 3\} \\ 72, & S = N \end{cases}$$

This coalition function can also be expressed by the vector

$$\begin{pmatrix} 0 \ (\{1\}) \\ 0 \ (\{2\}) \\ 0 \ (\{3\}) \\ 60 \ (\{1, 2\}) \\ 48 \ (\{1, 3\}) \\ 30 \ (\{2, 3\}) \\ 72 \ (\{1, 2, 3\}) \end{pmatrix}$$

Using the induction formula, the coefficients are

$$\begin{aligned}
d^v(\{1\}) &= d^v(\{2\}) = d^v(\{3\}) = 0, \\
d^v(\{1,2\}) &= v(\{1,2\}) - d^v(\{1\}) - d^v(\{2\}) \\
&= 60 - 0 - 0 = 60, \\
d^v(\{1,3\}) &= v(\{1,3\}) - d^v(\{1\}) - d^v(\{3\}) \\
&= 48 - 0 - 0 = 48, \\
d^v(\{2,3\}) &= v(\{2,3\}) - d^v(\{2\}) - d^v(\{3\}) = 30 \text{ and} \\
d^v(\{1,2,3\}) &= v(\{1,2,3\}) - d^v(\{1,2\}) - d^v(\{1,3\}) - d^v(\{2,3\}) \\
&\quad - d^v(\{1\}) - d^v(\{2\}) - d^v(\{3\}) \\
&= 72 - 60 - 48 - 30 - 0 - 0 - 0 \\
&= -66
\end{aligned}$$

and we obtain

$$\begin{aligned}
& d^v(\{1,2\})\, u_{\{1,2\}} + d^v(\{1,3\})\, u_{\{1,3\}} + d^v(\{2,3\})\, u_{\{2,3\}} + d^v(\{1,2,3\})\, u_N \\
= \; & 60 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + 48 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + 30 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} - 66 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
= \; & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 60 \\ 48 \\ 30 \\ 72 \end{pmatrix}
\end{aligned}$$

and hence the expected vector.

EXERCISE V.6. *Calculate the coefficients for the following games on* $N = \{1,2,3\}$:

- $v \in \mathbb{V}_N$ *is defined by* $v(\{1,2\}) = v(\{2,3\}) = v(\{1,2,3\}) = 1$ *and* $v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1,3\}) = 0$.
- $v \in \mathbb{V}_N$ *is defined by*

$$v(S) = \begin{cases} 0, & |S| \leq 1 \\ 8, & |S| = 2 \\ 9, & S = N \end{cases}$$

6. Topics and literature

The main topics in this chapter are

- Harsanyi dividend
- stability
- linear independence
- span
- basis
- coefficients

We recommend

7. Solutions

Exercise V.1

The vector describing the Maschler game is

$$\left(\underbrace{0}_{\{1\}},\underbrace{0}_{\{2\}},\underbrace{0}_{\{3\}},\underbrace{60}_{\{1,2\}},\underbrace{60}_{\{1,3\}},\underbrace{60}_{\{2,3\}},\underbrace{72}_{\{1,2,3\}}\right).$$

Exercise V.2

We obtain $v+w=(1,3,3)+(2,7,8)=(3,10,11)$ and $\alpha w=\frac{1}{2}(2,7,8)=\left(1,\frac{7}{2},4\right)$.

Exercise V.3

We have $(1,2)-(1,1)=(0,1)$. Thus, we need the coefficients 1 and -1.

Exercise V.4

No, they are not linearly independent. Consider $2(1,3,3)+3(2,1,1)=(8,9,9)$.

Exercise V.5

Take any $\ell\in\{1,...,k\}$. In order for $u_{T_\ell}(T)=1$ to hold, T would need to be a superset of T_ℓ. However, by $|T|\leq|T_\ell|$, T and T_ℓ would then need to be equal which they are not.

Exercise V.6

In general, we have

$$d^v(T):=\sum_{K\in 2^T\backslash\{\emptyset\}}(-1)^{|T|-|K|}v(K).$$

For the first game, we find

$$\begin{aligned}
d^v(\{1\}) &= d^v(\{2\}) = d^v(\{3\}) = 0, \\
d^v(\{1,2\}) &= (-1)^{2-1} v(\{1\}) + (-1)^{2-1} v(\{2\}) + (-1)^{2-2} v(\{1,2\}) = 1, \\
d^v(\{1,3\}) &= (-1)^{2-1} v(\{1\}) + (-1)^{2-1} v(\{3\}) + (-1)^{2-2} v(\{1,3\}) = 0, \\
d^v(\{2,3\}) &= (-1)^{2-1} v(\{2\}) + (-1)^{2-1} v(\{3\}) + (-1)^{2-2} v(\{2,3\}) = 1, \\
d^v(\{1,2,3\}) &= (-1)^{3-1} v(\{1\}) + (-1)^{3-1} v(\{2\}) + (-1)^{3-1} v(\{3\}) \\
&\quad + (-1)^{3-2} v(\{1,2\}) + (-1)^{3-2} v(\{1,3\}) + (-1)^{3-2} v(\{2,3\}) \\
&\quad + (-1)^{3-3} v(\{1,2,3\}) \\
&= 0 + 0 + 0 - 1 - 0 - 1 + 1 \\
&= -1
\end{aligned}$$

while the second leads to

$$\begin{aligned}
d^v(T) &= 0 \text{ für } |T| = 1, \\
d^v(T) &= d^v(\{1,2\}) = (-1)^{2-1} v(\{1\}) + (-1)^{2-1} v(\{2\}) + (-1)^{2-2} v(\{1,2\}) = 8 \text{ for } |T| = 2 \\
d^v(\{1,2,3\}) &= 3 \cdot (-1)^{3-2} v(\{1,2\}) + (-1)^{3-3} v(\{1,2,3\}) \\
&= -24 + 9 = -15.
\end{aligned}$$

8. Further exercises without solutions

CHAPTER VI

Axiomatizing the Shapley value

1. Introduction

This *is* a book on applications. Nevertheless, the reader should see the most prominent example of the axiomatization of a value, the Shapley value. We prepare the ground in section 2 – axiomatization means to find just the right set of axioms. If there are too many aioms, they contradict each other. Too few axioms are incapable of pointing to just one solution concept. If we strike the right balance, the axioms single out exactly one solution concpet.

The proof of the axiomatization theorem comes in two parts:

(1) We show that the Shapley value fulfills the four axioms (section 3.
(2) We prove that there is only one value fulfilling the four axiom (section 4). By the first part, this value needs to be the Shapley value.

Also, we present two other systems of axioms for the Shapley value (sections 5 and 6). The third axiomatization can be linked to a discussion on the concept of power-over (section 7).

Finally, we present the Banzhaf solution in section 8 which is an alternative to the Shapley value, in particular for simple games.

2. Too many axioms, not enough axioms

For any given set of axioms, we have three possibilities:

- There is no solution concept that fulfills all the axioms. That is, the axioms are contradictary.
- The axioms are compatible with several solution concepts.
- There is one and only one solution concept that fulfills the axioms. That is, the solution concept is axiomatized by this set of axioms.

Exercise VI.1. *Consider the following two axioms:*

(1) *Every player obtains the same payoff.*
(2) *Summing the players' payoffs yields* $v(N)$.
(3) *Every null player (with zero marginal contributions everywhere) obtains zero payoff.*

and the following two solutions:

(1) *Every player obtains* $v(N)/n$.
(2) *Every player obtains the* ρ*-value for the rank order* $(1, 2, ..., n)$.

Can you identify a set of contradictory axioms and can you identify a axioms fulfilled by both solution concepts?

Definition VI.1. *A solution concept σ (on $\mathbb{V}_N$ or on $\mathbb{V}$) is said be axiomatized by a set of axioms if σ fulfills all the axioms and if any solution concept to do so is identical with σ.*

The Shapley value is defined by

$$Sh_i(v) = \frac{1}{n!} \sum_{\rho \in RO_N} MC_i^\rho(v).$$

This formula tells us to sum up and average the marginal contributions for each rank order. The formula obeys some axioms and disobeys others. It turns out that the following four axioms are equivalent to the Shapley formula:

Definition VI.2. *Let σ be a solution function σ on $\mathbb{V}_N$. σ obeys*

- *the efficiency (or Pareto) axiom if $\sum_{i \in N} \sigma_i(v) = v(N)$ holds for all coalition functions $v \in \mathbb{V}_N$,*
- *the symmetry axiom if $\sigma_i(v) = \sigma_j(v)$ is true for all coalition functions $v \in \mathbb{V}_N$ and for any two symmetric players i and j,*
- *the null-player axiom if we have $\sigma_i(v) = 0$ for all coalition functions $v \in \mathbb{V}_N$ and for any null player i and*
- *the additivity axiom in case of $\sigma(v + w) = \sigma(v) + \sigma(w)$ for any two coalition functions $v, w \in \mathbb{V}_N$ with $N(v) = N(w)$.*

The main aim of this chapter is to prove

Theorem VI.1 (1. axiomatization of Shapley value). *The Shapley formula is axiomatized by the four axioms mentioned in the previous definition.*

3. The Shapley formula fulfills the four axioms

3.1. Efficiency axiom. The efficiency axiom holds for the Shapley value and even for the marginal contributions.

Definition VI.3 (ρ-solution). *For a player set N and a rank order $\rho \in RO_N$, the ρ-solution is given by*

$$(MC_1^\rho(v), ..., MC_n^\rho(v)).$$

Thus, let us assume any rank order $\rho \in RO_N$. We can savely assume $\rho = (1, ..., n)$. If the players come in a different order, we can rename them so as to obtain the order $(1, ..., n)$. We find

$$\begin{aligned}
\sum_{i \in N} MC_i^\rho(v) &= \sum_{i \in N} \left[v(K_i(\rho)) - v(K_i(\rho) \setminus \{i\})\right] \\
&= [v(\{\rho_1\}) - v(\emptyset)] \\
&\quad + [v(\{\rho_1, \rho_2\}) - v(\{\rho_1\})] \\
&\quad + [v(\{\rho_1, \rho_2, \rho_3\}) - v(\{\rho_1, \rho_2\})] \\
&\quad + ... \\
&\quad + \left[v\left(\{\rho_1, ..., \rho_{n-1}\}\right) - v\left(\{\rho_1, ..., \rho_{n-2}\}\right)\right] \\
&\quad + \left[v(\{\rho_1, ..., \rho_n\}) - v\left(\{\rho_1, ..., \rho_{n-1}\}\right)\right] \\
&= v(N) - v(\emptyset) \\
&= v(N).
\end{aligned}$$

LEMMA VI.1. *The ρ-solutions and the Shapley value fulfill the efficiency axiom.*

The efficiency of the ρ-solutions has been shown above. The efficiency of the Shapley value follows immediately:

$$\begin{aligned}
\sum_{i \in N} Sh_i(v) &= \sum_{i \in N} \frac{1}{n!} \sum_{\rho \in RO_N} MC_i^\rho(v) \\
&= \sum_{\rho \in RO_N} \frac{1}{n!} \sum_{i \in N} MC_i^\rho(v) \ \text{ (rearranging the summands)} \\
&= \sum_{\rho \in RO_N} \frac{1}{n!} v(N) \ \text{ (ρ-solutions are efficient)} \\
&= n! \frac{1}{n!} v(N) \\
&= v(N).
\end{aligned}$$

3.2. Symmetry axiom. Astonishingly, the symmetry axiom is not easy to show. We refer the reader to Osborne & Rubinstein (1994, S. 293). Intuitively, symmetry is obvious. After all,

- two players are symmetric if they contribute in a similar fashion and
- the Shapley formula's inputs are these marginal contributions.

3.3. Null-player axiom. A null player contributes nothing, per definition. The average of nothing is nothing. Therefore, the null-player axiom holds for the Shapley value. Just look at

$$\begin{aligned}\sum_{i\in N} Sh_i(v) &= \sum_{i\in N}\frac{1}{n!}\sum_{\rho\in RO_N} MC_i^\rho(v)\\ &= \sum_{i\in N}\frac{1}{n!}\sum_{\rho\in RO_N} 0\\ &= 0.\end{aligned}$$

3.4. Additivity axiom. In order to show additivity, note

$$\begin{aligned}&(v+w)(K)-(v+w)(K\backslash\{i\})\\ =\;& v(K)+w(K)-(v(K\backslash\{i\})+w(K\backslash\{i\}))\\ =\;& [v(K)-v(K\backslash\{i\})]+[w(K)-w(K\backslash\{i\})]\end{aligned}$$

for any two coalition functions $v, w \in \mathbb{V}_N$ any player $i \in N$ and any coalition $K \subseteq N$. Therefore, we find

$$\begin{aligned}&Sh_i(v+w)\\ =\;& \sum_{i\in N}\frac{1}{n!}\sum_{\rho\in RO_N} MC_i^\rho(v+w)\\ =\;& \sum_{i\in N}\frac{1}{n!}\sum_{\rho\in RO_N}[(v+w)(K_i(\rho))-(v+w)(K_i(\rho)\backslash\{i\})]\\ &\text{(definition of marginal contribution)}\\ =\;& \sum_{i\in N}\frac{1}{n!}\sum_{\rho\in RO_N}([v(K_i(\rho))-v(K_i(\rho)\backslash\{i\})]\\ &+[w(K_i(\rho))-w(K_i(\rho)\backslash\{i\})])\ \text{(see above)}\\ =\;& \sum_{i\in N}\frac{1}{n!}\sum_{\rho\in RO_N}[v(K_i(\rho))-v(K_i(\rho)\backslash\{i\})]\\ &+\sum_{i\in N}\frac{1}{n!}\sum_{\rho\in RO_N}[w(K_i(\rho))-w(K_i(\rho)\backslash\{i\})]\\ =\;& Sh_i(v)+Sh_i(w).\end{aligned}$$

4. ... and is the only solution function to do so

We now want to show that any solution function that fulfills the four axioms is the Shapley value. We follow the proof presented by Aumann (1989, S. 30 ff.). We remind the reader of two important facts.

- The unanimity games u_T, $T \neq \emptyset$, form a basis of the vector space $\mathbb{V}_N$ (see chapter IV, pp. 76) so that every coalition function v is a linear combination of these games:

$$v = \sum_{T\in 2^N\backslash\{\emptyset\}} \lambda_T(v)\, u_T. \tag{VI.1}$$

- For any game γu_T, $\gamma \in \mathbb{R}$, the players from $N \backslash T$ are the null players (compare exercise IV.6, S. 54).

Consider, now, any solution function σ that obeys the four axioms. We obtain

$$\begin{aligned}\sum_{i \in T} \sigma_i(\gamma u_T) &= \sum_{i \in T} \sigma_i(\gamma u_T) + \sum_{i \in N \backslash T} \sigma_i(\gamma u_T) \text{ (null-player axiom)} \\ &= (\gamma u_T)(N) \text{ (Pareto axiom)} \\ &= \gamma u_T(N) \\ &= \gamma.\end{aligned}$$

The null players (from $N \backslash T$) get zero payoff, the (symmetric!) T-players share γ :

$$\sigma_i(\gamma u_T) = \begin{cases} \frac{\gamma}{|T|}, & i \in T \\ 0, & i \notin T. \end{cases}$$

Let now v be any coalition function on N. Using the above results and applying the additivity axiom several times, we find

$$\begin{aligned}\sigma_i(v) &= \sigma_i\left(\sum_{T \in 2^N \backslash \{\emptyset\}} \lambda_T(v) u_T\right) \text{ (eq. VI.1)} \\ &= \sum_{T \in 2^N \backslash \{\emptyset\}} \sigma_i(\lambda_T(v) u_T) \text{ (additivity axiom)} \\ &= \sum_{T \in 2^N \backslash \{\emptyset\}} \begin{cases} \frac{\lambda_T(v)}{|T|}, & i \in T \\ 0, & i \notin T. \end{cases} \text{ (with } \gamma := \lambda_T(v)\text{)}\end{aligned}$$

Thus, the axioms determine the payoffs. Since the Shapley formula fulfills the axioms, we obtain the desired result

$$\sigma = Sh.$$

And we are done.

5. A second axiomatization via marginalism

The Shapley value is an average of the marginal contributions of the players. Thus, whenever we have two coalition functions v and w such that the marginal contributions (with respect to any given coalition) of a player is the same under v and under w, the player's Shapley value is the same. This fact is called the marginalism axiom:

DEFINITION VI.4 (marginalism axiom). *A solution function σ on $\mathbb{V}_N$ is said to obey the marginalism axiom if, for any player $i \in N$ and any two coalition functions $v, w \in \mathbb{V}_N$ with $N(v) = N(w)$,*

$$MC_i^K(v) = MC_i^K(w), K \subseteq N(v)$$

implies

$$\sigma_i(v) = \sigma_i(w).$$

The marginalism axiom is quite strong. Young (1985) has shown that the Shapley value can be axiomatized by just three axioms:

THEOREM VI.2 (2. axiomatization of Shapley value). *The Shapley formula is axiomatized by the symmetry axiom, the marginalism axiom and the efficiency axiom.*

6. A third axiomatization via balanced contributions

Finally, we want to consider the axiom of balanced contributions which is due to Myerson (1980). The basic idea is that players suffer equally if one of them withdraws from the game. We need some formal preliminary:

DEFINITION VI.5. *Let $v \in \mathbb{V}_N$ be a coalition function and let $S \subseteq N$, $S \neq \emptyset$ be a coalition. The restriction of v onto S is the coalition function*

$$\begin{array}{rcl} v|_S & : & 2^S \to \mathbb{R}, \\ K & \mapsto & v|_S(K) = v(K). \end{array}$$

Thus, $v|_S$ attributes the same worths as v but only to subsets of S.

DEFINITION VI.6 (axiom of balanced contributions). *A solution function σ on $\mathbb{V}$ is said to obey the axiom of balanced contributions if, for any coalition function v and any two players $i, j \in N(v) =: N$,*

$$\sigma_i(v) - \sigma_i\left(v|_{N\backslash\{j\}}\right) = \sigma_j(v) - \sigma_j\left(v|_{N\backslash\{i\}}\right)$$

holds.

The reader notes that we employ the solution function on $\mathbb{V}$, not on $\mathbb{V}_N$. After all, $v|_{N\backslash\{j\}}$ has one player less than game v. We will dwell on the interpretation of the balanced contributions in a minute. Before, let us note the axiomatization theorem:

THEOREM VI.3 (3. axiomatization of Shapley value). *The Shapley formula is axiomatized by the efficiency axiom and the axiom of balanced contributions.*

Balanced contributions is a very powerful axiom. Note, however, that we claim this axiom not just for a given player set N but for all its subsets also.

7. Balanced contributions and power-over

7.1. Introduction. The power of people and the power of some people over others have long been a central concern in sociology, politics, and psychology while Bartlett (1989) and Rothschild (2002) find a neglect of power apart from market power in mainstream economics. However, power seems to be an extraordinary elusive concept. As Bartlett (1989, pp. 9-10) observes, there exists a "multiplicity of concepts" of power, but no "widely accepted concept of power within either economics or its sister social sciences".

The thesis of this section is that there are basically three reasons for this lamentable state. First, power may be defined with reference to actions (actor 1 forces actor 2 to perform an act against 2's will) or with reference to payoffs (actor 1 benefits more than actor 2). This corresponds to the difference between I-power (with I standing for "influence") and P-power (with P denoting "prize" or "payoff") by Felsenthal & Machover (1998). Of course, I-power and P-power are closely related because actions result in payoffs and payoffs flow from actions.

An early and prominent definition of power is due to Max Weber (1968, p. 53):

> "Power is the probability that one actor within a social relationship will be in a position to carry out his own will despite resistance ... ."

Obviously, this is I-power. A Weberian P-power definition would be the following:

> "Power is the probability that one actor within a social relationship will obtain costly benefits from others."

Secondly, the multiplicity of power concepts also stems from the fact that power and power-over need to be distinguished. Consider James Coleman's (1990, p. 133) definition:

> "The power of an actor resides in his control of valuable events. The value of an event lies in the interests powerful actors have in that event. ... Power ... is not a property of the relation between two actors (so it is not correct in this context to speak of one actor's power over another, although it is possible to speak of the relative power of two actors)."

Most authors, however, prefer to understand power relatively, i.e., in terms of the power an actor 1 exercises over another actor 2. Proponents of this tradition are Max Weber (1968), Richard Emerson (1962), Dorwin Cartwright (1959, p. 196), and Vittorio Hösle (1997, p. 394-396) .

In this section, we will side with these authors and will talk about power in the sense of power-over. Our focus is on a third problem. According to some definitions, power is ubiquitous. For example, Viktor Vanberg (1982, p. 59, fn 48) observes that in every exchange relationship both sides do what they would not have done without the influence of the other party.

Indeed, if 1 offers 2 some money to perform a service and 2 obliges, does 1 have power over 2? Or, the other way around, does 2 have power over 1 because he "forces" 1 to give him money for some important (to 1) service. According to everyday usage, 1 exerts power over 2 if 1 obtains the service for "too little" money ("exploitation") while 2 exerts power over 1 if 2 asks for "too much" and 1 is in an urgent need for the service ("profiteering", "extortion", "usury").

In line with the above observation, we claim that every fruitful definition of power-over needs a reference point which may concern a "usual", "normal", or "moral" situation. We will argue for several and quite diverse reference points in section 7.2. It seems quite unavoidable that reference points contain some measure of arbitrariness and need to be defended rather specifically.

In section 7.3, we will try an alternative reference point that is not arbitrary. The idea of this reference point is simple. Actors may suffer (or gain) if other actors withdraw (where would you be without me?). In such a setting, 1 exerts power over 2 if 2 suffers more from a withdrawal by 1 than vice versa. However, we will find good reasons for this definition to fail. Indeed, if we use the Shapley value, withdrawal of 1 harms 2 as much as withdrawal of 2 harms 1 – this is the axiom of balanced contributions. While this may first seem counterintuitive, we will be able to indicate plausible mechanisms for this to come about.

The idea of this section is to tackle the reference-point issue by considering the difference between actual payoffs and payoffs according to some reference point. Of course, we will use cooperative game theory to define these payoffs.

The general idea of defining power by way of payoff differences can already be found in Johan Galtung (1969) who defines "violence ... as the cause of the difference between the potential and the actual". Less directly, Lukes (1986, p. 5) suggests "that to have power is to be able to make a difference to the world." Our difference approach captures these differences.

7.2. Payoff reflections of power-over .

7.2.1. *Payoff differences.* We want to measure power-over by looking at the payoff differences caused by the exercise of power of one player over another. In most examples, a player 1 exercises power over another player 2. We consider two coalition functions, v and w. Often, by v we mean a coalition function describing the actual social or economic situation where player 1 exercises power over player 2. w, on the other hand, describes what the players would get if, contrary to the actual state of affairs, player 2 were not subject to the power exerted by player 1. Formally, we usually get

$$D_1 := \varphi_1(v) - \varphi_1(w) > 0$$

and

$$D_2 := \varphi_2(v) - \varphi_2(w) < 0.$$

7.2.2. *Example: market power.* First, we consider the example of the gloves game where we assume one left-glove holder (player 1) and 4 right-glove holders (players 2 through 5). The left-glove holder is in a monopoly (or monopsony) position. The Shapley value is $\left(\frac{4}{5}, \frac{1}{20}, \frac{1}{20}, \frac{1}{20}, \frac{1}{20}\right)$. Assume that player 1 sells his left glove. He obtains the price of $\frac{4}{5}$. Each of the players 2 through 5 have $\frac{1}{4}$ chance to buy the glove for a price of $\frac{4}{5}$. Hence, each right-glove holder has an expected utility of $\frac{1}{4}\left(1 - \frac{4}{5}\right) = \frac{1}{20}$.

Let us now invoke the norm of equal splitting of gains between player 1 and player 2 to whom player 1 happens to sell the left glove. Then, payoffs are $\left(\frac{1}{2}, \frac{1}{2}, 0, 0, 0\right)$. There exists a coalition function w leading to these payoffs.

Then, player 1's power over player 2 is reflected by

$$\begin{aligned} D_1 &= \varphi_1(v) - \varphi_1(w) \\ &= \frac{4}{5} - \frac{1}{2} \\ &= \frac{3}{10} \end{aligned}$$

and

$$\begin{aligned} D_2 &= \varphi_2(v) - \varphi_2(w) \\ &= \frac{1}{20} - \frac{1}{2} \\ &= -\frac{9}{20}. \end{aligned}$$

7.2.3. *Example: emotional dependence.* As a second example, we consider the emotional dependence that may sometimes exist between a player M (man) and a player W (woman). They may both like to live together so that $v(M, W) > 0$. However, he may be more independent of her than the other way around. Then,

$$v(M) > v(W)$$

is a plausible assumption. (If the reader finds the example objectionable, she or he is welcome to reverse the roles.)

The Shapley values are given by

$$\begin{aligned}\varphi_M &= \frac{1}{2}v(M) + \frac{1}{2}[v(M,W) - v(W)] \\ &= \frac{1}{2}v(M,W) + \frac{1}{2}[v(M) - v(W)] \\ &> \frac{1}{2}v(M,W) + \frac{1}{2}[v(W) - v(M)] \\ &= \varphi_W.\end{aligned}$$

His payoff is higher than her's. Applying the egalitarian norm ($w(M) = w(W) = \frac{1}{2}v(M,W)$) we obtain $\varphi_M(w) = \frac{1}{2}v(M,W) = \varphi_W(w)$. We would therefore diagnose that he has power over her:

$$\begin{aligned}D_M &= \varphi_M(v) - \varphi_M(w) \\ &= \frac{1}{2}[v(M) - v(W)] \\ &> 0 \\ &> \frac{1}{2}[v(W) - v(M)] \\ &= D_W\end{aligned}$$

Both examples make clear that the problem about a reference point is not "solved". We rather choose to offer a taxonomy: If the reference point is some or other norm (or defined by some or other counterfactual), then we obtain this or that payoff difference. While this may seem an evasive strategy, we argue that power-over necessarily needs a reference point and that there is no unambiguous choice of such a point.

7.3. Action reflexions of power-over.

7.3.1. *Withdrawing and quitting.* Instead of invoking some quite arbitrary fairness norms, one might consider the differences

$$\varphi_1(v) - \varphi_1\left(v|_{N\backslash 2}\right)$$

and

$$\varphi_2(v) - \varphi_2\left(v|_{N\backslash 1}\right)$$

known from the axiom of balanced contributions. For player 1, $v|_{N\backslash 2}$ is the game v without player 2. In words: $\varphi_1(v) - \varphi_1\left(v|_{N\backslash 2}\right)$ measures the loss to player 1 if player 2 withdraws. We might try the following definition: Player 1 exerts power over player 2, if player 1 suffers less from a withdrawal by player 2 than vice versa.

Interestingly, this definition fails if we use the Shapley value: What 1 can do to 2 by withdrawing is exactly equal to what 2 can do to 1 by withdrawing. This is just what balanced contributions means.

7.3.2. *Example: Revisiting the gloves game.* Let us reconsider the gloves game. Again, we assume one left-glove holder (player 1) and 4 right-glove holders (players 2 through 5) (see subsection 7.2.2). It might seem that player 1's threat of withdrawal carries more weight than player 2's threat of withdrawal. However, this is not the case. The Shapley values are

$$\begin{aligned}
\left(\frac{4}{5},\frac{1}{20},\frac{1}{20},\frac{1}{20},\frac{1}{20}\right) \text{ for } N &= \{1,2,3,4,5\}, \\
\left(\frac{3}{4},\frac{1}{12},\frac{1}{12},\frac{1}{12}\right) \text{ for } N &= \{1,3,4,5\} \text{ and} \\
(0,0,0,0) \text{ for } N &= \{2,3,4,5\}
\end{aligned}$$

so that we have

$$\begin{aligned}
&\varphi_1(v) - \varphi_1\left(v|_{N\backslash 2}\right) \\
= &\frac{4}{5} - \frac{3}{4} \\
= &\frac{1}{20}
\end{aligned}$$

and

$$\begin{aligned}
&\varphi_2(v) - \varphi_2\left(v|_{N\backslash 1}\right) \\
= &\frac{1}{20} - 0.
\end{aligned}$$

The reason for the equality of these differences is this: Player 1 obtains a price of $\frac{4}{5}$ for his left glove in case of 4 potential buyers, but a price of $\frac{3}{4}$ in case of 3 potential buyers. So indeed, player 2's withdrawal does not do much damage to player 1. But player 2's disutility caused by player 1's withdrawal is small also. If player 1 is around, player 2 will have a small chance $(\frac{1}{4})$ of getting the glove and will also have to pay a high price $(\frac{4}{5})$. Therefore, in the presence of player 1, player 2 gets the payoff $1 - \frac{4}{5} = \frac{1}{5}$ with a chance of $\frac{1}{4}$ only. The small payoff of $1/20$ is lost when player 1 withdraws.

While payoff differences with respect to the threat of withdrawal are not useful for defining power-over, they can be used to theorize about the action players have to take. In the gloves example, it is the balanced contributions that allow player 1 to charge a high price for his left glove.

7.3.3. *Example: Revisiting emotional dependence.* We also reconsider the emotional-dependence example (see section 7.2.3) and obtain her payoff difference as

$$\begin{aligned}
& \varphi_W(v) - \varphi_W\left(v|_{N\backslash M}\right) \\
= & \left[\frac{1}{2}v(M,W) + \frac{1}{2}v(W) - \frac{1}{2}v(M)\right] - v(W) \\
= & \frac{1}{2}\left[v(M,W) - v(W) - v(M)\right].
\end{aligned}$$

In case of superadditivity, his threat of withdrawal (divorce, say) is effective and she suffers from it. However, for player M we get the same result:

$$\varphi_M(v) - \varphi_M\left(v|_{N\backslash W}\right) = \varphi_W(v) - \varphi_W\left(v|_{N\backslash M}\right).$$

Again, we can use this equality to infer actions: Just because of $v(M) > v(W)$, he can make her do the washing-up. But taking her washing-up into account, she suffers less from a break-down of the relationship and his loss of her would be more serious than in a "fair" partnership.

7.4. Negative sanctions and the threat to withdraw. The equality of the threats to withdraw may be particularly astonishing for negative sanctions and coercion (see Willer 1999, pp. 24). Indeed, if a robber (player 1) points his gun to my, player 2's, head, it may seem impossible for me to "withdraw". However, we need to look more closely.

It is important to note that withdrawing is analyzed within the given game v. The question of whether a player can quit a game or opt out is a totally different one. For example, I normally do not need to partake in a market game but sometimes I cannot help being part of a game as in our gun-and-money game.

First, we need to define the coalition function. For the coalition $\{1,2\}$, $v(1,2) = 0$ seems plausible. I hand over some money $c > 0$ to the robber so that his gain is my loss. We then have $\varphi_1(v) = c = -\varphi_2(v)$ which fulfills the efficiency axiom. (Of course, I may be traumatized by the experience and he may be afraid of being caught and arrested in which case $v(1,2)$ should be negative.)

One may be tempted to put $v(2) = 0$ since I do not lose any money if the robber is not there. However, what I can achieve on my own still depends on what the robber does (withdrawal is not quitting!). If I do not hand over the money peacefully, he may injure me. We define the worth for a coalition K as the minimum of what the other players, $N\backslash K$, can inflict on K. We let i represent the pain of being injured and obtain $v(2) = -i < 0$.

Similarly, $v(1)$ is the minimum of what I can inflict on the robber. I can run away and force him to injure me. Then, he will be in fear of prosecution for injury; let f stand for this fear so that we have $v(1) = -f$.

Now, because of $v(1) = v|_{N\backslash 2}(1)$ and $\varphi_1\left(v|_{N\backslash 2}\right) = v(1)$, my running away or his injuring me leads to the payoff differences

$$\begin{aligned} & \varphi_1(v) - \varphi_1\left(v|_{N\backslash 2}\right) \\ = & \underbrace{c}_{\substack{\text{money} \\ \text{obtained}}} - \underbrace{-f}_{\substack{\text{disutility from fear of} \\ \text{prosecution for injury}}} \end{aligned}$$

and

$$\begin{aligned} & \varphi_2(v) - \varphi_2\left(v|_{N\backslash 1}\right) \\ = & \underbrace{-c}_{\substack{\text{money given} \\ \text{to robber}}} - \underbrace{-i}_{\text{disutility from injury}} \end{aligned}$$

The equality between these two differences can now be used to calculate the money I will have to hand over to the robber. It is given by

$$c = \frac{i - f}{2}.$$

The less the robber's fear of prosecution for injury and the higher my unwillingness to suffer injury, the higher the robber's loot. For c to be nonnegative, we need $i \geq f$; my fear of injury has to be higher than the robber's fear of prosecution.

7.5. Revisiting Weber's definition of power. For the Shapley value, the threat of withdrawal from a cooperative agreement has to be symmetric between the two players. In the gloves game, this symmetry determines the price of gloves; in the emotional-dependence example it leads to her doing the washing up; and in the case of robbery, the robber's gain obtains.

Of course, the holder of the non-scarce commodity would prefer a fair price of $\frac{1}{2}$, the dependent woman would like to share the burden of housework evenly, and the victim of robbery would prefer to hold on to his money. However, the holder of the scarce commodity, the man in the dependency example and the robber manage to "realize their own will ... against the resistance" of the other party. We just cited Max Weber in order to indicate that we consider these three examples instances of power in his sense.

In fact, a research program suggests itself: Whenever we have a seemingly asymmetric power-over relationship we should look out for Weberian power by equalizing the payoff differences with respect to the threat of withdrawal. For example, power-over relationships may exist between parents and children, God and humans, a king and his subjects, a bureaucrat and people obtaining permission, master and slave, etc.. Which actions lead to balanced contributions?

8. The Banzhaf solution

8.1. The Banzhaf formula. The Banzhaft solution is due to Banzhaf (1965) who applied it to weighted majority games. The Banzhaf formula is given by

$$Ba_i(v) = \frac{1}{2^{n-1}} \sum_{\substack{K \subseteq N, \\ i \notin K}} [v(K \cup \{i\}) - v(K)], i \in N.$$

Similar to the Shapley value, an average of marginal contributions is calculated. However, while Shapley considers all rank orders, Banzhaf proposes to look at all coalitions which (do not) contain a given player i. We can find

$$\left|2^{N \backslash \{i\}}\right| = 2^{|N \backslash \{i\}|} = 2^{n-1}$$

of these coalitions.

Thus, under the Shapley value, every rank order has the same probability while the Banzhaf index attributes the same probability for each coalition that contains a specfic player.

Exercise VI.2. *Given* $N = \{1, 2, 3\}$, *write down the coalitions that do not contain player* i.

The Banzhaf formula can be applied to any game but the main field of application concerns simple games. Then, the Banzhaf formula is also called Banzhaft power index or Banzhaf index.

Restricting attention to simple games, we can focus on pivotal coalitions. We remind the reader of the definition found in chapter IV:

Definition VI.7 (pivotal coalition). *For a simple game* v, $K \subseteq N$ *is a pivotal coalition for* $i \in N$ *if* $v(K) = 0$ *and* $v(K \cup \{i\}) = 1$. *The number of* i*'s pivotal coalitions is denoted by* $\eta_i(v)$,

$$\eta_i(v) := |\{K \subseteq N : v(K) = 0 \text{ and } v(K \cup \{i\}) = 1\}|.$$

We have $\eta(v) := (\eta_1(v), ..., \eta_n(v))$ *and* $\bar{\eta}(v) := \sum_{i \in N} \eta_i(v)$. *We sometimes omit the game and write* η_i $(\eta, \bar{\eta})$ *rather than* $\eta_i(v)$ $(\eta(v), \bar{\eta}(v))$.

Thus, a player i is pivotal for a coalition K if $v(K) = 0$ and $v(K \cup \{i\}) = 1$ hold. Player i's number of pivotal coalitions is denoted by $\eta_i(v)$ (or η_i).

Exercise VI.3. *Find* η_i *for a null player and for a dictator.*

Now, the Banzhaf index for player i can be rewritten as

$$Ba_i(v) = \frac{\eta_i}{2^{n-1}}.$$

Exercise VI.4. *Calculate the Banzhaf payoffs for player* 1 *in case of* $N = \{1, 2, 3\}$ *and* $u_{\{1,2\}}$. *What do you find for* $N = \{1, 2, 3, 4\}$ *and* $u_{\{1,2,3\}}$*?*

EXERCISE VI.5. *Find the Banzhaf payoffs for* $N = \{1, 2, 3, 4\}$ *and the apex game* h_1 *defined by*

$$h_1(K) = \begin{cases} 1, & 1 \in K \text{ and } K \backslash \{1\} \neq \emptyset \\ 1, & K = N \backslash \{1\} \\ 0, & \text{sonst} \end{cases}$$

Does the Banzhaf solution fulfill Pareto efficiency?

8.2. The Banzhaf axiomatization. While the Banzhaf index violates Pareto efficiency in general, it always fulfills the other three Shapley axioms. Indeed, the following theorem can be shown:

THEOREM VI.4 (axiomatization of the Banzhaf value). *The Banzhaf formula is axiomatized by null-player axiom, the symmetry axiom, the marginalism axiom and the merging axiom.*

You know all these axioms except the merging axiom. It means that if you merge two players into one player, then this new player obtains the sum of what the two constituent players got.

DEFINITION VI.8 (merging players). *For a game* (N, v) *and two players* $i, j \in N$, $i \neq j$, *the merged game* (N_{ij}, v_{ij}) *is given by* $N_{ij} = (N \backslash \{i, j\}) \cup \{ij\}$ *and*

$$v_{ij}(K) = \begin{cases} v(K), & K \subseteq N \backslash \{ij\} \\ v((K \backslash \{ij\}) \cup \{i, j\}), & ij \in K \end{cases}$$

for all $K \subseteq N_{ij}$.

DEFINITION VI.9 (merging axiom). *A solution function* σ *is said to obey the merging axiom if we have*

$$\sigma_i(v) + \sigma_j(v) = \sigma_{ij}(N_{ij}, v_{ij})$$

for any merged game in the sense of the definition above.

Consider the gloves game $v_{\{1,2\},\{3\}}$. Its Shapley payoffs are $Sh\left(v_{\{1,2\},\{3\}}\right) = \left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right)$ while the Banzhaf formula yields $Ba\left(v_{\{1,2\},\{3\}}\right) = \left(\frac{1}{4}, \frac{1}{4}, \frac{3}{4}\right)$.

Let us now assume that players 1 and 2 merge. The new player 12 obtains the Shapley payoff $\frac{1}{2} > \frac{1}{6} + \frac{1}{6}$. Intuitively, he players 1 and 2 (from the same market side) do not compete against each other any more so that their joint payoff increases while player 3 suffers. In contrast the Banzhaf payoffs are $\frac{1}{2}$ for both 12 and 3. In line with the merging axiom, we have $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. However, player 3's payoff reduces so that there is some indication of decreased competition between the left-hand glove owners even for this value.

If players 2 and 3 merge, the new player 23 is a dictator with Shapley value 1 and Banzhaf value 1. Again, the Banzhaf value obeys the merging axiom while the Shapley value does not.

9. Topics and literature

The main topics in this chapter are

- axiomatization
- balanced contributions
- marginalism
- power-over
- P-power and I-power

We introduce the following mathematical concepts and theorems:

- t

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We recommend

10. Solutions

Exercise VI.1

The set of all three axioms is contradictory. Just consider the unanimity game $u_{\{1\}}$ for $N = \{1, 2\}$. According to the third axiom, we should have $\sigma_2\left(u_{\{1\}}\right) = 0$, while the second axom then yields $\sigma_1\left(u_{\{1\}}\right) = 1 - \sigma_2\left(u_{\{1\}}\right) = 1$. However, the first axiom claims $\sigma_1\left(u_{\{1\}}\right) = \sigma_2\left(u_{\{1\}}\right)$.

Both solution concepts fulfill axioms 2. Using the same unanimity game as above, the first solution concept yields the payoffs $\sigma_1\left(u_{\{1\}}\right) = \sigma_2\left(u_{\{1\}}\right) = \frac{1}{2}$ while the rank order $\rho = (1, 2)$ leads to the rank-order value $(1, 0)$.

Exercise VI.2

Player 1 does not belong to four coalitions: $\emptyset, \{2\}, \{3\}, \{2, 3\}$.

Exercise VI.3

For a null player, we find $\eta_i = 0$, while $\eta_i = 2^{n-1}$ characterizes a dictator.

Exercise VI.4

Player 1 has the two pivotal coalitions, $\{2\}$ and $\{3\}$. Therefore, his Banzhaf index is $\frac{2}{4} = \frac{1}{2}$.

Exercise VI.5

For player 1, every coalition is pivatal except $\emptyset$ and $\{2, 3, 4\}$. Therefore, we find $Ba_1(h_1) = \frac{6}{8} = \frac{3}{4}$.

Player 2's pivatal coalitions are $\{1\}$ and $\{2, 3\}$ and he therefore obtains $Ba_2(h_1) = \frac{2}{8} = \frac{1}{4}$. By symmetry, we obtain $Ba_3(h_1) = Ba_4(h_1) = \frac{1}{4}$. Therefore, the sum of Banzhaf payoffs exceeds the worth of the grand coalition:

$$\frac{3}{4} + 3 \cdot \frac{1}{4} = \frac{3}{2} > 1 = h_1(N).$$

The Banzhaf index is not Pareto efficient.

11. Further exercises without solutions

(including Banzhaf)
The Shapley value on partitions

Part C

The Shapley value on partitions

In the second part of our book, we introduce the Shapley value and other simple solution concepts. In this third part, we now get to more complicated problems where the players are structured in some way or other. We assume that players split up in disjunct groups called components (of a partition). Components might stand for groups of people that

- work together and create worth, for example people trading goods with each other or people working in firms (chapter VII) or
- bargain together where unions are the prime example (chapter VIII).

In chapter IX, we combine both sorts of partition. The fact that workers belong to a firm is expressed by a working-together partiton while a second partition stands for the union that a worker may or may not belong to.

CHAPTER VII

The outside option values

1. Introduction

Let us reconsider the Shapley value and the core for the gloves game. The core represents the competitive solution where the holders of the scarce commodity (the right-glove owners in case of $|R| < |L|$) obtain a payoff of 1. This result holds for $|L| = 100$ and $|R| = 99$ as well as for $|L| = 100$ and $|R| = 1$. The following table reports the core payoffs for an owner of a right glove in a market with r right-glove owners and l left-glove owners:

		number l of left-glove owners				
		0	1	2	3	4
number r	1	0	$\in [0,1]$	1	1	1
of	2	0	0	$\in [0,1]$	1	1
right-glove	3	0	0	0	$\in [0,1]$	1
owners	4	0	0	0	0	$\in [0,1]$

Shapley & Shubik (1969, p. 342) denounce the "violent discontinuity exhibited by ... the core".

In contrast, the Shapley value is sensitive to the relative scarcity of the gloves. The following table, taken from Shapley & Shubik (1969, S. 344), tells the Shapley values for the right-glove owner, again depending on the number of right and left gloves:

		number l of left-glove owners				
		0	1	2	3	4
number r	1	0	0,500	**0,667**	0,750	0,800
of	2	0	0,167	0,500	0,650	**0,733**
right-glove	3	0	0,083	0,233	0,500	0,638
owners	4	0	0,050	0,133	0,271	0,500

This table clearly shows how the payoff increases with the number of players on the other market side. Shapley & Shubik (1969, p. 344) show that the Shapley value of the gloves game converges to the core: When replicating the game (i.e., increasing the number of left and right gloves by way of multiplication), the Shapley values converge toward 0 or 1 in case of $l \neq r$ (for $l = r$ we get a core payoff $\frac{1}{2}$). Consider, for example, $r = 1$ and $l = 2$ (bold face) and then, by using the factor 2, $r = 2$ and $l = 4$. You see that the payoff for the scarce-resource holder increases. The convergence can also be seen from the following table:

replication factor	$n = 3, r = 1$	$n = 4, r = 1$
1	0.6666...	0.75
10	0.8822...	0.9407...
100	0.9816...	0.9927...

Note that the Shapley value attributes a positive value to all players unless $|L| = 0$ holds or $|R| = 0$. However, in case of $|L| > |R|$, some left-glove owners will not be able to strike a deal. They should then get a pay-off of zero. Therefore, the Shapley value is an ex-ante value, indicating the expected payoff to an agent in the gloves game before it is clear whether or not he will find a trading partner.

In this chapter, we are interested in an ex-post value that should give us an idea about the payoff for glove holders once they have, or have not, found a trading partner. In particular, this value could be used to make predictions about the price of a left (or right) glove. While the Shapley value does not attempt to predict a price, the values presented in this chapter are candidates for that purpose.

The trading-partner distribution can be modelled by coalition structures. A coalition structure is a partition on the set of players; the sets making up the partition are called components. Building on the Shapley value, several partitional values (or values for coalition structures) have been presented in the literature, most notably by Aumann & Drèze (1974) and Owen (1977). There is an important interpretational difference between the Aumann-Dreze (AD) value and the Owen value. For Aumann and Dreze, players are organized in (active) components in order to do business together. Then the players within each component should arguably get its worth, as in the Aumann-Dreze value (AD-value). This is the property of component efficiency. The idea of the Owen value is that players form bargaining components (unions etc.) that offer the service of all their members or no service at all. In this chapter, we have the Aumann-Dreze interpretation in mind. The Owen value is the subject matter of the next chapter.

By component efficiency, the AD-value seems a good candidate for predicting the price of a left glove. Of course, we have to specify a partition before we can apply the AD-value. Turning to the gloves game, we often assume maximal-pairs partitions. These are partitions that host $\min(|L|, |R|)$ components, each containing one left-glove holder and one right-glove owner. If $|L| > |R|$, a maximal-pairs partition contains other components as well, with elements from L only. A left-glove and a right-glove owner who make up one component of the partition, receive an AD-value of $1/2$ each, irrespective of how many other left-hand or right-hand gloves are present.

The AD-payoffs do not accord well with our intuition about competition. More specifically, they do not take account of outside options, i.e. the number of left and right gloves outside the component in question. The outside-option value (oo-value, for short) W due to Wiese (2007) and the outside-option value Ca introduced by Casajus (2009) are component-efficient value that produce results that are more sensitive to the relative scarcity of gloves. Assume player set $N = \{1, 2, 3\}$ and the gloves game $v_{\{1\},\{2,3\}}$. Now let $\mathcal{P} = \{\{1, 2\}, \{3\}\}$ be a maximal-pairs partition. We find

$$\begin{aligned} AD\left(v_{\{1\},\{2,3\}}, \mathcal{P}\right) &= \left(\frac{1}{2}, \frac{1}{2}, 0\right), \\ W\left(v_{\{1\},\{2,3\}}, \mathcal{P}\right) &= \left(\frac{2}{3}, \frac{1}{3}, 0\right), \\ Ca\left(v_{\{1\},\{2,3\}}\right) &= \left(\frac{3}{4}, \frac{1}{4}, 0\right) \end{aligned}$$

The oo-values attributes a higher payoff to player 1 than to player 2 thus reflecting the outside opportunities of player 1 ($v(\{1, 3\}) = 1 > 0 = v(\{2, 3\})$).

In spirit, the bargaining set (a concept we will not go into) is close to the outside-option values. (In the above example, the bargaining set yields $(0, 60, 0)$, a somewhat "extreme" solution.) In fact, I find Maschler's (1992, pp. 595) introducing remarks pertinent to these value:

> During the course of negotiations there comes a moment when a certain coalition structure is "crystallized". The players will no longer listen to "outsiders", yet each [component] has still to adjust the final share of proceeds. (This decision may depend on options outside the [component], even though the chances of defection are slim).

Arguably, there are many economic and political situations where we need these properties. Apart from market games (as the gloves game), one might think of the power within a government coalition. This power rests with the parties involved (component efficiency) but the power of each party within the government depends on other governments that might possibly form (outside options).

Close to the AD-approach, the oo-values obey component efficiency, symmetry and additivity. However, we argue that these values cannot possibly obey the null-player axiom. Consider $N = \{1, 2, 3\}$ and the unanimity game $u_{\{1,2\}}$ which maps the worth 1 to coalitions $\{1, 2\}$ and $\{1, 2, 3\}$ and the worth 0 to all other coalitions. We now look at the coalition structure $\mathcal{P}_1 = \{\{1, 3\}, \{2\}\}$. By component efficiency, we get $\sigma_1^{oo}\left(u_{\{1,2\}}, \mathcal{P}_1\right) + \sigma_3^{oo}\left(u_{\{1,2\}}, \mathcal{P}_1\right) = 0 = \sigma_2^{oo}\left(u_{\{1,2\}}, \mathcal{P}_1\right)$. Player 3 is a null player; his contribution to any coalition is zero. Yet, his payoff cannot be zero under σ^{oo}. The reason is this: Player 1 has outside options. By joining forces with player

2 (thus violating the existing coalition structure) he would have claim to a payoff of 1/2. Within the existing coalition structure, he will turn to player 3 to satisfy at least part of this claim. But then, player 3's payoff is negative.

Most solution concepts found in the literature do obey the null-player axiom. A noticeable excepts is the solidarity value concocted by Nowak & Radzik (1994). Consider the unanimity game $N = \{1, 2, 3\}$ and the unanimity game $u_{\{1,2\}}$. The two productive players do not obtain $\frac{1}{2}$ (their Shapley value but only $\frac{7}{18}$; they leave $\frac{4}{18}$ for null player 3, for charity reasons.

It should also be clear that a component-efficient value that respects outside options cannot always coincide with the value for some "stable" partition. In our example, stable partitions might be given by $\mathcal{P}_2 = \{\{1, 2\}, \{3\}\}$ or $\mathcal{P}_3 = \{\{1, 2, 3\}\}$. By component efficiency the sum of payoffs for all three players is zero for $\mathcal{P}_1$ but 1 for $\mathcal{P}_2$ and $\mathcal{P}_3$.

Some readers might object to a negative payoff for player 3 by pointing to the possibility that player 3 departs from coalition $\{1, 3\}$ to obtain the zero payoff. However, for the purpose of determining the outside-option value, the coalition structure $\mathcal{P}$ is given. The stability of $\mathcal{P}$ is another – separate – issue that we will with in subsection **??**. Also, it is easy to show that negative payoffs need not bother us if we consider the gloves game and a maximal-pairs partitions.

It has been noted that the oo-values are close the AD-value and the Shapley value. Indeed, they are generalizations of both these values.

This chapter is organized as follows: In section 2 basic definitions (partitions, partitional games) are given. Section 3 presents important axioms for partitional values. We briefly introduce the Aumann-Dreze value in section 4 before presenting the outside-option values due to Wiese (with an application to the gloves game) and due to Casajus (with an application to the elections in Germany for the Bundestag 2009) in sections 5 and 6, respectively. We discuss the differences between these values in section 7. In the final section 8, we turn to the question of stable partitions.

2. Solution functions for partitional games

2.1. Partitions. Partitioning a set means to define subsets such that every element from the set is an element from exactly one subset. Consider the set $\{1, 2, 3, 4\}$.

$$\{\{1, 2\}, \{3\}, \{4\}\}$$

is an example of a partition of that set while

$$\begin{aligned} &\{\{1, 2\}, \{4\}\} \text{ or} \\ &\{\{1, 2\}, \{2, 3\}, \{4\}\} \end{aligned}$$

are not.

DEFINITION VII.1 (partition). *Let N be a set (of players). A system of subsets*

$$\mathcal{P} = \{C_1, ..., C_k\}$$

is called a partition if

- $C_j = N$,
- $C_j \cap C_{j'} = \emptyset$ *for all* $j \neq j'$ *from* $\{1, ..., k\}$ *and*
- $C_j \neq \emptyset$ *for all* $j = 1, ..., k$

hold. The subsets $C_j \subseteq N$ are called components.

The set of all partitions on N is denoted by $\mathfrak{P}_N$. The component hosting player i is denoted by $\mathcal{P}(i)$.

Sometimes, we need to compare partitions.

DEFINITION VII.2. *A partition $\mathcal{P}_1$ is called finer than a partition $\mathcal{P}_2$ if $\mathcal{P}_1(i) \subseteq \mathcal{P}_2(i)$ holds for all $i \in N$. In that case, $\mathcal{P}_2$ is called coarser than $\mathcal{P}_1$. The finest partition is called the atomic partition and given by $\{\{1\}, ..., \{n\}\}$. The coarsest partition is called the trivial partition and equal to $\{N\}$.*

EXERCISE VII.1. *Is $\mathcal{P}_1$ finer or coarser than $\mathcal{P}_2$?*

(1) $\mathcal{P}_1 = \mathcal{P}_2 = \{\{1, 2\}, \{3, 4\}, \{5\}\}$,
(2) $\mathcal{P}_1 = \{\{1, 2\}, \{3, 4\}, \{5\}\}$, $\mathcal{P}_2 = \{\{1, 2, 3\}, \{4, 5\}\}$,
(3) $\mathcal{P}_1 = \{\{1, 2\}, \{3, 4\}, \{5\}\}$, $\mathcal{P}_2 = \{\{1, 2\}, \{3\}, \{4\}, \{5\}\}$.

2.2. Partitional games. We are now set to define partitional games.

DEFINITION VII.3 (partitional game). *For any player set N, every coalition function $v \in \mathbb{V}_N$ and any partition $\mathcal{P} \in \mathfrak{P}_N$, $(v, \mathcal{P})$ is called a partitional game. The set of all partitional games on N is denoted by $\mathbb{V}_N^{part}$ and the set of all partitional games for all player sets N by $\mathbb{V}^{part}$.*

We need to extend the definition of a solution function:

DEFINITION VII.4 (solution function for partitional games). *A function σ that attributes, for each partitional game $(v, \mathcal{P})$, a payoff to each of v's players,*

$$\sigma(v, \mathcal{P}) \in \mathbb{R}^{|N(v)|},$$

is called a solution function (on $\mathbb{V}^{part}$).

3. Important axioms for partitional values

Solution functions σ on $(N, \mathfrak{P}_N)$ might obey one or several of the following axioms. We concentrate on the axioms that we make use of in this chapter. We encounter additional ones in the next chapter.

Definition VII.5 (component-efficiency axiom). *A solution function σ (on $\mathbb{V}^{part}$) is said to obey the component-efficiency axiom if*

$$\sum_{i\in C}\sigma_i(v,\mathcal{P})=v(C)$$

holds for all partitional games $(v,\mathcal{P})\in\mathbb{V}^{part}$ and all $C\in\mathcal{P}$.

Component effciency is a natural requirement for partitions if we have the "work together and create worth" interpretation in mind. The next axiom says that a player's payoff depends on his component only, not on the way the players outside his component are structured.

Definition VII.6 (component-focus axiom). *Consider a player $i\in N$ and two partitions $\mathcal{P}$ and $\mathcal{P}'$ from $\mathfrak{P}_N$ that obey $\mathcal{P}(i)=\mathcal{P}'(i)$. A solution function σ (on $\mathbb{V}^{part}$) is said to obey the component-focus axiom if*

$$\sigma_i(v,\mathcal{P})=\sigma_i\left(v,\mathcal{P}'\right)$$

holds for all coalition functions $v\in\mathbb{V}_N$.

Exercise VII.2. *Does the axiomof component focus imply*

$$\sigma_i(v,\mathcal{P})=\sigma_i(v,\{\mathcal{P}(i),N\backslash\mathcal{P}(i)\})?$$

While the axiomatizations to follow do not make use of the component-focus axiom, the three new values in this chapter (AD-value, the Casajus and the Wiese value) fulfill it.

We need a symmetry axiom where symmetry has to refer to the coalition function and to the partition.

Definition VII.7 ($\mathcal{P}$-symmetry). *Two players i and j from N are called $\mathcal{P}$-symmetric if they symmetric and if $\mathcal{P}(i)=\mathcal{P}(j)$ holds.*

Definition VII.8 (symmetry axiom). *A solution function σ is said to obey the symmetry axiom if we have*

$$\sigma_i(v,\mathcal{P})=\sigma_j(v,\mathcal{P})$$

for all partitional games $(v,\mathcal{P})\in\mathbb{V}^{part}$ and for any two $\mathcal{P}$-symmetric players i and j.

As argued above, a component-efficient value that takes outside options into account, cannot possibly satisfy the null-player axiom:

Definition VII.9 (null-player axiom). *A solution function σ is said to obey the null-player axiom if we have*

$$\sigma_i(v,\mathcal{P})=0$$

for all partitional games $(v,\mathcal{P})\in\mathbb{V}^{part}$ and for every null player $i\in N$.

A much milder requirement is the grand-coalition null-player axiom introduced by Casajus (2009):

DEFINITION VII.10 (grand-coalition null-player axiom). *A solution function σ is said to obey the grand-coalition null-player axiom if we have*

$$\sigma_i(v, \{N\}) = 0$$

for all partitional games $(v, \{N\}) \in \mathbb{V}^{part}$ and for every null player $i \in N$.

Of course, we also have an additivity axiom:

DEFINITION VII.11 (additivity axiom). *A solution function σ is said to obey the additivity axiom if we have*

$$\sigma(v + w, \mathcal{P}) = \sigma(v, \mathcal{P}) + \sigma(w, \mathcal{P})$$

for any two coalition functions $v, w \in \mathbb{V}$ with $N(v) = N(w)$ and any partition $\mathcal{P} \in \mathfrak{P}_{N(v)}$.

4. The Aumann-Dreze value: formula and axiomatization

Once we know how to calculate the Shapley value, it is simple to obtain the Aumann-Dreze payoffs. Just proceed in two steps:

(1) Restrict the coalition function to the components.
(2) Calculate the Shapley value for the restricted function.

DEFINITION VII.12 (Aumann-Dreze value). *The Aumann-Dreze value on $\mathbb{V}^{part}$ is the solution function AD given by*

$$AD_i(v, \mathcal{P}) := Sh_i\left(v|_{\mathcal{P}(i)}\right), i \in N$$

The Aumann-Dreze value is an obvious extension of the Shapley value:

LEMMA VII.1. *We have $AD(v, \{N\}) = Sh(v)$.*

EXERCISE VII.3. *Calculate the Aumann-Dreze payoffs for $\mathcal{P} = \{\{1\}, \{2, 3\}\}$ and the coalition functions*

- $u_{\{1,2\}}$ *and*
- $v_{\{1,2\},\{3\}}$.

The axiomatization for the Aumann-Dreze value is very close to the Shapley axiomatization:

THEOREM VII.1 (Aumann-Dreze axiomatization). *The Aumann-Dreze value is the unique solution function on $\mathbb{V}^{part}$ that fulfills the symmetry axiom, the component-efficiency axiom, the null-player axiom and the additivity axiom.*

The Aumann-Dreze value rests on the premise that every component is an island. There are not interlinkages between players in a component and those outside.

5. The outside-option value due to Wiese

5.1. Definition and properties. The Wiese outside-option value uses a rank-order definition. Assume a partition $\mathcal{P}$, a rank order ρ and a player i. Player i belongs to the component $\mathcal{P}(i)$ and also to the set $K_i(\rho)$. If player i appears, is he the last player of his component, i.e., have all the other players from $\mathcal{P}(i)$ appeared before him? Formally, this is true if and only if

$$\mathcal{P}(i) \subseteq K_i(\rho)$$

holds.

Exercise VII.4. *Indicate the players that complete their components for the partition* $\mathcal{P} = \{\{1,2,3\},\{4,5\},\{6\}\}$ *and the rank order* $\rho = (3,5,6,1,2,4)$*!*

While the Aumann-Dreze value ignores any effect of players outside a component on those inside, the outside-option values model these effects. The Wiese (2007) value has an interpretation in terms of rank orders.

Definition VII.13 (Wiese value). *The Wiese value on* $\mathbb{V}^{part}$ *is the solution function* W *given by*

$$W_i(v,\mathcal{P}) := \frac{1}{n!} \sum_{\rho \in RO_N} \begin{cases} v(\mathcal{P}(i)) - \sum_{j \in \mathcal{P}(i)\setminus\{i\}} MC_j(v,\rho), & \mathcal{P}(i) \subseteq K_i(\rho), \\ MC_i(v,\rho), & \textit{otherwise,} \end{cases}$$

The reader notes that player i's payoff does not depend on the partition $\mathcal{P}$ in general, but only on $\mathcal{P}(i)$. In looking at a rank order ρ, player i gets her marginal contribution $MC_i(v,\rho)$ if she is not the last player in her component in ρ, i.e., if $\mathcal{P}(i)$ is not included in $K_i(\rho)$. If i is the last player in her component, she gets the worth of this component minus the payoffs (marginal contributions $MC_j(v,\rho)$) to the other players in her component.

The above formula lends itself to an interpretation very close to the one given for the Shapley value. For both formulae, we consider that all players arrive in a random order. For the Shapley value, the player's receive their marginal contribution with respect to the players arriving before them. In our formula, matters are a bit more complicated. For every rank order ρ, exactly one player i from $\mathcal{P}(i)$ is not followed by other players from her component. The other players from $\mathcal{P}(i)\setminus\{i\}$ get their marginal contributions as in the Shapley case. This marginal contribution will not always concern players from $\mathcal{P}(i)$ exclusively. Some of the players in $K_j(\rho)$, $j \in \mathcal{P}(i)\setminus\{i\}$, may well be outside $\mathcal{P}(i) = \mathcal{P}(j)$ so that outside options are taken into account. Player i, who is the last player in her component, obtains the worth of her component net of the marginal contributions awarded to the other players in her component.

The construction makes clear that the Wiese value is component efficient. Since the axiomatization for this is not very nice, we confine ourselves to state some important properties.

THEOREM VII.2 (properties of the Wiese value). *The Wiese value obeys the symmetry axiom, the component-efficiency axiom, the grand-coalition null-player axiom and the additivity axiom. It violates the null-player axiom.*

The Wiese value is a generalization of the Shapley value in two senses:

LEMMA VII.2. *We have* $W(v, \{N\}) = Sh(v)$.

For a proof, consider the trivial partition $\mathcal{P} = \{N\}$ and a player $i \in N$. Note that $N = \mathcal{P}(i)$ is a subset of $K_i(\rho)$ only if i is the last player in ρ. In that case, we have $v(\mathcal{P}(i)) - \sum_{j \in \mathcal{P}(i) \setminus \{i\}} MC_j(v, \rho) = MC_i(v, \rho)$ by (component) efficiency.

LEMMA VII.3. *Let* v *be a simple game and* $\mathbb{W}(v)$ *its set of winning coalitions. Let there be a veto player* $i_{veto} \in N$, *i.e.,* $i_{veto} \in W$ *for all* $W \in \mathbb{W}(v)$. *Let* $\mathcal{P}$ *be a partition of* N *such that* $\mathcal{P}(i_{veto}) \in \mathbb{W}(v)$. *Then,* $W_{i_{veto}}(v, \mathcal{P}) = Sh_i(v)$.

We do not provide a proof, but invite the reader to consult Wiese (2007).

5.2. Application: the gloves game.

5.2.1. *Every player holds one glove, only.* The Wiese value for a right-glove owner whose component also contains a left-glove owner is given in the following table:

		no. of left-glove holders				
		0	1	2	3	4
no. of	1	0	0.500	0.667	0.750	0.800
right-	2	0	0.333	0.500	0.633	0.717
glove	3	0	0.250	0.367	0.500	0.614
holders	4	0	0.200	0.283	0.386	0.500

It seems clear that the value is an ex-post value while retaining the sensitivity to the relative scarcity. Thus, if a right-glove owner manages to sell his glove, he can expect the price given in that table. The reader may also note that in case of one right-glove owner, only, this agent obtains the Shapley value, in accordance with lemma VII.3.

In private communication, Joachim Rosenmüller conjectured that the outside-option value of the gloves game converges to the core. (After all, the Shapley value does.) The following examples corroborate this conjecture:

replication factor	$n = 3, r = 1$	$n = 4, r = 1$
1	0.6666...	0.75
10	0.8531...	0.9278...
100	0.9734...	0.9904...

As yet, a proof has not been found.

5.2.2. *The generalized gloves game.* Exercise IV.16 (p. 58) alerts us to the fact that burning gloves may be a profitable strategy if payoffs are evaluated with the core.

Consider the situation of farmers. They may well benefit from a bad harvest that hits all of them. However, we might be surprised to find a single farmer who benefits from a bad harvest striking himself only but not the other farmers. In this sense the core exhibits an extreme outcome.

It is clear that a Shapley-payoff recipient will never burn a glove. After all, his marginal benefit can never increase by such an action. How does the Wiese value fare in that respect?

Let us now consider the endowment economy (see the general definition on p. 60)

$$\mathcal{E} = \left(N, \{L, R\}, \left(\omega_L^i, \omega_R^i\right)_{i\in N}, \min\right)$$

where player $i \in N$ has ω_L^i left and ω_R^i right gloves. The corresponding endowment coalition function is defined by

$$v^{\mathcal{E}}(K) = \min\left(\sum_{i\in K}\omega_L^i, \sum_{i\in K}\omega_R^i\right).$$

For example, let $\mathcal{E}$ be specified by

$$\begin{aligned}
\omega_L^1 &= 1, \omega_R^1 = 0,\\
\omega_L^2 &= 2, \omega_R^2 = 0,\\
\omega_L^3 &= 1, \omega_R^3 = 0,\\
\omega_L^4 &= 0, \omega_R^4 = 1,\\
\omega_L^5 &= 0, \omega_R^5 = 1,\\
\omega_L^6 &= 0, \omega_R^6 = 1.
\end{aligned}$$

This game is obviously very close to $v_{\{1,2,3\},\{4,5,6\}}$. Player 2 holds two gloves while all the other players hold one glove each, with players 1 to 3 holding left gloves and players 4 to 6 holding right gloves. For the maximal-pairs partition

$$\mathcal{P} = \{\{1,4\}, \{2,5\}, \{3,6\}\}$$

we obtain the Wiese payoff

$$\left\{\frac{5}{12}, \frac{31}{60}, \frac{5}{12}, \frac{7}{12}, \frac{29}{60}, \frac{7}{12}\right\}$$

while the Wiese payoff is

$$\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}$$

for the gloves game $v_{\{1,2,3\},\{4,5,6\}}$.

We have three observations. First, player 2 benefits from her additional endowment although her component's worth is 1 in both cases. Second, by component efficiency, player 5 suffers from the increased endowment of player 2. Third, players 4 and 6 who hold right gloves, benefit from the increase in left gloves. These observations can be generalized:

PROPOSITION VII.1. *Let ω and $\hat{\omega}$ be two endowments and i, j $(i \neq j)$ two players from N. Let $\omega^k = \hat{\omega}^k$ for all $k \neq i$, $\omega_R^i = \hat{\omega}_R^i$ and $\omega_L^i < \hat{\omega}_L^i$. We denote the corresponding endowment games by v_ω and $v_{\hat{\omega}}$, respectively. For any partition $\mathcal{P}$, we get*

- $$W_i\left(v_\omega, \mathcal{P}\right) \leq W_i\left(v_{\hat{\omega}}, \mathcal{P}\right),$$
- *if* $\mathcal{P}(i) = \{i, j\}$ *and* $\omega_L^i + \omega_L^j \geq \omega_R^i + \omega_R^j$,
$$W_j\left(v_\omega, \mathcal{P}\right) \geq W_j\left(v_{\hat{\omega}}, \mathcal{P}\right),$$
- *if* $\mathcal{P}(j) \neq \mathcal{P}(i)$, $\omega_R^j \geq \omega_R^k$, *and* $\omega_L^j \leq \omega_L^k$ *for all* $k \in \mathcal{P}(j)$,
$$W_j\left(v_\omega, \mathcal{P}\right) \leq W_j\left(v_{\hat{\omega}}, \mathcal{P}\right),$$

The first assertion states that a player whose endowment is increased (player 2 in the above example) can never be hurt by this increase. This result is in contrast to results for the core where a player may benefit from burning a glove. The second assertion is a direct conclusion from the first, together with component efficiency. The third generalizes the observation about players 4 and 6 above: Since player j holds less left gloves and more right gloves than any other player in his component, he will benefit more from a higher endowment of left gloves outside his component than the other players in his component. For a proof, consult the working paper "The outside-option value - axiomatization and application to the gloves game" on the webpage http://www.uni-leipzig.de/~micro/wopap.html.

6. The outside-option value due to Casajus

6.1. The splitting axiom. The splitting axiom is the central axiom for the outside-option value concocted by Casajus (2009):

DEFINITION VII.14 (splitting axiom). *Consider two partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $\mathcal{P}_1$ is finer than $\mathcal{P}_2$. If two players i and j belong to the same component of the finer partition ($j \in \mathcal{P}_1(i)$), we have*

$$\sigma_i\left(v, \mathcal{P}_2\right) - \sigma_i\left(v, \mathcal{P}_1\right) = \sigma_j\left(v, \mathcal{P}_2\right) - \sigma_j\left(v, \mathcal{P}_1\right)$$

for all partitional games $(v, \mathcal{P}) \in \mathbb{V}^{part}$.

Casajus makes a good case for this axiom: "Splitting a structural coalition affects all players who remain in the same structural coalition in the same way. As the value is already meant to reflect the outside options of the players, one could argue that the gains/losses of splitting/separating should be distributed equally within a resulting structural coalition."

We come back to the splitting axiom later.

6.2. Axiomatization of the Casajus value. The Casajus value does not, as far as we know, admit a rank-order definition. Instead it builds on the Shapley values in the most simple fashion:

DEFINITION VII.15 (Casajus value). *The Casajus value on* $\mathbb{V}^{part}$ *is the solution function* Ca *given by*

$$Ca_i(v) := Sh_i(v) + \frac{v(\mathcal{P}(i)) - \sum_{j\in\mathcal{P}(i)} Sh_j(v)}{|\mathcal{P}(i)|}$$

According to this value, the players obtain the Shapley value which then has to be made component-efficient. If the sum of the Shapley values in a component happens to equal the component's worth, the Casajus value equals the Shapley value. If the sum of a component's Shapley values exceed the component's worth, the difference, averaged over all the players in the component, has to be "paid" by every player.

THEOREM VII.3 (axiomatization of Casajus value). *The Casajus formula is axiomatized by the symmetry axiom, the component-efficiency axiom, the grand-coalition null-player axiom, the additivity axiom and the splitting axiom.*

EXERCISE VII.5. *Determine the Casajus value for* $N = \{1, 2, 3\}$ *and the unanimity game* $u_{\{1,2\}}$. *Consider both* $\mathcal{P} = \{\{1, 3\}, \{2\}\}$ *and* $\mathcal{P} = \{\{1, 2\}, \{3\}\}$.

6.3. Application: elections in Germany for the Bundestag 2009.

6.3.1. *Political parties.* In 2009, 27 parties were present in one or several or all of the 16 German Länder. Among these, we find

- SPD – Sozialdemokratische Partei Deutschlands (16 lists)
- CDU – Christlich Demokratische Union Deutschlands (15 lists – not in Bavaria)
- FDP – Freie Demokratische Partei (16 lists)
- DIE LINKE – Die Linke (16 lists)
- GRÜNE – Bündnis 90/Die Grünen (16 lists)
- CSU – Christlich-Soziale Union in Bayern (1 list only – Bavaria)
- NPD – Nationaldemokratische Partei Deutschlands (16 lists)
- MLPD – Marxistisch-Leninistische Partei Deutschlands (16 lists)
- PIRATEN – Piratenpartei Deutschland (15 lists, not in Saxony)
- DVU – Deutsche Volksunion (12 lists)
- REP – Die Republikaner (11 lists)
- ödp – Ökologisch-Demokratische Partei (8 lists)
- BüSo – Bürgerrechtsbewegung Solidarität (7 lists)
- Die Tierschutzpartei – Mensch Umwelt Tierschutz (6 lists)

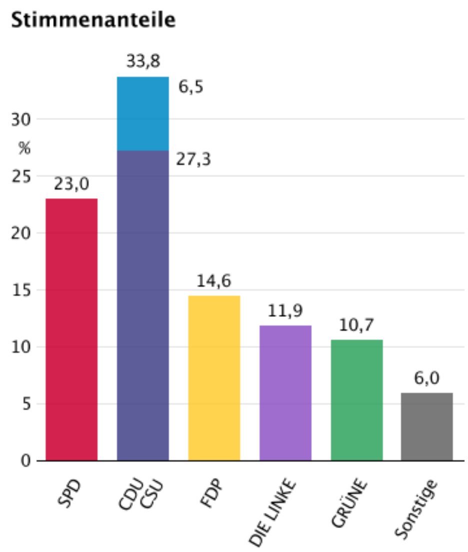

FIGURE 1

6.3.2. *Results.* The election for the 17$^{\text{th}}$ German Bundestag took playe on September, 27$^{\text{th}}$, 2009 and brought forth some extreme results:

- The participation rate (70.78%) was the lowest ever recorded in the Federal Republic of Germany.
- The Christian democrats and the liberals collected the number of votes necessary to form a government coalition.
- The liberals, the lefts and the greens obtained the best results in their party histories.
- The parties of the ruling grand coalition (Christian democrats, social democrtes) lost in big way:
  - The social democrats witnessed their worst result in any election for the Bundestag.
  - The Christian democrats saw their worst election result since 1949.

The vote distribution can be seen from the following table:

The vote distribution leads to the seat distribution seen in the following diagram:

6.3.3. *Coalitions functions and actual political outcome.* Which parties can form government coalitions? The Christian democrats and the liberales ruled out a coalition with the leftist party. So did Frank-Walter Steinmeier on behalf of the social democrats.

The liberals excluded a coalition with the greens and the social democrats (traffic-light coalition: red - yellow - green). The green party excluded the Jamaica coalition (black - yellow - green).

We suggest to consider three assumptions:

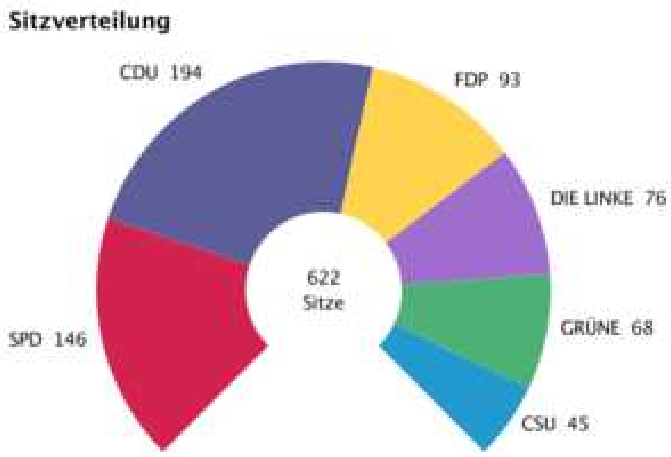

FIGURE 2

- assumption 1: Black - yellow and black - red are possible coalitions, only.
- assumption 2: Apart from the two coalitions mentioned in assumption 1, red - yellow -green and black - yellow - green are also possible
- assumption 3: All government coalitions are feasible except that the left party will not be seen in a coalition with the christian democrats or the liberals.

Thus, we have three different coalition functions:

Under assumption 1, we find the coalition function

$$v(K) = \begin{cases} 1, & \text{CDU} \in K, \text{SPD} \in K \\ 1, & \text{CDU} \in K, \text{FDP} \in K \\ 0, & \text{otherwise} \end{cases}$$

with the Shapley payoffs

$$\text{Sh}_{\text{CDU}} = \frac{2}{3}, \text{Sh}_{\text{SPD}} = \frac{1}{6}, \text{Sh}_{\text{FDP}} = \frac{1}{6},$$

the Casajus payoffs for the black - yellow coalition

$$\chi_{\text{CDU}} = \frac{3}{4}, \chi_{\text{SPD}} = 0, \chi_{\text{FDP}} = \frac{1}{4}$$

and the Casajus payoffs for the black - red coalition

$$\chi_{\text{CDU}} = \frac{3}{4}, \chi_{\text{SPD}} = \frac{1}{4}, \chi_{\text{FDP}} = 0$$

Taking the seat distribution into account, assumptions 2 and 3 do not change the above coalition function:

- The green party is a null player within a Jamaica (black - yellow - green) coalition.

- The traffic-light (red - yellow - green) coalition does not avail of 50% of the seats in the Bundestag.

Therefore, the promises made by the liberals and greens proved not to be expensive ex-post.

The actual government coalition has the Christian democrats form a government coalition with the liberal party. The actual distribution of ministries taken over by these parties approximates the Casajus values. 11 portfolios are in the hands of CDU/CSU and 5 in the hands of the liberals with $\frac{5}{16}$ being slightly above $\frac{4}{16} = \frac{1}{4}$.

6.3.4. *Coalitions functions and the Sonntagsfrage.* German demographers regularly ask potential voters about their actual inclinations. On February, 19$^{\text{th}}$, 2010, a few months after the 2009 elections, Infratest dimap reported these results:

	distribution of votes	... of seats
SPD	27	28
CDU	34	36
Left	10	10
FDP	10	10
Green	15	16

After the Oskar Lafontaine (a very prominent member of the left party and a former social democrat disliked by many social democrats) withdraws from politics, some social democrats are ready to review their willingness to form a coalition with the left party.

Therefore, one might reconsider assumption 3 from above. We now obtain the coalition function

$$v(K) = \begin{cases} 1, & \text{CDU} \in K, \text{SPD} \in K \\ 1, & \text{CDU} \in K, \text{Green} \in K \\ 1, & \text{SPD} \in K, \text{Green} \in K, \text{FDP} \in K \\ 1, & \text{SPD} \in K, \text{Green} \in K, \text{Left} \in K \\ 0, & \text{otherwise} \end{cases}$$

the Shapley payoffs

$$\text{Sh}_{\text{CDU}} = \frac{22}{60}, \text{Sh}_{\text{SPD}} = \frac{17}{60}, \text{Sh}_{\text{FDP}} = \frac{2}{60}, \text{Sh}_{\text{Linke}} = \frac{2}{60}, \text{Sh}_{\text{Green}} = \frac{17}{60}$$

and the Casajus payoffs

- for the grand coalition:

$$\chi_{\text{CDU}} = \frac{39}{72}, \chi_{\text{SPD}} = \frac{33}{72},$$

- for the black-green coalition:

$$\chi_{\text{CDU}} = \frac{39}{72}, \chi_{\text{Green}} = \frac{33}{72},$$

- for the black-green-liberal coalition:
$$\chi_{\text{SDP}} = \frac{30}{72}, \chi_{\text{Green}} = \frac{30}{72}, \chi_{\text{FDP}} = \frac{12}{72}$$
- for the red-red-green coalition:
$$\chi_{\text{SDP}} = \frac{30}{72}, \chi_{\text{Green}} = \frac{30}{72}, \chi_{\text{Left}} = \frac{12}{72}$$
- and, finally, for the Jamaica coalition
$$\chi_{\text{CDU}} = \frac{34}{72}, \chi_{\text{Green}} = \frac{28}{72}, \chi_{\text{FDP}} = \frac{10}{72}.$$

Thus, the Christian democrats are free to choose the social democrats or the green party as a coalition partner. Both have no better alternative than to go along.

7. Contrasting the Casajus and the Wiese values

7.1. The splitting axiom. We try to find out under what circumstances the Wiese value violates the splitting axiom. Consider the game on $N = \{1, 2, 3\}$ partly given by

$$\begin{aligned} v(i) &= 0, i = 1, 2, 3, \\ v(N) &= 1. \end{aligned}$$

The Shapley values for players 1 and 2 are

$$\begin{aligned} W_1(v, \{N\}) &= Sh_1(v) = \frac{2 + v(1,2) + v(1,3) - 2v(2,3)}{6}, \\ W_2(v, \{N\}) &= Sh_2(v) = \frac{2 + v(1,2) + v(2,3) - 2v(1,3)}{6} \end{aligned}$$

Consider the grand coalition $N = \{1, 2, 3\}$ and assume that players 1 and 2 split off. Then we obtain the partition

$$\mathcal{P} = \{\{1, 2\}, \{3\}\}$$

and the Wiese payoffs

$$\begin{aligned} W_1(v, \mathcal{P}) &= \frac{-2 + 2v(1,2) + v(2,3)}{6}, \\ W_2(v, \mathcal{P}) &= \frac{-2 + 2v(1,2) + v(1,3)}{6}. \end{aligned}$$

The splitting axiom claims that players 1 and 2 should benefit (or be hurt) equally. It holds for the Casajus value where we find

$$\begin{aligned} Ca_1(v, \{N\}) - Ca_1(v, \mathcal{P}) &= Sh_1(v) - \left(Sh_1(v) + \frac{v(\{1,2\}) - Sh_1(v) - Sh_2(v)}{2}\right) \\ &= Sh_2(v) - \left(Sh_2(v) + \frac{v(\{1,2\}) - Sh_1(v) - Sh_2(v)}{2}\right) \\ &= Ca_2(v, \{N\}) - Ca_2(v, \mathcal{P}) \end{aligned}$$

The splitting axiom is not fulfilled by the Wiese value. In fact, we have

$$W_1(v, \{N\}) - W_1(v, \mathcal{P}) < W_2(v, \{N\}) - W_2(v, \mathcal{P})$$

if an only if

$$v(1,3) - v(3) < v(2,3) - v(3)$$

holds. Thus, splitting away from player 3 hurts player 1 less than player 2 iff player 1's marginal contribution with respect to player 3 is less than player 2's marginal contribution.

One could argue that this is quite a sensible outcome. Assume that the above inequality holds, i.e., player 2's marginal contribution with respect to player 3 is higher than player 1's contribution. The splitting axiom used for the Casajus value implies that player 1 has to pay damages to player 2 so that both are harmed equally. In the final analysis, the question seems to be whether outside options are as important as inside opportunities. The Casajus value says "yes" while the Wiese value says "not quite".

7.2. Why make the last player the residual claimant? Noting that the Wiese value makes the last player in a component the residual claimant, Casajus (2009, p. 56) asks why not take the first or any other position. Indeed, let us define a series of values W^k for $k = 0, 1, ..., |\mathcal{P}(i)| - 1$ by

$$W_i^k(v, \mathcal{P}) = \frac{1}{n!} \sum_{\rho \in RO_N} \begin{cases} v(\mathcal{P}(i)) - \sum_{j \in \mathcal{P}(i) \backslash \{i\}} MC_j(v, \rho), & |\mathcal{P}(i) \backslash K_i(\rho)| = k, \\ MC_i(v, \rho), & \text{otherwise}, \end{cases}$$

The have $W = W^0$. Generalizating lemma VII.2 (p. 109), we have $W^k(v, \{N\}) = Sh$ for $k \in \{0, 1, ..., |\mathcal{P}(i)| - 1\}$.

Let us do the same exercise as in the previous subsection, this time for W^1. We find

$$\begin{aligned} 6W_1^1(v, \mathcal{P}) &= v(1,2) - MC_2(v, (1,2,3)) + v(1,2) - MC_2(v, (1,3,2)) \\ &\quad + MC_1(v, (2,1,3)) + MC_1(v, (2,3,1)) \\ &\quad + v(1,2) - MC_2(v, (3,1,2)) \\ &\quad + MC_1(v, (3,2,1)) \\ &= v(1,2) - [v(1,2) - v(1)] + v(1,2) - [v(1,2,3) - v(1,3)] \\ &\quad + [v(1,2) - v(2)] + [v(1,2,3) - v(2,3)] \\ &\quad + v(1,2) - [v(1,2,3) - v(1,3)] \\ &\quad + [v(1,2,3) - v(2,3)] \\ &= 3v(1,2) + 2v(1,3) - 2v(2,3) \end{aligned}$$

and hence

$$\begin{aligned} W_1^1(v,\mathcal{P}) &= \frac{3v(1,2)+2v(1,3)-2v(2,3)}{6} \text{ and} \\ W_2^1(v,\mathcal{P}) &= \frac{3v(1,2)-2v(1,3)+2v(2,3)}{6} \end{aligned}$$

by component efficiency.

We now get

$$W_1^1(v,\{N\}) - W_1^1(v,\mathcal{P}) < W_2^1(v,\{N\}) - W_2^1(v,\mathcal{P})$$

if an only if

$$v(2,3) - v(3) < v(1,3) - v(3)$$

holds. Thus, splitting away from player 3 hurts player 1 less than player 2 iff (and although) player 1's marginal contribution with respect to player 3 is larger than player 2's marginal contribution. Thus, we have the opposite result as in the previous section.

8. Attacking the stability problem

In the introduction, we consider the the player set $N = \{1,2,3\}$, the unanimity game $u_{\{1,2\}}$, and the coalition structure $\mathcal{P} = \{\{1,3\},\{2\}\}$. We argue that outside-option values attribute a negative payoff to player 3. Indeed, we find

$$\begin{aligned} Ca_3\left(u_{\{1,2\}},\mathcal{P}\right) &= Sh_3\left(u_{\{1,2\}}\right) + \frac{u_{\{1,2\}}(\{1,3\}) - Sh_1\left(u_{\{1,2\}}\right) - Sh_3\left(u_{\{1,2\}}\right)}{|\{1,3\}|} \\ &= 0 + \frac{0-\frac{1}{2}-0}{2} = -\frac{1}{4} \end{aligned}$$

and

$$W_3\left(u_{\{1,2\}},\mathcal{P}\right) = \frac{1}{6}\left(\underbrace{0-0}_{(1,2,3)} + \underbrace{0-0}_{(1,3,2)} + \underbrace{0-\frac{1}{2}}_{(2,1,3)} + \underbrace{0}_{(2,3,1)} + \underbrace{0}_{(3,1,2)} + \underbrace{0}_{(3,2,1)}\right) = -\frac{1}{12}.$$

Of course, player 3 will not be happy with this payoff. After all, she can obtain the payoff zero by forming a component by herself. In other words, the above partition is not stable.

Adapting the definition proposed by Hart & Kurz (1983), we define stability in the following manner:

Definition VII.16. *A coalition structure $\mathcal{P}$ is stable for a solution function σ (on $\mathbb{V}^{part}$) if there is no coalition L such that all players from L profit from forming a component, i.e. if for all $L \neq \emptyset$ we have*

$$\sigma_i(v,\mathcal{P}) \geq \sigma_i(v,\{L, N\backslash L\}) \text{ for some } i \in L,$$

or, equivalently, if for all $L \neq \emptyset$ we have

$$\sigma_i(v,\mathcal{P}) \geq \sigma_i(v, N\backslash L) \text{ for some } i \in L.$$

Our definition is simpler than the orignal one. The reason is that the outside option value for a player in component C is not influenced by how the players outside C are partitioned (this is the component-focus axiom).

EXERCISE VII.6. *Consider the apex game* h_1 *for* $N = \{1, .., 4\}$. *Find the Casajus values and examine the stability of the partitions* $\mathcal{P}' = \{N\}$, $\mathcal{P}'' = \{\{1\}, \{2, 3, 4\}\}$, $\mathcal{P}''' = \{\{1, 2\}, \{3, 4\}\}$, *and* $\mathcal{P}'''' = \{\{1, 2, 3\}, \{4\}\}$.

We will present a few results on stability, which are due to Casajus (2009, p. 56), Tutic (2010 (?)), and Wiese (2007):

THEOREM VII.4. *The following stability results can be shown:*

- *Stable coalition structures for the Casajus outside-option value exist for all coalition functions.*
- *Stable coalition structures for the Wiese outside-option value do not exist for all coalition functions.*
- *Stable coalition structures for the Wiese outside-option value exist for all symmetric and for all convex games.*
- *Any stable partition* $\mathcal{P}$ *fulfills* $W_i(v, \mathcal{P}) \geq v(\{i\})$ *for all players* $i \in N$.
- *Consider a weighted majority game* $[g_1, ..., g_n]$ *and the corresponding coalition function* $v \in \mathbb{V}_N$. *Let* i_{veto} *be a veto player, i.e., let* $g_{i_{veto}} \geq \sum_{i \in N, i \neq i_{veto}} g_i$. *If* $\mathcal{P}(i_{veto})$ *is a winning coalition,* $\mathcal{P}$ *is stable.*

9. Topics and literature

The main topics in this chapter are

- outside-option values
- Casajus value
- Wiese value
- component efficiency
- splitting axiom

We introduce the following mathematical concepts and theorems:

- t

-

We recommend.

10. Solutions

Exercise VII.1

We find:

(1) $\mathcal{P}_1$ is both finer and coarser than $\mathcal{P}_2$.
(2) $\mathcal{P}_1$ is neither finer nor coarser than $\mathcal{P}_2$.
(3) $\mathcal{P}_1$ is coarser than $\mathcal{P}_2$, but not finer.

Exercise VII.2

Yes.

Exercise VII.3

We have $u_{\{1,2\}}(1) = v_{\{1,2\},\{3\}}(1) = 0$ and hence $AD_1\left(u_{\{1,2\}}, \mathcal{P}\right) = AD_1\left(v_{\{1,2\},\{3\}}, \mathcal{P}\right) = 0$. For the unanimity game, we find $AD_3\left(u_{\{1,2\}}, \mathcal{P}\right) = 0$ for null player 3 and $AD_2\left(u_{\{1,2\}}, \mathcal{P}\right) = u_{\{1,2\}}(\{2,3\}) - 0 = 0$ by component efficiency. Turning to the gloves game, we obtain

$$\begin{aligned} AD_2\left(v_{\{1,2\},\{3\}}, \mathcal{P}\right) &= Sh_2\left(v_{\{1,2\},\{3\}}\big|_{\{2,3\}}\right) \\ &= Sh_2\left(v_{\{2\},\{3\}}\right) \\ &= \frac{1}{2} \\ &= AD_3\left(v_{\{1,2\},\{3\}}, \mathcal{P}\right). \end{aligned}$$

Exercise VII.4

The players 6, 2 and 4 complete their components.

Exercise VII.5

The Shapley value for the unanimity game $u_{\{1,2\}}$ is $Sh\left(u_{\{1,2\}}\right) = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$ so that we get player 1's Casajus value

$$Ca_1\left(u_{\{1,2\}}, \{\{1,3\}, \{2\}\}\right) = \frac{1}{2} + \frac{0 - \left(\frac{1}{2} + 0\right)}{2} = \frac{1}{4}.$$

The other players' payoffs can be obtained by component efficiency. Finally, we have

$$Ca\left(u_{\{1,2\}}, \{\{1,3\}, \{2\}\}\right) = \left(\frac{1}{4}, 0, -\frac{1}{4}\right).$$

For the other partition, we find

$$Ca\left(u_{\{1,2\}}, \{\{1,2\}, \{3\}\}\right) = \left(\frac{1}{2}, \frac{1}{2}, 0\right).$$

For example, you could have applied component efficiency to player 3 and then $\mathcal{P}$-symmetry to the other two players.

Exercise VII.6

The Casajus values are

$$\begin{aligned}
Ca\left(h_1, \mathcal{P}'\right) &= \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right), \\
Ca\left(h_1, \mathcal{P}''\right) &= \left(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \\
Ca\left(h_1, \mathcal{P}'''\right) &= \left(\frac{2}{3}, \frac{1}{3}, 0, 0\right), \text{ and} \\
Ca\left(h_1, \mathcal{P}''''\right) &= \left(\frac{5}{9}, \frac{2}{9}, \frac{2}{9}, 0\right).
\end{aligned}$$

Up to symmetry, these four partitions are the only serious candidates for stable partitions. $\mathcal{P}'$ is not stable because the coalitions $\{1,2\}$ or $\{2,3,4\}$ can profitably deviate. $\mathcal{P}''''$ is not stable because of $\mathcal{P}'''$. $\mathcal{P}''$ and $\mathcal{P}'''$ are stable.

11. Further exercises without solutions

(1) Assume two men, Max (M) and Onno (O), who both love Ada (A). Their coalition function is given

$$v(K) = \begin{cases} 0, & |K| \leq 1 \\ 6, & K = \{M, A\} \\ 4, & K = \{O, A\} \\ 1, & K = \{M, O\} \\ 2, & K = \{M, O, A\} \end{cases}$$

- Calculate the AD payoffs and the outside options values due both to Casajus and Wiese for the partition $\mathcal{P} = \{\{M, A\}, \{O\}\}$!
- Comment!

(2) A capitalist employs two workers 1 and 2. The firm's coalition function is given by $N = \{K, 1, 2\}$ and

$$\begin{aligned} v(\{K\}) &= 10, \\ v(\{1\}) &= v(\{2\}) = v(\{1,2\}) = 0, \\ v(\{K,1\}) &= v(\{K,2\}) = 16, \\ v(N) &= 19 \end{aligned}$$

Find the players' payoffs by applying suitable solution concepts for
- full employment,
- partial employment (worker 2 is fired).

Comment!

(3) Consider the player set $N = \{m, w1, w2\}$ where m stands for a man and $w1$ and $w2$ for two women. The government's viewpoint on marriages, homosexual marriages and polygamy is expressed by the coalition functions v given by

$$\begin{aligned} v(\{m\}) &= v(\{w1\}) = v(\{w2\}) = 0, \\ v(\{m, w1\}) &= v(\{m, w2\}) = 5, \\ v(\{w1, w2\}) &= 3, \\ v(N) &= -2. \end{aligned}$$

- Is v monotonic, superadditive or essential?
- Which solution concept would you like to apply? How about the
  - core
  - the Shapley value,
  - the AD-value,
  - the outside-option value (due to either Casajus or Wiese)?

(4) Using the axioms, derive the Shapley payoffs and the AD-payoffs for the coalition function given by $N = \{1, 2, 3, 4\}$ and

$$v(K) = \begin{cases} 0, & K \in \{\{1\}, \{2\}, \{3\}\} \\ 10, & K \in \{\{4\}, \{1,4\}, \{2,4\}, \{3,4\}\} \\ 60, & K \in \{\{1,2\}, \{1,3\}, \{2,3\}\} \\ 72, & K = \{1,2,3\} \\ 70, & K \in \{\{1,2,4\}, \{1,3,4\}, \{2,3,4\}\} \\ 82, & K = N \end{cases}$$

and the partition $\mathcal{P} = \{\{1,2,3\}, \{4\}\}$!

CHAPTER VIII

The union value

1. Introduction

The components in this chapter are bargaining groups. They players in such a component put their aggregate contributions in the balance. A priori, it is unclear whether that is a good idea. For example, German citizens form a component within the European Union. It seems that the average German stands a smaller chance of becoming a EU commissioner than an Irish person.

- We find that the productive players in a unanimity game profit when they dissociate themselves from other productive players.
- Left-glove owners may benefit from forming a cartel of left-glove holders.

The main idea behind the Owen, or union, value is this. We consider two games. First, the components play against each other leading to some aggregate payoff for each of them. Second, within each component, the players bargain about their share of the component's aggregate payoff.

We proceed as follows. In the next section, we explain how some rank orders are not consistent with some partitions. We present the union value in section 3 and its axiomatizations in section 4. Examples in section 5 conclude the chapter.

The Owen value is a generalization of the Shapley value. This will become obvious for the trivial partition $\mathcal{P} = \{N\}$ (one bargaining block containing all players) and for the atomic partition $\mathcal{P} = \{\{1\}, \{2\}, ..., \{n\}\}$ (every player bargains for himself). In section 6, we show that the Shapley value can be obtained as the mean of Owen values for different partitions.

2. Partitions and rank orders

Before presenting the union value, we need to do some preparatory groundwork. First of all, we remind the reader of definition VIII.1 (p. 123): For a component $\mathcal{P}$ of the player set N, the component containing player $i \in N$ is denoted by $\mathcal{P}(i) \in 2^N$. Second, we need to define $\mathcal{P}(R)$ for a player set $R \subseteq N$.

Definition VIII.1 (subpartition). *Let $\mathcal{P} = \{C_1, ..., C_k\}$ be a partition of N. Partition $\mathcal{P}_1$ is called a subpartition of $\mathcal{P}_2$ if $\mathcal{P}_1 \subseteq \mathcal{P}_2$ holds. The set* *of components containing any player from $R \subseteq N$ is given by*

$$\mathcal{P}(R) := \{C \subseteq N : \exists i \in R \text{ such that } C = \mathcal{P}(i)\}$$

According to the above definition, we have $C \cap R \neq 0 \Leftrightarrow C \in \mathcal{P}(R)$ for all $C \in \mathcal{P}$. Differently put, $\mathcal{P}(R)$ is a subpartition of $\mathcal{P}$ (i.e., $\mathcal{P}(R)$ contains nothing but components from $\mathcal{P}$) and the smallest subpartition that places all players from R in components. We get from a partition $\mathcal{P}$ to $\mathcal{P}(R)$ by deleting those components that do not contain R-players.

Exercise VIII.1. *Express $\mathcal{P}(T)$ and $\mathcal{P}(i) \cap T$ in your own words.*

Definition VIII.2 (union of components). *Let $\mathcal{P} = \{C_1, ..., C_k\}$ be a partition of N. We denote the union of R-components by*

$$\bigcup \mathcal{P}(R) := \bigcup_{i \in R} \mathcal{P}(i).$$

Thus, $\mathcal{P}(R)$ is a set of subsets of N while $\bigcup \mathcal{P}(R)$ is a subset of N. Alternatively, $\bigcup \mathcal{P}(R)$ is the set with partition $\mathcal{P}(R)$.

Exercise VIII.2. *Consider $\mathcal{P} = \{\{1\}, \{2\}, \{3,4\}, \{5,6,7\}\}$ and find $\mathcal{P}(\{2,5\})$ and $\bigcup \mathcal{P}(\{2,5\})$.*

Do you see that $\mathcal{P}(i)$ is a subset of N while $\mathcal{P}(\{i\})$ is the set that contains $\mathcal{P}(i)$, $\mathcal{P}(\{i\}) = \{\mathcal{P}(i)\}$? Also, $\mathcal{P}(R)$ is a subpartition of $\mathcal{P}$ while $\mathcal{P}(i)$ is not. Do not worry your head off if you do not understand. In any case, have a close look at the following exercise.

Exercise VIII.3. *Determine $\mathcal{P}(2)$, $\mathcal{P}(\{2,3\})$, $\mathcal{P}(\{2\})$ and $\mathcal{P}(N \backslash \{2,3\})$ for $N = \{1, ..., 4\}$ and the partitions*

- $\mathcal{P} = \{\{1\}, \{2\}, \{3,4\}\}$ *and*
- $\mathcal{P} = \{\{1\}, \{2,3\}, \{4\}\}$!

Are any of the resulting expression partitions?

We now turn to the final and most important bit of formal language. For a given partition $\mathcal{P} \in \mathfrak{P}_N$, we want to consider those rank orders $\rho \in RO_n$ that leave the players of each component together. Consider, for example, the partition $\mathcal{P} = \{\{1\}, \{2\}, \{3,4\}\}$. The rank order $\rho = (3,1,2,4)$ tears the component $\{3,4\}$ apart while the rank order $\rho = (3,4,1,2)$ does not.

Definition VIII.3 (consistent rank orders). *A rank order $\rho \in RO_n$ is called consistent with a partition $\mathcal{P} \in \mathfrak{P}_N$, if, for every component C from $\mathcal{P}$, there exist an index j and a number $\ell \in \{0, ..., n-j\}$ such that*

$$C = \{\rho_j, \rho_{j+1}, ..., \rho_{j+\ell}\}$$

holds. The set of all rank oders on N that are consistent with a partition $\mathcal{P}$ are denoted by $RO_n^{\mathcal{P}}$ or $RO^{\mathcal{P}}$.

The $RO_n^{\mathcal{P}}$ is contained in the set RO_n. Starting with RO_n, we get to $RO_n^{\mathcal{P}}n$ by deleting those rank oders that tear apart players belonging to the same component.

EXERCISE VIII.4. *Which of the following rank oders are consistent with the partition* $\mathcal{P} = \{\{1\}, \{2\}, \{3,4\}, \{5,6,7\}\}$*?*

- $\rho = (1,2,3,4,5,6,7)$
- $\rho = (2,1,4,5,6,7,3)$
- $\rho = (1,5,2,3,4,6,7)$
- $\rho = (1,4,3,7,5,6,2)$

EXERCISE VIII.5. *Which rank orders from* RO_7 *are consistent with*

- $\mathcal{P} = \{\{1,2,3,4,5,6,7\}\}$ *or*
- $\mathcal{P} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}\}$?

You certainly remember

$$|RO_n| = n!$$

We derive this formula on p. 42. How many rank orders are consistent with a partition

$$\mathcal{P} = \{S_1, ..., S_k\}?$$

Note.

- We have $k!$ possibilities to rank the components S_1 through S_k.
- Within component S_j, there are $|S_j|!$ possibilities to rank its players.

Thus, we find

$$\left|RO_n^{\mathcal{P}}\right| = k! \cdot |S_1|! \cdot ... \cdot |S_k|!$$

and hence a second reason why $\left|RO_n^{\{\{1,2,...,n\}\}}\right| = \left|RO_n^{\{\{1\},\{2\},...,\{n\}\}}\right|$ (see exercise VIII.5) holds.

3. Union-value formula

The union partition stands for groups of players who put their aggregate marginal contribution into the balance.

DEFINITION VIII.4 (Owen value). *The Owen value on* $\mathbb{V}^{part}$ *is the solution function* Ow *given by*

$$Ow_i(v, \mathcal{P}) = \frac{1}{|RO_n^{\mathcal{P}}|} \sum_{\rho \in RO_n^{\mathcal{P}}} [v(K_i(\rho)) - v(K_i(\rho))], i \in N.$$

Thus, in contrast to the Shapley value, we consider the rank orders that are consistent with the partition $\mathcal{P}$, only, rather than all rank orders.

Let us consider the player set $N = \{1,2,3\}$, the gloves game $v_{\{1,2\},\{3\}}$. Right gloves are scarce and the Shapley payoffs are $\left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right)$. Let us now assume that the left-glove owners form a cartel so that we are dealing with the partition $\mathcal{P} = \{\{1,2\},\{3\}\}$. We have four rank orders consistent with $\mathcal{P}$:

$$(1,2,3),(2,1,3),(3,1,2) \text{ and } (3,2,1).$$

Thus, we obtain the Owen payoffs

$$\begin{aligned}
Ow_1\left(v_{\{1,2\},\{3\}},\mathcal{P}\right) &= \frac{1}{4}\left(\underbrace{0}_{(1,2,3)} + \underbrace{\diagup}_{(1,3,2)} + \underbrace{0}_{(2,1,3)} + \underbrace{\diagup}_{(2,3,1)} + \underbrace{1}_{(3,1,2)} + \underbrace{0}_{(3,2,1)}\right) = \frac{1}{4}, \\
Ow_2\left(v_{\{1,2\},\{3\}},\mathcal{P}\right) &= \frac{1}{4}\left(\underbrace{0}_{(1,2,3)} + \underbrace{\diagup}_{(1,3,2)} + \underbrace{0}_{(2,1,3)} + \underbrace{\diagup}_{(2,3,1)} + \underbrace{0}_{(3,1,2)} + \underbrace{1}_{(3,2,1)}\right) = \frac{1}{4}, \\
Ow_3\left(v_{\{1,2\},\{3\}},\mathcal{P}\right) &= \frac{1}{4}\left(\underbrace{1}_{(1,2,3)} + \underbrace{\diagup}_{(1,3,2)} + \underbrace{1}_{(2,1,3)} + \underbrace{\diagup}_{(2,3,1)} + \underbrace{0}_{(3,1,2)} + \underbrace{0}_{(3,2,1)}\right) = \frac{2}{4}.
\end{aligned}$$

In this case, unionization pays.

Do you see that $\mathcal{P} = \{\{1,2,...,n\}\}$ and $\mathcal{P} = \{\{1\},\{2\},...,\{n\}\}$ lead to the same Owen values?

4. Axiomatization

The Owen value is a solution function σ on $(N,\mathfrak{P}_N)$ that obeys

- the efficiency axiom,
- the symmetry axiom (payoff equality for $\mathcal{P}$-symmetric players),
- the null-player axiom, and
- the additivity axioms.

These axioms do not suffice to pin down the Owen value. We introduce additional axioms which need some preparation. The symmetry axiom for components claims that symmetric components should obtain the same aggregate payoff. Thus, this axiom is well in line with the two games underlying the Owen value, the game between components first and the game within components second.

Definition VIII.5 (component symmetry). *Consider a partition $\mathcal{P} \in \mathfrak{P}_N$. Two components C and C' from $\mathcal{P}$ are called symmetric if*

$$v\left(\bigcup \mathcal{P}(K) \cup C\right) = v\left(\bigcup \mathcal{P}(K) \cup C'\right)$$

holds for all $K \subseteq N \backslash (C \cup C')$.

Definition VIII.6 (symmetry axiom for components). *A solution function (on $\mathbb{V}^{part}$) σ is said to obey symmetry between components if*

$$\sigma_C(v,\mathcal{P}) = \sigma_{C'}(v,\mathcal{P})$$

holds for all symmetric components C and C' from $\mathcal{P}$.

Owen (1977) suggests a nice axiomatization:

THEOREM VIII.1 (Axiomatization of the Owen value). *The Owen formula is the unique solution function that fulfills the symmetry axiom, the symmetry axiom for components, the efficiency axiom, the null-player axiom and the additivity axiom.*

Let us revisit the gloves game $v_{\{1,2\},\{3\}}$ and the partition $\mathcal{P} = \{\{1,2\},\{3\}\}$ (see section 3). Both components are needed to produce the worth of 1. Therefore, the symmetry axiom for components yields

$$Ow_1\left(v_{\{1,2\},\{3\}},\mathcal{P}\right) + Ow_2\left(v_{\{1,2\},\{3\}},\mathcal{P}\right) = Ow_3\left(v_{\{1,2\},\{3\}},\mathcal{P}\right)$$

efficiency then leads to

$$\begin{aligned} Ow_3\left(v_{\{1,2\},\{3\}},\mathcal{P}\right) &= 1 - \left(Ow_1\left(v_{\{1,2\},\{3\}},\mathcal{P}\right) + Ow_2\left(v_{\{1,2\},\{3\}},\mathcal{P}\right)\right) \\ &= 1 - Ow_3\left(v_{\{1,2\},\{3\}},\mathcal{P}\right) \end{aligned}$$

and hence to $Ow_3\left(v_{\{1,2\},\{3\}},\mathcal{P}\right) = \frac{1}{2}$. Finally, the symmetry between players 1 and 2 produces $Ow_1\left(v_{\{1,2\},\{3\}},\mathcal{P}\right) = Ow_2\left(v_{\{1,2\},\{3\}},\mathcal{P}\right) = \frac{1}{4}$.

5. Examples

5.1. Unanimity games. We now develop a general formula for unanimity games. First of all, we disregard any component C with $C \subseteq N\backslash T$. These null components do not influence the payoffs. Thus, we focus on components that host at least one T-player and on the partition $\mathcal{P}(T)$. Each component in $\mathcal{P}(T)$ has the same probability $\frac{1}{|\mathcal{P}(T)|}$ to be the last component. Within each of these components, every $i \in T$ player has the same probability $\frac{1}{|\mathcal{P}(i)\cap T|}$ to complete T.

Thus, the Owen value yields the following payoffs for a unanimity game u_T, $T \neq \emptyset$:

$$Ow_i\left(u_T,\mathcal{P}\right) = \begin{cases} \frac{1}{|\mathcal{P}(T)|}\frac{1}{|\mathcal{P}(i)\cap T|}, & i \in T \\ 0, & \text{otherwise} \end{cases}$$

Every T-player obtains a positive payoff, even if not all T-players belong to a single component.

Assume that a player $i \in T$, for whom $|\mathcal{P}(i) \cap T| \geq 2$ holds, breaks off and forms a component all by himself. In that case,

- the number of T-components increases from $|\mathcal{P}(T)|$ to $|\mathcal{P}(T)| + 1$ while
- the number of T-players in i's component decreases from $|\mathcal{P}(i) \cap T| \geq 2$ to 1.

Then, his payoff weakly increases as can be seen from

$$\frac{1}{|\mathcal{P}(T)|}\frac{1}{|\mathcal{P}(i)\cap T|} \leq \frac{1}{|\mathcal{P}(T)|+1}\frac{1}{1}$$

which is equivalent to

$$\frac{|\mathcal{P}(T)|+1}{|\mathcal{P}(T)|} \leq |\mathcal{P}(i) \cap T|$$

where equality holds for $|\mathcal{P}(T)| = 1$ and $|\mathcal{P}(i) \cap T| = 2$, only.

5.2. Symmetric games. The Shapley values are identical for players in a symmetric game. The simple reason is that players are symmetric in a symmetric game. However, symmetric players may well not be $\mathcal{P}$-symmetric. Consider $N = \{1, 2, 3\}$, $\mathcal{P} = \{\{1, 2\}, \{3\}\}$ and the coalition function v given by

$$v(S) = \begin{cases} 0, & |S| \leq 1 \\ \alpha, & |S| = 2 \\ 1, & |S| = 3 \end{cases}$$

for any $\alpha \in \mathbb{R}$. To calculate player 1's Owen payoff, we consider the following table.

rank order	marginal contribution for player 1
1-2-3	0
2-1-3	α
3-1-2	α
3-2-1	$1-\alpha$
sum	$1+\alpha$
Owen payoff	$\frac{1+\alpha}{4}$

Since players 1 and 2 are $\mathcal{P}$-symmetric, we have $Ow_2(v, \mathcal{P}) = Ow_1(v, \mathcal{P}) = \frac{1+\alpha}{4}$. Efficiency yields

$$\begin{aligned} Ow_3(v, \mathcal{P}) &= 1 - Ow_1(v, \mathcal{P}) - Ow_2(v, \mathcal{P}) \\ &= 1 - 2 \cdot \frac{1+\alpha}{4} = \frac{1}{2} - \frac{1}{2}\alpha. \end{aligned}$$

Thus, we obtain $Ow_3(v, \mathcal{P}) \neq Ow_1(v, \mathcal{P})$ unless $\alpha = \frac{1}{3}$ happens to hold.

5.3. Apex games. Unionization does not pay for powerful players in a unanimity game. However, the weak players in an apex game win by forming a union.

EXERCISE VIII.6. *Find the Owen payoffs for the n-player apex game h_1 and the partition $\mathcal{P} = \{\{1\}, \{2, ..., n\}\}$.*

If the unimportant players form several components, the apex player obtains a positive payoff. For example, if the players 2 to n form two components, the apex player obtains the marginal payoff 1 in one out of three cases – therefore, we have $Ow_1(v, \mathcal{P}) = \frac{1}{3}$.

EXERCISE VIII.7. *Can you find a partition $\mathcal{P} = \{\{1\}, C_1, C_2\}$ such that a player $j \in \{2, ..., n\}$ obtains a higher payoff than $\frac{1}{n-1}$?*

6. The Shapley value is an average of Owen values

We plan to present a probabilistic generalization of the Owen value. Instead of looking at a particular partition, we assume a probability distribution on the set of all partitions.

6.1. Probability distribution. In this section, we introduce probability distributions on the set of partitions $\mathfrak{P}_N$. This important concept merits a proper definition, where $[0,1]$ is short for $\{x \in \mathbb{R} : 0 \leq x \leq 1\}$:

Definition VIII.7 (probability distribution). *Let M be a nonempty set. A probability distribution on M is a function*

$$prob : 2^M \to [0,1]$$

such that

- $prob(\emptyset) = 0$,
- $prob(A \cup B) = prob(A) + prob(B)$ *for all* $A, B \in 2^M$ *obeying* $A \cap B = \emptyset$ *and*
- $prob(M) = 1$.

Subsets of M are also called events. For $m \in M$, we often write $prob(m)$ rather than $prob(\{m\})$. If a $m \in M$ exists such that $prob(m) = 1$, $prob$ is called a trivial probability distribution and can be identified with m. We denote the set of all probability distributions on M by $Prob(M)$.

Exercise VIII.8. *Throw a fair dice. What is the probability for the event A, "the number of pips (spots) is 2", and the event B, "the number of pips is odd". Apply the definition to find the probability for the event "the number of pips is 1, 2, 3 or 5".*

Thus, a probability distribution associates a number between 0 and 1 to every subset of M. (This definition is okay for finite sets M but a problem can arise for sets with M that are infinite but not countably infinite. For example, in case of $M = [0,1]$, a probability cannot be defined for every subset of M, but for so-called measurable subsets only. However, it is not easy to find a subset of $[0,1]$ that is not measurable. Therefore, we do not discuss the concept of measurability.)

6.2. Symmetric probability distribution. We now consider probability distributions $prob$ on $M = \mathfrak{P}_N$. Following Casajus (2010), let us consider those probability distributions that are unaffected by the labeling of the players. We call these probability distributions "symmetric". For example, the probability distribution $prob$ on $\mathfrak{P}(\{1,2,3\})$ given by

$$prob(\{\{1,2\},\{3\}\}) = \frac{1}{2} = prob(\{\{1\},\{2,3\}\})$$

is not symmetric because of $prob(\{\{2\},\{1,3\}\}) = 0$. Also, defining $prob$ by

$$prob(\{\{1,2\},\{3\}\}) = 1$$

does not yield a symmetric probability distribution, again because of $prob(\{\{2\},\{1,3\}\}) = 0$.

In contrast, the probability distributions $prob_1$, $prob_2$, and $prob_3$ given by

$$\begin{aligned} prob_1(\{\{1,2\},\{3\}\}) &= prob_1(\{\{1\},\{2,3\}\}) = prob_1(\{\{2\},\{1,3\}\}) = \frac{1}{3}, \\ prob_2(\{\{1,2,...,n\}\}) &= 1, \text{ and} \\ prob_3(\{\{1\},\{2\},...,\{n\}\}) &= 1 \end{aligned}$$

are symmetric.

We now like to present the formal definition proposed by Casajus (2010). Consider a bijection $\pi : N \to N$. For example, for $N = \{1,2,3\}$, a bijection π is defined by

$$\begin{aligned} \pi(1) &= 3, \\ \pi(2) &= 1, \text{ and} \\ \pi(3) &= 2. \end{aligned}$$

For a partition $\mathcal{P}$, $\pi(\mathcal{P})$ is the partition $\{\pi(C) : C \in \mathcal{P}\}$.

EXERCISE VIII.9. *Let* $\mathcal{P} = \{\{1,2\},\{3\}\}$. *Find* $\pi(\mathcal{P})$ *for the above bijection* π*!*

DEFINITION VIII.8 (symmetric probability distribution). *Let prob be a probability distribution on* $\mathfrak{P}_N$. *prob is called symmetric if every bijection* $\pi : N \to N$ *yields*

$$prob(\mathcal{P}) = prob(\pi(\mathcal{P})).$$

Let us applying the definition to the probability distributions $prob_1$, $prob_2$, and $prob_3$ given above. $prob_1$ is symmetric because there exist three partitions with

- one player in a singleton component and
- the two other players sharing a component

and these three partitions have the same probability ($\frac{1}{3}$).

Do you see that $\pi(\{1,2,...,n\}) = \{1,2,...,n\}$ for every bijection π. Also, every partition π keeps the atomic partition intact.

6.3. The probabilistic Owen value.

DEFINITION VIII.9 (probabilistic Owen value). *The probabilistic Owen value on* $\mathbb{V}^{part}$ *is the solution function Ow given by*

$$Ow_i(v, prob) = \sum_{\mathcal{P} \in \mathfrak{P}_N} prob(\mathcal{P}) \, Ow_i(v, \mathcal{P}), i \in N,$$

where $prob \in Prob\,(\mathfrak{P}_N)$ *is a probability distribution on the set of partitions of* N*.*

Casajus (2010) shows the following result:

THEOREM VIII.2. *For any symmetric probability distribution prob on* $\mathfrak{P}_N$*, we have*

$$Ow\,(v, prob) = Sh\,(v)\,.$$

7. Topics and literature

The main topics in this chapter are

- union value

We introduce the following mathematical concepts and theorems:

- t

-

We recommend

8. Solutions

Exercise VIII.1

$\mathcal{P}(T)$ is the set of $\mathcal{P}$'s components that contain at least one T-player. $\mathcal{P}(i) \cap T$ is the set of T-players in i's component.

Exercise VIII.2

We have $\mathcal{P}(\{2,5\}) = \{\{2\}, \{5,6,7\}\}$ and $\bigcup \mathcal{P}(\{2,5\}) = \{2,5,6,7\}$.

Exercise VIII.3

For the first partition, we obtain $\mathcal{P}(2) = \{2\}, \mathcal{P}(\{2,3\}) = \{\{2\}, \{3,4\}\}$, $\mathcal{P}(\{2\}) = \{\{2\}\}$ and $\mathcal{P}(N\backslash\{2,3\}) = \{\{1\}, \{3,4\}\}$, the second partition yields $\mathcal{P}(2) = \{2,3\}, \mathcal{P}(\{2,3\}) = \{\{2,3\}\}, \mathcal{P}(\{2\}) = \{\{2,3\}\}$ and $\mathcal{P}(N\backslash\{2,3\}) = \{\{1\}, \{4\}\}$. $\mathcal{P}(\{2,3\})$, $\mathcal{P}(\{2\})$ and $\mathcal{P}(N\backslash\{2,3\})$ are subsets of the partitions and partitions in their own right, albeit of different sets.

Exercise VIII.4

The first and the last rank order are consistent with $\mathcal{P}$. The second rank order tears component $\{3,4\}$ apart and the third rank order does not leave the component $\{5,6,7\}$ intact.

Exercise VIII.5

For both partitions, we find $RO_n^{\mathcal{P}} = RO_n$.

Exercise VIII.6

The apex player's marginal payoff is zero if his one-man component is first and also if his component is last. Therefore, we have $Ow_1(h_1, \mathcal{P}) = 0$ and, by $\mathcal{P}$-symmetry $Ow_j(h_1, \mathcal{P}) = \frac{1}{n-1}$ for all players $j = 2, ..., n-1$.

Exercise VIII.7

If all unimportant players $j \in \{2, ..., n\}$ are gathered in one component, each of them obtains $\frac{1}{n-1}$. In partition $\mathcal{P} = \{\{1\}, C_1, C_2\}$, component C_1 gets the payoff $\frac{1}{3}$ (why?) which is also the payoff to some player j which is the only player in that component – $C_1 = \{j\}$. This player has a higher payoff than $\frac{1}{3}$ whenever n exceeds 4 :

$$\frac{1}{3} > \frac{1}{n-1} \Leftrightarrow n \geq 5.$$

Exercise VIII.8

We have $prob\,(A) = \frac{1}{6}$ and $prob\,(B) = \frac{1}{2}$ for the two events and, by $A \cap B = \emptyset$, $prob\,(A \cup B) = prob\,(A) + prob\,(B) = \frac{1}{6} + \frac{1}{2} = \frac{4}{6}$.

Exercise VIII.9

We have

$$\begin{aligned} \pi\,(\mathcal{P}) &= \{\pi\,(C) : C = \{1,2\} \text{ or } C = \{3\}\} \\ &= \{\pi\,(\{1,2\})\,, \pi\,(\{3\})\} \\ &= \{\{1,3\}\,, \{2\}\} \end{aligned}$$

9. Further exercises without solutions

(1) Assume two men, Max (M) and Onno (O), who both love Ada (A). Their coalition function is given

$$v\,(K) = \begin{cases} 0, & |K| \leq 1 \\ 6, & K = \{M, A\} \\ 4, & K = \{O, A\} \\ 1, & K = \{M, O\} \\ 2, & K = \{M, O, A\} \end{cases}$$

- Calculate the Owen payoffs for the partition $\mathcal{P} = \{\{M, O\}\,, \{A\}\}$!
- Comment!

CHAPTER IX

Unions and unemployment benefits

1. Introduction

There is more structure on labor markets than supply and demand functions for labor reveal. First, some workers are employed while others are unemployed. Second, some workers, employed or not, form a union. The aim of this chapter is to analyze the interconnections between employment and unionization. We will also see how unemployment benefits drive the interplay of employment and unionization.

To fix ideas, consider one capitalist (player 1) who may employ 1 or 2 workers (players 2 and 3). If both workers are employed, we are dealing with the (trivial) partition $\mathcal{P}_{AD} = \{\{1,2,3\}\}$. On the other hand, $\mathcal{P}_{AD} = \{\{1,2\},\{3\}\}$ reflects that worker 3 is unemployed.

Besides the AD-partition (modelling unemployment), we will introduce the union partition which serves to model unionization. For our simple example, the relevant union partitions are $\mathcal{P}_u = \{\{1\},\{2\},\{3\}\}$ and $\mathcal{P}_u = \{\{1\},\{2,3\}\}$, the first indicating the absence of a union and the second unionization (workers 2 and 3 form a union). While the agents in an AD-partition work together, the union components bargain as a group.

In order to address the problems of unionization and unemployment, we need a value that depends on both the AD-partition and the union partition. Therefore, we will blend the outside-option value and the Owen value. The resulting value is called union outside-option value.

Our model is not the first to try a cooperative application to labor market issues. In a recent paper, Bae (2005) uses the Shapley value to analyze the merger incentives of firms and their unions. The merging of unions means that workers of both unions join the productive process. In terms of our model, this union merger would be reflected in an appropriate AD- rather than Owen partition. Indeed, the author uses AD-payments without any outside-option argument.

In spirit, our setup is close to the modelling by Berninghaus, Güth, Lechler & Ramser (2001). They consider the question of whether parties are better off bargaining on their own (decentralized bargaining) or together with others (collective bargaining). However, instead of a partitional approach, these authors use the Nash bargaining solution and propose the following procedure. If the two players merge (collective bargaining), there is only one

Nash bargaining game; if they do not merge, two separate Nash bargaining games are considered.

In our approach, there is only one game to play. This, in our mind, allows for interdependencies between the "two" bargaining processes. For example, if two workers offer their services to a capitalist, one might expect that the payoff for each of them depends on the productivity of the other. This is indeed what we will find.

Together with unionization, we analyze unemployment benefits. These have to be understood in a broad sense and may include (the monetary equivalent of) the benefits of leisure. Of course, unemployment benefits can be negative if the financial hand-out is low and if unemployed agents suffer from boredom or the stigma of being unemployed.

While the outside-option value is the basic input, the outer structure of our model is non-cooperative and has three stages. 1. On the basis of the unemployment benefits, the workers decide on unionization. 2. The capitalist makes an employment offer to the workers, individually or to both. 3. The workers, who foresee the wages, decide on whether to accept employment or not. In contrast to principal–agent models, the principal does not propose wages. Instead, wages rest on the productivities of the workers, the outside options, and the unionization.

As might be expected, unemployment benefits do not only define the payoff for unemployed workers but influence the payoff for the employed ones. Although an employed worker does not receive unemployment benefits, his payoff ("wage") is a positive function of unemployment benefits. This fact has often been noted in the labor-market literature (see, for example, Snower 1995, p. 626).

However, unemployment benefits also determine employment. We find that unemployment benefits may drive people out of work. In this chapter, there are two reasons for this to happen. Either unemployment benefits drive up wages by increasing workers' threat point so that employment is not worthwhile for the capitalist. Or, unemployment benefits are so high that workers prefer not to work although the capitalist would be ready to offer employment (voluntary unemployment).

We also look into the question of how unionization influences wages and employment. The economic effects of trade unions have been analyzed for a long while, at least since the seminal works by Dunlop (1944) (the union as an economic organization maximizing the wage bill), Ross (1948) (the union as a political institution fighting for fairness and equity), and Freeman & Medoff (1984) (the union as a two-faced institution, provoking inefficiency (high wages and unemployment) on one hand and promoting productivity and better workplace conditions on the other hand). More recent appraisals are Blanchflower & Bryson (2004), Kaufman (2002), and Turnbull (2003).

Of course, our model cannot do justice to all the exhaustive theoretical and empirical work cited in these surveys. What we try to do is to shed some light on these issues from the point of view of cooperative game theory. The rather complex setup (outside options, two partitions) makes impossible a general approach. Rather, we will have to content outselves with a specific three-player example along the lines of the above partitions.

With respect to wages, we find that a worker will prefer to be part of a union if the other worker (also a union member!) is unemployed and outside options are important. Indeed, unions prevent the capitalist from exploiting the industrial reserve. Our model is supported by the often observed "union/nonunion relative wage differential" (for an early survey, see Lewis 1986) only, if unemployed workers keep on being union members. If there is no unemployment, overstaffing (to be made precise later) makes unionization worthwhile for the employed.

Our prediction about the effect of unionization on employment is ambiguous. If workers are free to choose whether to form unions or not, they will not unionize if doing so is detrimental to employment.

In summary, we argue that our approach is well-suited for the problem at hand. In particular, it is an improvement over the cooperative models cited above. It can address the complicated interlinkages between unionization, unemployment benefits, and unemployment (industrial reserve) in a novel framework.

We proceed as follows. We present the union outside-option value in section 2. We apply this value in section 3. Section 4 concludes this chapter.

2. The union outside-option value

Values return payoff vectors for coalition functions and partitions. You remember that

- a value on N is a function $\sigma : \mathbb{V}_N \to \mathbb{R}^n$;
- a (partitional) value on $(N, \mathfrak{P}_N)$ is a function $\sigma : \mathbb{V}_N \times \mathfrak{P}_N \to \mathbb{R}^n$;
- a (bi-partitional) value on $(N, \mathfrak{P}_N, \mathfrak{P}_N)$ is a function $\sigma : \mathbb{V}_N \times \mathfrak{P}_N \times \mathfrak{P}_N \to \mathbb{R}^n$.

The Shapley value is an example for a function on N while the AD-value, the outside-option values and the Owen value are partitional values on $(N, \mathfrak{P})$. In this chapter, we develop a bi-partitional value that blends a generalized Wiese value with the Owen value.

In order to prepare the generalized Wiese value, we present an alternative, rank-order definition, of the AD-value:

Lemma IX.1 (A rank-order definition of the AD-value). *The AD-value is is given by*

$$AD_i(v,\mathcal{P})=\frac{1}{n!}\sum_{\rho\in RO}MC_i^{K_i(\rho)\cap\mathcal{P}(i)}(v),i\in N$$

According to this alternative characterization of the AD-value, we consider all rank orders but disregard all players outside player i's component.

Definition IX.1 (generalized Wiese value). *The generalized Wiese value is the solution function W given by*

$$\begin{aligned}&W_i(v,\mathcal{P}_{AD},\lambda)\\=\;&\frac{1}{n!}\sum_{\rho\in RO}\begin{cases}v(\mathcal{P}_{AD}(i))-\sum_{j\in\mathcal{P}_{AD}(i)\backslash\{i\}}MC_j(\rho,\mathcal{P}_{AD},\lambda), & \mathcal{P}_{AD}(i)\subseteq K_i(\rho),\\ MC_i(\rho,\mathcal{P}_{AD},\lambda), & \textit{otherwise,}\end{cases}\end{aligned}$$

$\lambda\in[0,1]$ *and*

$$MC_i(\rho,\mathcal{P}_{AD},\lambda)=\lambda MC_i^{K_i(\rho)}(v)+(1-\lambda)MC_i^{K_i(\rho)\cap P_{AD}(i)}(v).$$

We have dealt with the special case of $\lambda:=1$ in chapter VII. Again, player i gets a marginal contribution, here $MC_i(\rho,\mathcal{P}_{AD},\lambda)$, if he is not the last player in his component in ρ, i.e., if $\mathcal{P}_{AD}(i)$ is not included in $K_i(\rho)$. A last player is the residual claimant who gets the worth of his component after repaying the other players' marginal contributions.

The difference between the Wiese value presented in chapter VII and this generalized version lies in the parameter λ. In case of $\lambda=0$, we get the AD-value as a special case which can be seen

- from the above lemma and
- from component efficiency which ensures that the last player also gets his marginal contribution.

Positive values of λ reflect outside opportunities where marginal contributions to coalitions outside $\mathcal{P}(i)$ get a positive weight. Low values of λ reflect the inability to use outside options as a serious threat in bargaining. For example, employment of yet another worker may not be feasible or substitution of the presently employed by the presently unemployed may not be possible. For the trivial partition $\mathcal{P}_{AD}=\{N\}$ we obtain $W_i(v,\{N\},\lambda)=Sh_i(v)=AD_i(v,\{N\})$ for all $\lambda\in[0,1]$.

We now remind the reader of the Owen value which is given by

$$Ow_i(v,\mathcal{P}_u)=\frac{1}{|RO_{\mathcal{P}_u}|}\sum_{\rho\in RO_{\mathcal{P}_u}}MC_i^{K_i(\rho)}(v),i\in N.$$

Finally, we can present the union outside-option value. It is obtained by merging the union and the outside-option values in the obvious manner:

DEFINITION IX.2 (union outside-option value). *The union outside-option value is the solution function OW given by*

$$\begin{aligned} & OW_i\left(v,\mathcal{P}_{AD},\lambda,\mathcal{P}_u\right) \\ = & \frac{1}{\left|RO_{\mathcal{P}_u}\right|}\sum_{\rho\in RO_{\mathcal{P}_u}}\left\{\begin{array}{ll} v\left(\mathcal{P}_{AD}\left(i\right)\right)-\sum_{j\in\mathcal{P}(i)\backslash i}MC_j\left(\rho,\mathcal{P}_{AD},\lambda\right), & \mathcal{P}_{AD}\left(i\right)\subseteq K_i\left(\rho\right), \\ MC_i\left(\rho,\mathcal{P}_{AD},\lambda\right), & \textit{otherwise,}\end{array}\right., i\in N\end{aligned}$$

$\lambda\in[0,1]$ *and*

$$MC_i\left(\rho,\mathcal{P}_{AD},\lambda\right)=\lambda MC_i^{K_i(\rho)}\left(v\right)+\left(1-\lambda\right)MC_i^{K_i(\rho)\cap\mathcal{P}_{AD}(i)}\left(v\right).$$

3. A simple labour market

3.1. Partitions and payoffs. Turning to the 3-player example from the introduction, we consider three AD-partitions, $\mathcal{P}_{AD}=\{\{1,2,3\}\}$, $\mathcal{P}_{AD}=\{\{1,2\},\{3\}\}$, and $\mathcal{P}_{AD}=\{\{1\},\{2\},\{3\}\}$. In the first, the capitalist (player 1) employs both workers (players 2 and 3), in the second, player 3 is unemployed, and in the third, both are unemployed. We also deal with two union partitions, $\mathcal{P}_u=\{\{1\},\{2\},\{3\}\}$ and $\mathcal{P}_u=\{\{1\},\{2,3\}\}$. The second indicates that workers 2 and 3 form a union. Thus, we have 6 partition combinations.

To fix ideas, we set $v(N):=100$ (any positive value or a variable would do) and let $a_2:=v(\{1,2\})$ and $a_3:=v(\{1,3\})$. We assume $a_2>a_3\geq 0$, i.e., worker 2 is more productive than worker 3 in a one-worker firm. If workers are not employed, they receive unemployment benefit, $u\geq 0$. Hence, $v(\{2\})=v(\{3\})=u$ and $v(\{2,3\})=2u$. Since we want to concentrate on unionization and unemployment benefits, we let $v(\{1\}):=0$, assuming zero normal profits for the capitalist. Superadditivity (which we do not, in general, assume) implies

$$\begin{aligned} 2u &\leq 100, \\ a_2+u &\leq 100, \text{ and} \\ a_3+u &\leq 100. \end{aligned}$$

We first report the values and then (see the next section) solve a three-stage model.

Result 1: For the six partition combinations, the union outside-option value yields the following payoffs:

$\mathcal{P}_{AD}$	$\mathcal{P}_u$	φ^{u-oo}
$\{\{1,2,3\}\}$	$\{\{1\},\{2\},\{3\}\}$	$\begin{pmatrix} A := \frac{100}{3} + \frac{a_2}{6} + \frac{a_3}{6} - u \\ B := \frac{100}{3} + \frac{a_2}{6} - \frac{a_3}{3} + \frac{u}{2} \\ C := \frac{100}{3} - \frac{a_2}{3} + \frac{a_3}{6} + \frac{u}{2} \end{pmatrix}$
$\{\{1,2,3\}\}$	$\{\{1\},\{2,3\}\}$	$\begin{pmatrix} D := 50 - u \\ E := 25 + \frac{a_2}{4} - \frac{a_3}{4} + \frac{u}{2} \\ F := 25 - \frac{a_2}{4} + \frac{a_3}{4} + \frac{u}{2} \end{pmatrix}$
$\{\{1,2\},\{3\}\}$	$\{\{1\},\{2\},\{3\}\}$	$\begin{pmatrix} G := \frac{a_2}{2} + \frac{1}{6}\lambda(a_3 - u) - \frac{u}{2} \\ H := \frac{a_2}{2} - \frac{1}{6}\lambda a_3 + \frac{1}{6}u(3+\lambda) \\ = \frac{a_2}{2} - \frac{1}{6}\lambda(a_3 - u) + \frac{u}{2} \\ I := u \end{pmatrix}$
$\{\{1,2\},\{3\}\}$	$\{\{1\},\{2,3\}\}$	$\begin{pmatrix} J := \frac{a_2}{2} - \frac{u}{2} \\ K := \frac{a_2}{2} + \frac{u}{2} \\ L := u \end{pmatrix}$
$\{\{1\},\{2\},\{3\}\}$	$\{\{1\},\{2\},\{3\}\}$ or $\{\{1\},\{2,3\}\}$	$\begin{pmatrix} M := 0 \\ N := u \\ P := u \end{pmatrix}$

In particular, we find:

Result 1a: If the capitalist wants to employ one worker only, he will choose the more productive worker 2. For moderate unemployment benefits ($u < a_3$), worker 2's wage is higher in the presence of a union than without a union. In order to accept employment, worker 2 needs to be sufficiently productive and unemployment benefits need to be sufficiently low.

Result 1b: The incentives of the capitalist to employ worker 3 on top of worker 2 depend on whether or not the workers form a union. If they do, worker 3 will be employed whenever his marginal contribution exceeds unemployment benefits. If there is no union, the capitalist might be prepared to employ a worker 3 even if that worker's marginal contribution is negative. In case of low average marginal contributions ($\frac{1}{2}(100 - a_3) + \frac{1}{2}(100 - a_2) < 50$) the workers prefer to be unionized.

The values in the above table are obtained by straightforward calculations. The first row corresponds to the Shapley value (all workers employed, no union). The Owen value (all workers employed, workers form a union) is seen in row 2. In the last row, the capitalist does not employ any worker so that no output is produced. Then the capitalist and the workers are paid their reservation payoff, a profit of 0 and the unemployment benefit u, respectively. Rows 3 and 4 refer to the case where only worker 2 is employed. For these cases, the (union) outside-option value has been devised. The appendix provides an example of how the payoff is to be calculated.

Note that the bargaining power of any agent is expressed by the two partitions and by λ. In particular, the capitalist cannot make a take-it-or-leave-it offer to the worker(s) in order to lower their payoffs to the reservation level.

We can use the values to theorize about the players' preferences. By simple comparisons (note the letters standing for the payoffs), and explicating Result 1a, we find:

- By $a_2 > a_3$, the capitalist prefers to have worker 2 rather than worker 3 as his only employee (compare profits G and J, respectively, with the symmetrical profits obtained by interchanging workers 2 and 3).
- If worker 2 is the only employee and if the workers are not unionized, worker 2's payoff

$$H = \frac{a_2}{2} - \frac{1}{6}\lambda\left(a_3 - u\right) + \frac{1}{2}u$$

  reveals that the capitalist can use worker 3 to lower worker 2's wage. This mechanism will work,
  - if there is a high degree of flexibility and outside options (λ is high),
  - if worker 3 is productive (if he were employed), and
  - if unemployment benefits are moderate.

  In terms of Marx (1985, pp. 657), worker 3 forms the industrial reserve.

  If, however, unemployment benefits are not moderate ($u > a_3$), the capitalist suffers from the outside option (dealing with worker 3). Indeed, worker 2 might say to the capitalist that the capitalist would need to deal with worker 3 unless he, worker 2, would be prepared to put up employment.

  Interestingly, unionization prevents the use of the industrial reserve by the capitalist. This can be seen from worker 2's payoff $K = \frac{a_2}{2} + \frac{1}{2}u$. A comparison of H with K shows that unions make worker 2 more willing to accept employment in case of $a_3 > u$.

Turning to Result 1b, we find:

- If the workers form a union, the capitalist wants to employ worker 3 on top of worker 2 whenever worker 3's marginal contribution $100 - a_2$ exceeds unemployment benefit u ($D > J$).

  If there is not union, the capitalist might be willing to employ worker 3 even if that worker has a negative marginal contribution. Indeed, we find

$$A > G \Leftrightarrow 100 - a_2 > \frac{1}{2}\left[u\left(3 - \lambda\right) - a_3\left(1 - \lambda\right)\right]$$

where moderate unemployment benefits can make the right-hand term negative. In that case, the third worker is not employed for his productiveness but is brought into the firm in order to increase the capitalist's bargaining power vis-a-vis worker 2. This function is especially important to the capitalist if he cannot use the unemployed worker 3 as industrial reserve. Formally, we have $\frac{\partial\left[\frac{1}{2}[u(3-\lambda)-a_3(1-\lambda)]\right]}{\partial\lambda} = \frac{1}{2}(a_3 - u)$, where $a_3 > u$ (moderate unemployment benefits) and low values of λ make employment of worker 3 more probable.

- Since both $E > B$ and $F > C$ are equivalent to $\frac{1}{2}a_2 + \frac{1}{2}a_3 > 50$, both employed workers prefer unionization if their average productivity in a one-worker firm is sufficiently high, or differently put, if the average marginal contribution of the additional worker is sufficiently low ($\frac{1}{2}(100 - a_3) + \frac{1}{2}(100 - a_2) < 50$). Thus, overstaffing makes unionization worthwhile for the employed. In contrast, the capitalist prefers unionization if the workers are relatively unproductive. A comparable result has been presented by Horn & Wolinsky (1988, p.488) in the context of a non-cooperative model. They assume $a_2 = a_3$ and find an incentive to unionize in case of $a_2 > 50$ which is a special case of our result. The reader is invited to consult the appendix for details.

3.2. The sequential model.

3.2.1. *Game sequence.* We now turn to a model consisting of three stages. First, the workers decide on unionization. Here, we apply the Pareto principle so that one worker alone can decide about unionization if the other is indifferent. Second, the capitalist makes an employment offer to worker 2, worker 3, both, or none. (Wages are determined later.) Finally, the workers accept employment or decline. If any worker declines, no workers are employed. This is not restrictive. Since the capitalist can foresee the workers' payoffs and decisions, he will make acceptable offers. In order to maintain tractability, we assume $\lambda := 1$.

3.2.2. *Solving for subgame-perfect equilibria.* If the capitalist plans to employ one worker only (stage 2), he will choose the more productive worker 2 by $a_2 > a_3$ (see profits G and J). Thus, at stage 2, the capitalist chooses between

- employing worker 2, only,
- employing both workers, and
- employing none.

By $a_2 > a_3$, both workers will accept employment if worker 3 accepts (see payoffs $B > C$ and $E > F$). Solving the model requires simple but tedious case distinctions which we will relegate to the appendix.

3.3. Employment (stages 2 and 3).

3.3.1. *Voluntary unemployment.* From the point of view of social policy, it is an important question whether unemployment benefits affect unemployment and voluntary unemployment. As De Vroey (2004, pp. 13) points out, several alternative definitions of voluntary unemployment coexist. Definitions of voluntary unemployment make use of counterfactual thought experiments: Would the unemployed worker like to be employed in lieu of another actually employed one? Since we deal with small numbers of heterogeneous workers, we propose the following working definition: an unemployed worker is voluntarily unemployed if employing him - on top of the actually employed workers - would lead to an unattractive wage rate, i.e., a wage rate lower than his unemployment benefit.

Result 2: Voluntary unemployment may happen.

In order to show Result 2, we look at the special case depicted in figure 1. "Nu" stands for "no union" and "Nu2" refers to one of several cases (see appendix). The agents preferences are indicated in their respective lines and are noted as a function of unemployment benefits u. For example, to the left of the first vertical line, the capitalist will prefer to employ worker 2 instead of employing no worker. To the left of the second vertical line, he prefers to employ both workers rather than none. To the left of the last vertical line, the capitalist would rather employ both workers than worker 2, only. The preferences for worker 2 (who will get an offer if only one worker gets an offer) and of both workers (identical to the preferences of worker 3) are depicted in a similar way. The fourth line ("accepted offer") summarizes these lines into statements about employment.

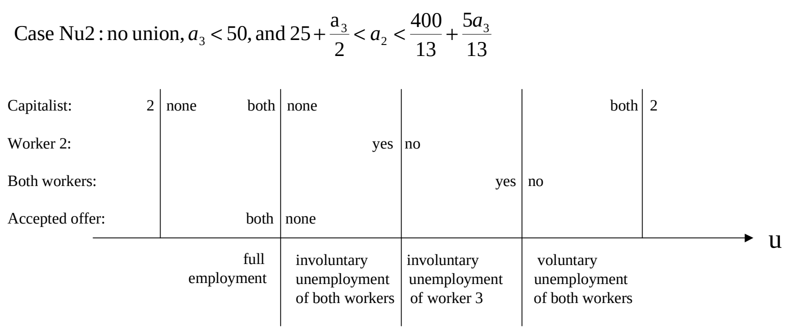

Figure 1: Preferences and outcomes in case Nu2

In the actual case, the capitalist is able to achieve his preferred outcome (both workers left of the second vertical line, none to the right of this line) because both workers are prepared to put up employment whenever the capitalist is ready to offer employment to both.

We see that involuntary and voluntary unemployment can well happen. Between the second and third vertical line, worker 2 would be prepared to accept employment of him alone and both workers would be prepared to accept employment of both. Here, we have involuntary unemployment of both workers. The area between the third and fourth vertical line is difficult to classify. Worker 3 is involuntarily unemployed. However, worker 2 is not prepared to be employed on his own while he is willing to be employed if both workers were active. To the right of the fourth line (at high levels of unemployment benefits), we have voluntary unemployment of both workers.

3.3.2. *Employment and unionization.* We now present the results about the effects of unemployment benefits and unionization on employment.

Result 3: As a general picture, unemployment is an increasing function of the level of unemployment benefits. Depending on parameters, unions can be harmful or beneficial for employment.

In order to show that unemployment benefits create unemployment, we refer to two figures. Figure 2 is based on $a_3 = 20$ (an example for $a_3 < 50$) and figure 3 on $a_3 = 60$ (an example for $a_3 > 50$). In these figures, unemployment benefit u is plotted against $a_2 \geq a_3$. Obviously, unemployment benefits are detrimental to employment.

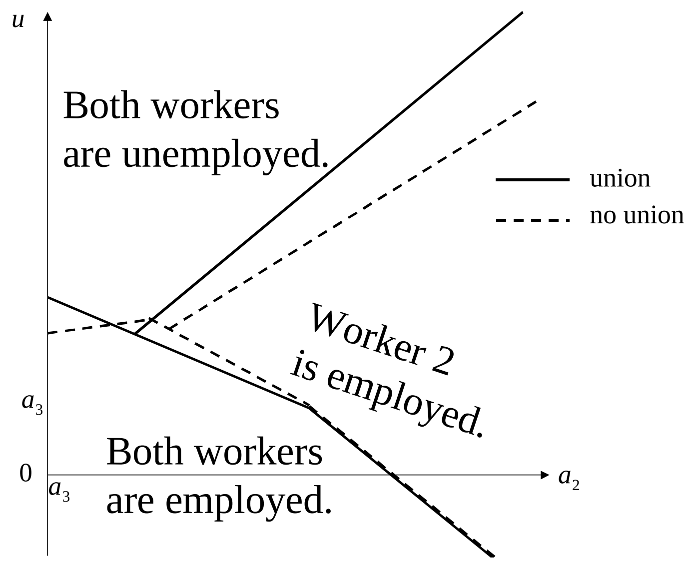

Figure 2: Employment and unions for $a_3 = 20$

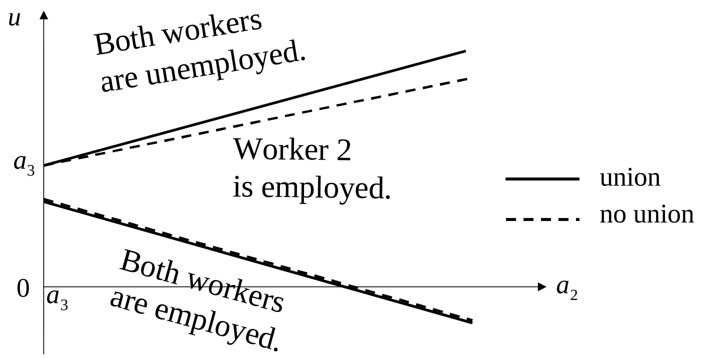

Figure 3: Employment and unions for $a_3 = 60$

The effect of unionization on employment is quite unclear. For example, in figure 2, in the leftmost triangle bordering the u-axis, we have unemployment without unions and full employment in case of unions. For the very small triangle to the right of this triangle, we have unemployment in case of, and full employment without, unions.

3.4. Union choice (stage 1). We now turn to stage 1 of our model, i.e., to the question of whether workers will want to unionize. There are two somewhat distinct reasons for unionization (or for deciding against unions). Workers make their union-choices in order to be employed (i.e., in order to obtain a salary instead of unemployment benefits) or in order to increase their salary. Thus, we distinguish between the employment and the salary motive.

Result 4: Workers are unanimous in their union choice and unions can never be blamed for unemployment. Workers decide on unions for both employment and salary motives. Unions tend to be beneficial for (employed!) workers if there is overstaffing or unemployment.

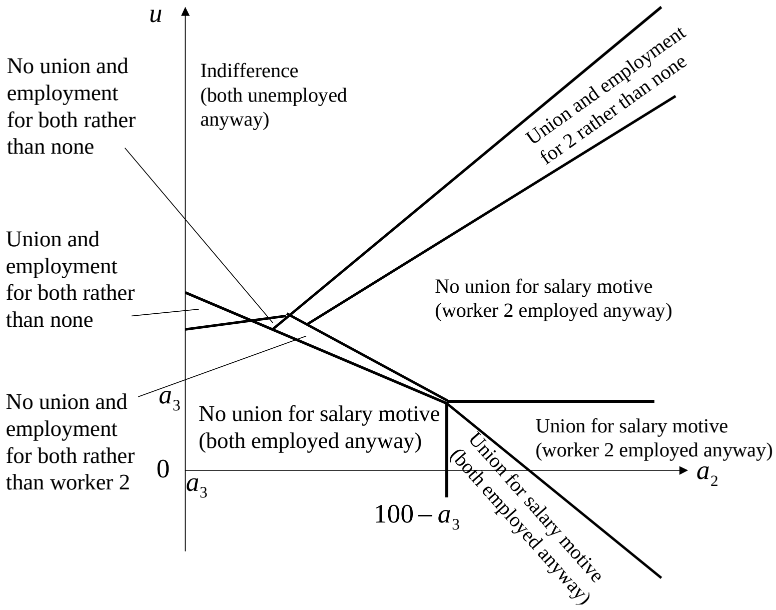

Figure 4: Union choice for $a_3 = 20$

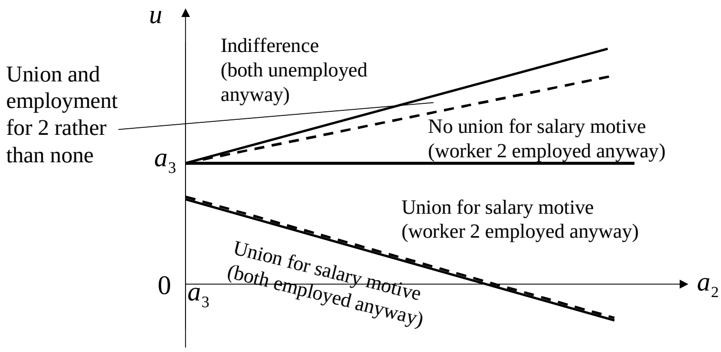

Figure 5: Union choice for $a_3 = 60$

Figures 4 (based on $a_3 = 20$) and 5 (based on $a_3 = 60$) inform about the choice of unions by the two workers. If one worker (worker 3) is indifferent towards unionization (because he is unemployed in either case), while the other (worker 2) has a definite preference, we assume that the latter one's preferences count. We also find that if both workers are employed, their preferences coincide. Therefore, unions can never be blamed for unemployment from the point of view of stage 1.

In Result 1a, we note that moderate unemployment benefits ($u < a_3$) make worker 2 - as the only employee - prefer a union. This is reflected in both figures. According to Result 1b, overstaffing ($\frac{1}{2}(100 - a_3)+\frac{1}{2}(100 - a_2) <$

50) imply that both (employed!) workers prefer a union. Indeed, the equivalent formulation is $a_2 > 100 - a_3$ which holds everywhere in figure 5 (which is based on $a_3 > 50$) and to the right of $100 - a_3$ in figure 4.

4. Conclusions

With respect to employment maximization, the model takes a negative view on (high) unemployment benefits and a differentiated view on unionization. In fact, depending on the parameters, unionization may increase or decrease employment. If unions lead to higher wages, these wages may depress employment because the capitalist is not prepared to pay high wages. However, unemployment may also result from an unwillingness of workers to accept employment. In that case, high wages affected by a union may actually be beneficial for employment.

Interestingly, endogenous unionization has positive effects on employment. A worker who foresees that the existence of a union leads to his being unemployed will not join. Note, however, that in our model all workers a unionized or none. In real-world labor markets, some percentage of the workers (employed or unemployed) are union members, only. Then, union members may lobby for high wages that prove detrimental for the employment of other, non-union workers.

The model may also provide an indication of when obligatory unions (all the workers are obliged to join) can be expected to increase wages. If a substantial industrial reserve exists, a union provides protection against the potential competition by the unemployed. If (almost) all workers are employed, unions are beneficial if there is overstaffing, i.e., if there are some workers that might be laid off without much harm to productivity.

Our model uses a non-core cooperative solution concept which can readily be criticized. Why do workers not earn their marginal product? Why do markets not clear? In our mind, there are two justifications for applying the union outside-option value. First of all, it encompasses a lot of social structure (employment, unions) that would be very difficult to model in a non-cooperative manner. Attempts in this direction have been presented by Horn & Wolinsky (1988) and Jun (1989), both using the Rubinstein bargaining procedure. These authors concentrate on the union aspect but do not take unemployment or unemployment benefits into account. Another interesting paper by Davidson (1988) assumes a Cournot oligopoly. Here, workers are homogeneous and outside options and unemployment benefits have no role to play.

Second, the use of non-core concepts may be taken to reflect labour market rigidities. While the industrial reserve does indeed lower wages, it cannot do so in a perfectly competitive fashion.

Future research could be persued along the following lines. In this chapter, we have a single employer. Our method could also be used with several employers in order to provide a cooperative analogue to the above mentioned Cournot approach by Davidson (1988).

5. Appendix

5.1. Calculating G in Result 1. In order to confirm $G = \frac{a_2}{2} + \frac{1}{6}\lambda(a_3 - u) - \frac{u}{2}$ in the table of Result 1, assume $\mathcal{P}_{AD} = \{\{1,2\},\{3\}\}$ and $\mathcal{P}_u = \{\{1\},\{2\},\{3\}\}$. We then obtain

$$\begin{aligned}
MC_1((1,2,3),\mathcal{P}_{AD},\lambda) &= MC_1((1,3,2),\mathcal{P}_{AD},\lambda) \\
&= \lambda MC_1^{\{1\}}(v) + (1-\lambda) MC_1^{\{1\}\cap\mathcal{P}_{AD}(1)}(v) \\
&= MC_1^{\{1\}}(v) = 0 - 0 = 0 \\
MC_1((3,1,2),\mathcal{P}_{AD},\lambda) &= \lambda MC_1^{\{3,1\}}(v) + (1-\lambda) MC_1^{\{3,1\}\cap\mathcal{P}_{AD}(1)}(v) \\
&= \lambda MC_1^{\{3,1\}}(v) + (1-\lambda) MC_1^{\{1\}}(v) \\
&= \lambda(a_3 - u) + (1-\lambda)(0-0) = \lambda(a_3 - u) \\
MC_2((2,1,3),\mathcal{P}_{AD},\lambda) &= MC_2((2,3,1),\mathcal{P}_{AD},\lambda) \\
&= \lambda MC_2^{\{2\}}(v) + (1-\lambda) MC_2^{\{2\}\cap\mathcal{P}_{AD}(2)}(v) \\
&= MC_2^{\{2\}}(v) = u - 0 = u \\
MC_2((3,2,1),\mathcal{P}_{AD},\lambda) &= \lambda MC_2^{\{3,2\}}(v) + (1-\lambda) MC_2^{\{3,2\}\cap\mathcal{P}_{AD}(2)}(v) \\
&= \lambda MC_2^{\{3,2\}}(v) + (1-\lambda) MC_2^{\{2\}}(v) \\
&= \lambda(2u - u) + (1-\lambda)(u - 0) = u
\end{aligned}$$

and

$$\begin{aligned}
&\varphi_1^{u-oo}(v,\mathcal{P}_{AD},\lambda,\mathcal{P}_u) \\
= &\frac{1}{6}\Bigg(\underbrace{MC_1((1,2,3),\mathcal{P}_{AD},\lambda)}_{(1,2,3)} + \underbrace{MC_1((1,3,2),\mathcal{P}_{AD},\lambda)}_{(1,3,2)} \\
&+\underbrace{v(\{1,2\}) - MC_2((2,1,3),\mathcal{P}_{AD},\lambda)}_{(2,1,3)} + \underbrace{v(\{1,2\}) - MC_2((2,3,1),\mathcal{P}_{AD},\lambda)}_{(2,3,1)} \\
&+\underbrace{MC_1((3,1,2),\mathcal{P}_{AD},\lambda)}_{(3,1,2)} + \underbrace{v(\{1,2\}) - MC_2((3,2,1),\mathcal{P}_{AD},\lambda)}_{(3,2,1)}\Bigg) \\
= &\frac{1}{6}\Bigg(\underbrace{0}_{(1,2,3)} + \underbrace{0}_{(1,3,2)} + \underbrace{a_2 - u}_{(2,1,3)} + \underbrace{a_2 - u}_{(2,3,1)} + \underbrace{\lambda(a_3 - u)}_{(3,1,2)} + \underbrace{a_2 - u}_{(3,2,1)}\Bigg) \\
= &\frac{1}{6}(3a_2 - 3u + \lambda(a_3 - u)) = \frac{a_2}{2} + \frac{1}{6}\lambda(a_3 - u) - \frac{u}{2} = G
\end{aligned}$$

5.2. Stages 2 and 3 in case of no unions. For any given u, we first assume that workers decide against unions (stage 1). The capitalist

- prefers to employ both workers rather than worker 2, only, in case of

$$A > G \Leftrightarrow u < 100 - a_2 =: \gamma_{23>2},$$

- prefers to employ worker 2, only, rather than none in case of

$$G > M \Leftrightarrow u < \frac{3}{4}a_2 + \frac{1}{4}a_3 =: \gamma_{2>0},$$

- and prefers to employ both workers rather than none in case of

$$A > M \Leftrightarrow u < \frac{100}{3} + \frac{1}{6}(a_2 + a_3) =: \gamma_{23>0}.$$

Worker 2 is ready to accept employment as the only worker if

$$H > N \Leftrightarrow u < \frac{3}{2}a_2 - \frac{1}{2}a_3 =: \omega_2$$

holds. Both workers are prepared to put up employment if worker 3 is ready, i.e., if

$$C > P \Leftrightarrow u < \frac{200}{3} - \frac{2}{3}a_2 + \frac{1}{3}a_3 =: \omega_{23}.$$

We obtain the following partition in $a_2 - a_3$ space:

Nu1	$a_3 < 50, a_3 < a_2 < 25 + \frac{1}{2}a_3$	$\gamma_{2>0} < \omega_2 < \gamma_{23>0} < \omega_{23} < \gamma_{23>2}$
Nu2	$a_3 < 50, 25 + \frac{1}{2}a_3 < a_2 < \frac{400}{13} + \frac{5}{13}a_3$	$\gamma_{2>0} < \gamma_{23>0} < \omega_2 < \omega_{23} < \gamma_{23>2}$
Nu3	$a_3 < 50, \frac{400}{13} + \frac{5}{13}a_3 < a_2 < 40 + \frac{1}{5}a_3$	$\gamma_{2>0} < \gamma_{23>0} < \omega_{23} < \omega_2 < \gamma_{23>2}$
Nu4	$a_3 < 50, 40 + \frac{1}{5}a_3 < a_2 < \frac{800}{17} + \frac{1}{17}a_3$	$\gamma_{2>0} < \omega_{23} < \gamma_{23>0} < \gamma_{23>2} < \omega_2$
Nu5	$a_3 < 50, \frac{800}{17} + \frac{1}{17}a_3 < a_2 < \frac{400}{7} - \frac{1}{7}a_3$	$\omega_{23} < \gamma_{2>0} < \gamma_{23>0} < \gamma_{23>2} < \omega_2$
Nu6	$a_3 < 50, \frac{400}{7} - \frac{1}{7}a_3 < a_2 < 100 - a_3$	$\omega_{23} < \gamma_{23>2} < \gamma_{23>0} < \gamma_{2>0} < \omega_2$
Nu7	$a_2 + a_3 > 100$	$\gamma_{23>2} < \omega_{23} < \gamma_{23>0} < \gamma_{2>0} < \omega_2$

5.3. Stages 2 and 3 in case of unions. Assume that the workers have formed a union in stage 2. The capitalist

- prefers to employ both workers rather than worker 2, only, in case of

$$D > J \Leftrightarrow u < 100 - a_2 =: \gamma_{23>2}^{\text{union}},$$

- prefers to employ worker 2, only, rather than none in case of

$$J > M \Leftrightarrow u < a_2 =: \gamma_{2>0}^{\text{union}},$$

- and prefers to employ both workers rather than none in case of

$$D > M \Leftrightarrow u < 50 =: \gamma_{23>0}^{\text{union}}.$$

Worker 2 is ready to accept employment as the only worker if

$$K > N \Leftrightarrow u < a_2 =: \omega_2^{\text{union}}$$

holds. Both workers are prepared to put up employment if worker 3 is ready, i.e., if

$$F > P \Leftrightarrow u < 50 - \frac{1}{2}(a_2 - a_3) =: \omega_{23}^{\text{union}}.$$

We find the following partition in $a_2 - a_3$ space:

U1	$a_3 < 50, a_3 < a_2 < \frac{100}{3} + \frac{1}{3}a_3 < 50$	$\gamma_{2>0}^{\text{union}} = \omega_2^{\text{union}} < \omega_{23}^{\text{union}} < \gamma_{23>0}^{\text{union}} < \gamma_{23>2}^{\text{union}}$
U2	$a_3 < 50, \frac{100}{3} + \frac{1}{3}a_3 < a_2 < 50$	$\omega_{23}^{\text{union}} < \gamma_{2>0}^{\text{union}} = \omega_2^{\text{union}} < \gamma_{23>0}^{\text{union}} < \gamma_{23>2}^{\text{union}}$
U3	$a_3 < 50, a_2 > 50$ $a_2 + a_3 < 100$	$\omega_{23}^{\text{union}} < \gamma_{23>2}^{\text{union}} < \gamma_{23>0}^{\text{union}} < \gamma_{2>0}^{\text{union}} = \omega_2^{\text{union}}$
U4	$a_2 + a_3 > 100$	$\gamma_{23>2}^{\text{union}} < \omega_{23}^{\text{union}} < \gamma_{23>0}^{\text{union}} < \gamma_{2>0}^{\text{union}} = \omega_2^{\text{union}}$

5.4. Stage 1: Will the workers form a union? We merge the Nu-partition with the U-partition to find out whether unionization is profitable for the workers. We find that a relatively simple partition suffices to answer this question:

1	$a_3 < 50, a_3 < a_2 < 25 + \frac{1}{2}a_3 < 50$	$\text{Nu1} \cap \text{U1}$
2	$a_3 < 50, 25 + \frac{1}{2}a_3 < a_2 < \frac{100}{3} + \frac{1}{3}a_3 < 50$	$(\text{Nu2} \cup \text{Nu3}) \cap \text{U1}$
3	$a_3 < 50, \frac{100}{3} + \frac{1}{3}a_3 < a_2 < 40 + \frac{1}{5}a_3 < 50$	$\text{Nu3} \cap \text{U2}$
4	$a_3 < 50, 40 + \frac{1}{5}a_3 < a_2 < \frac{800}{17} + \frac{1}{17}a_3 < 50$	$\text{Nu4} \cap \text{U2}$
5	$a_3 < 50, \frac{800}{17} + \frac{1}{17}a_3 < a_2 < 100 - a_3$	$(\text{Nu5} \cup \text{Nu6}) \cap (\text{U2} \cup \text{U3})$
6	$a_2 > 100 - a_3$	$\text{Nu7} \cap \text{U4}$

Comparing the payoffs for the two workers in all these parameter regions, we obtain figures 2 through 5 in the main text.

5.5. The Horn-Wolinsky model. Horn & Wolinsky (1988) assume to workers, workers A and B who jointly produce $x + y$ while either one of them alone produces x, only. Thus, we have

$$a_2 = a_3 = x.$$

The authors find that unionization pays in case of $y < x$. Letting $x+y = 100$ (which is not a serious assumption) and transferring this result into our notation yields

$$\begin{aligned} 100 - x &< x \text{ and} \\ x &> 50. \end{aligned}$$

This is exactly our result for the special case $a_2 = a_3$.

Shapley values on networks

Part D

Shapley values on networks

In this part of the course, we aim to define and apply solutions that depend on a network rather than a partition (as in the previous part). A network represents the relationships that may exist between any two players. For example, players may know each other so that they can cooperative. We present the most famous value, the Myerson value in chapter X. It is based on symmetric (undirected) links. We pursue this approach in chapter **??** that interprets the ideas put forward by the sociologist Granovetter: Are weak links between agents more important than strong links?

Chapters XI and XII work with asymmetric (directed) links. In terms of the interpretation, we look at permission and use structures and analyze the payoff consequences of hierarchies.

CHAPTER X

The network value

1. Introduction

The main idea of the first chapter of this part is to modify a coalition function by a network. We then apply the Shapley solution to the modified coalition function. The resulting value is called the network or the Myerson value. This chapter leans heavily on the monography by Slikker & Nouweland (2001).

Consider the upper network $\mathcal{L}_1$ presented in fig. 1. Players 1 and 3 are not directly linked so that the modified coalition function $v^{\mathcal{L}_1}$ is, inter alia, given by

$$v^{\mathcal{L}_1}(\{1,3\}) = v(\{1\}) + v(\{3\}).$$

The same is true, of course, for the lower network $\mathcal{L}_2$: $v^{\mathcal{L}_2}(\{1,3\}) = v(\{1\}) + v(\{3\})$. In contrast, we find

$$\begin{aligned} v^{\mathcal{L}_1}(\{1,2,3\}) &= v(\{1,2,3\}) \\ v^{\mathcal{L}_2}(\{1,2,3\}) &= v(\{1,2\}) + v(\{3\}) \end{aligned}$$

The Myerson value for a coalition function v and for a network $\mathcal{L}$ is the Shapley for $v^{\mathcal{L}}$. It turns out that it is definable by the axiom of balanced contributions. If the link between players 2 and 3 is destroyed (moving from the upper to the lower network), players 2 and 3 are harmed equally.

2. Links, networks, and subnetworks

We summarize the links between pairs of players in so-called networks. On the set $\{1,2,3,4\}$, player 1 may be linked with all the other players who

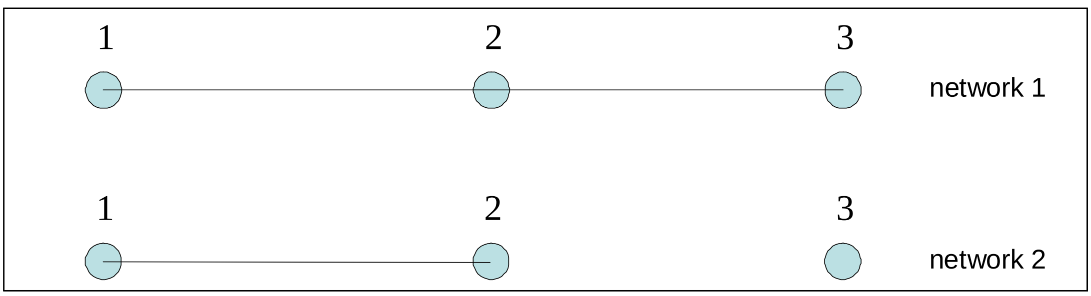

FIGURE 1. A simple network

do not have direct links with each other. This network is described by

$$\{12, 13, 14\}.$$

DEFINITION X.1 (network). *Let N be a set (of players). The set of all subsets with exactly two elements is called the full network and is denoted by $\mathcal{L}^{full}$,*

$$\mathcal{L}^{full} = \{\{i,j\} : i,j \in N, i \neq j\}.$$

Elements ℓ from $\mathcal{L}^{full}$ are called links. $\mathcal{L} \subseteq \mathcal{L}^{full}$ is called a network on N. The set of all networks on N is denoted by $\mathfrak{L}_N$. By $\mathfrak{L}$ we denote the set of networks on any player set N. $\mathcal{L}_1 \subseteq \mathcal{L}^{full}$ is called a subnetwork of $\mathcal{L}_2 \subseteq \mathcal{L}^{full}$ if $\mathcal{L}_1 \subseteq \mathcal{L}_2$ holds.

In case of $\{i,j\} \in \mathcal{L}$ the players i and j are called directly linked. We also write ij instead of $\{i,j\}$.

The set $\mathcal{L}(i) := \{\ell \in \mathcal{L} : i \in \ell\} \subseteq \mathcal{L}$ is the set of all links entertained by player i.

Let R be a subset of N. The links on R induced by a network $\mathcal{L}$ is denoted by $\mathcal{L}(R)$ and defined by

$$\mathcal{L}(R) := \{\{i,j\} : i,j \in R, \{i,j\} \in \mathcal{L}\}.$$

EXERCISE X.1. *Consider $N = \{1,2,3,4\}$ and define the network $\mathcal{L}$ where player 2 is directly linked to players 1 and 3. Determine $\mathcal{L}(1)$, $\mathcal{L}(2)$ und $\mathcal{L}(4)$.*

Both $\mathcal{L}(i)$ and $\mathcal{L}(R)$ are subnetworks of $\mathcal{L}$. We get $\mathcal{L}(i)$ from $\mathcal{L}$ by deleting all the links that do not contain i. $\mathcal{L}(R)$ is obtained from $\mathcal{L}$ by deleting all the links containing one or two players outside R. Therefore, $\mathcal{L}(i)$ and $\mathcal{L}(\{i\})$ differ. $\mathcal{L}(i)$ is nonempty whenever there is a link $\ell \in \mathcal{L}$ such that $i \in \ell$. In contrast, $\mathcal{L}(\{i\})$ is always empty:

$$\begin{aligned} \mathcal{L}(\{i\}) &= \{\{i,j\} : i,j \in \{i\}, \{i,j\} \in \mathcal{L}\} \\ &= \emptyset \end{aligned}$$

Admittedly, all this is somewhat confusing. Can you solve the following exercise?

EXERCISE X.2. *Assume an arbitrary network $\mathcal{L}$ on N. Can you find other expressions for*

- $\mathcal{L}(N)$,
- $\mathcal{L}(\{1,2\})$ *(case distinction!) and*
- $\bigcup_{i \in N} \mathcal{L}(i)$?

Our notation of networks is somewhat parallel to the notation chosen for partitions. The following table juxtaposes important symbols used for partitions and networks on player set N:

partition		network	
symbol	meaning	symbol	meaning
$\mathcal{P}$	partition	$\mathcal{L}$	undirected graph
$\mathcal{P}(i)$	i's component	$\mathcal{L}(i)$	set of i's links
$\mathfrak{P}_N$	set of partitions on N	$\mathfrak{L}_N$	set of networks on N
$\mathcal{P}(R)$	set of components with R-players	$\mathcal{L}(R)$	set of links on R

3. Trails and connectedness

Networks are more complicated than partitions.

Definition X.2 (connectedness). *Let $\mathcal{L}$ be a network on N. A trail in $\mathcal{L}$ from i to j (a $i-j$ trail) is a network $\{i_0 i_1, ..., i_{k-1} i_k\} \subseteq \mathcal{L}$ where $i = i_0$ and $j = i_k$. It is also denoted by $T(i \to j) = \langle i = i_0, ..., j = i_k \rangle$. The set of such trails is denoted by $\mathbb{T}(i \to j)$.*

Players i and j are called connected or linked if an $i-j$ trail exists or if $i = j$ holds. i and j are indirectly connected if they are connected but not directly connected. i and j are called connected within a subset $K \subseteq N$ if they are connected by an $i-j$ trail that is a subnetwork of $\mathcal{L}(K)$. A subset $K \subseteq N$ is called internally connected if any two players i and j from K are connected within K.

A network $\mathcal{L}$ is called connected if every pair of players is connected.

Exercise X.3. *Consider a network $\mathcal{L}$ on N and a player $i \in N$ with $\mathcal{L}(i) = \emptyset$. We call such a player isolated. Show: If a network $\mathcal{L} \neq \emptyset$ admits such a player, the network is not connected.*

Exercise X.4. *Use the definition of a trail to define the direct and indirect connectedness.*

4. Networks and their partitions

We can generate partitions from graphs. In order to do so, we need to know what an equivalence relation is. The following subsection shows that partitions give rise to, and can be derived from, equivalence relations.

Let us consider three examples

4.1. Relations and equivalence classes. Our aim is to consider relations on the goods space $\mathbb{R}^\ell_+$. However, we begin with three examples from outside preference theory.

Example X.1. *For any two inhabitants from Leipzig, we ask whether*

- *one is the father of the other or*
- *they are of the same sex.*

Example X.2. *For the set of integers $\mathbb{Z}$ (the numbers ..., -2, -1, 0, 1, 2, ...) , we consider the difference and examine whether this difference is an even number (i.e., from ..., $-2, 0, 2, 4$, ...).*

All three examples define relations, the first two on the set of the inhabitants from Leipzig, the last on the set of integers. Often, relations are expressed by the symbol $\sim$. To take up the last example on the set of integers, we have $5 \sim -3$ (the difference $5-(-3)=8$ is even) and $5 \nsim 0$ (the difference $5-0=5$ is odd).

DEFINITION X.3 (relation). *A relation on a set M is a subset of $M \times M$. If a tuple $(a,b) \in M \times M$ is an element of this subset, we often write $a \sim b$.*

Relations have, or have not, specific properties:

DEFINITION X.4 (properties of relations). *A relation $\sim$ on a set M is called*

- *reflexive if $a \sim a$ holds for all $a \in M$,*
- *transitive if $a \sim b$ and $b \sim c$ imply $a \sim c$ for all $a, b, c \in M$,*
- *symmetric if $a \sim b$ implies $b \sim a$ for all $a, b \in M$,*
- *asymmetric if $a \sim b$ implies $b \nsim a$ (i.e., not $b \sim a$),*
- *antisymmetric if $a \sim b$ and $b \sim a$ imply $a = b$ for all $a, b \in M$, and*
- *complete if $a \sim b$ or $b \sim a$ holds for all $a, b \in M$.*

LEMMA X.1. *On the set of integers $\mathbb{Z}$, the relation $\sim$ defined by*

$$a \sim b :\Leftrightarrow a - b \text{ is an even number}$$

is reflexive, transitive, and symmetric, but neither antisymmetric nor complete.

"$:\Leftrightarrow$" means that the expression left of the colon is defined by the expression to the right of the equivalence sign.

PROOF. We have $a - a = 0$ for all $a \in \mathbb{Z}$ and hence $a \sim a$; therefore, $\sim$ is reflexive. For transitivity, consider any three integers a, b, c that obey $a \sim b$ and $b \sim c$. Since the sum of two even numbers is even, we find that

$$\begin{aligned} & (a-b)+(b-c) \\ = \;& a-c \end{aligned}$$

is also even. This proves $a \sim c$ and concludes the proof of transitivity. Symmetry follows from the fact that a number is even if and only if its negative is even.

$\sim$ is not complete which can be seen from $0 \nsim 1$ and $1 \nsim 0$. Finally, $\sim$ is not antisymmetric. Just consider the numbers 0 and 2. $\square$

EXERCISE X.5. *Which properties have the relations "is the father of" and "is of the same sex as"? Fill in "yes" or "no":*

property	*is the father of*	*is of the same sex as*
reflexive		
transitive		
symmetric		
asymmetric		
antisymmetric		
complete		

DEFINITION X.5 (equivalence relation). *Let $\sim$ be a relation on a set M which obeys reflexivity, transitivity and symmetry. Then, any two elements $a, b \in M$ with $a \sim b$ are called equivalent and $\sim$ is called an equivalence relation. By an equivalence class of $a \in M$, we mean the set*

$$[a] := \{b \in M : b \sim a\}.$$

Our relation on the set of integers (even difference) is an equivalence relation. We have two equivalence classes:

$$\begin{aligned} [0] &= \{b \in M : b \sim 0\} = \{..., -2, 0, 2, 4, ...\} \text{ and} \\ [1] &= \{b \in M : b \sim 1\} = \{..., -3, -1, 1, 3, ...\} \end{aligned}$$

EXERCISE X.6. *Continuing the above example, find the equivalence classes* $[17]$, $[-23]$, *and* $[100]$. *Reconsider the relation "is of the same sex as". Can you describe its equivalence classes?*

Generalizing the above example, $a \sim b$ implies $[a] = [b]$ for every equivalence relation. Here comes the proof. Consider any $a' \in [a]$. We need to show $a' \in [b]$. Now, $a' \in [a]$ means $a' \sim a$. Together with $a \sim b$, transitivity implies $a' \sim b$ and hence $a' \in [b]$. We have shown $[a] \subseteq [b]$. The converse, $[b] \subseteq [a]$, can be shown similarly.

The following lemma uses the above result and the observation $a \in [a]$ which is true by reflexivity.

LEMMA X.2. *Let $\sim$ be an equivalence relation on a set M. Then, we have*

$$\begin{aligned} \bigcup_{a \in M} [a] &= M \text{ and} \\ [a] &\neq [b] \Rightarrow [a] \cap [b] = \emptyset. \end{aligned}$$

Thus, equivalence classes form a partition of the underlying set.

The other direction holds also: Once we have a partition, we can define an equivalence relation whose equivalence classes are equal to the components of the partition. Just say that two elements are related if they belong to the same component.

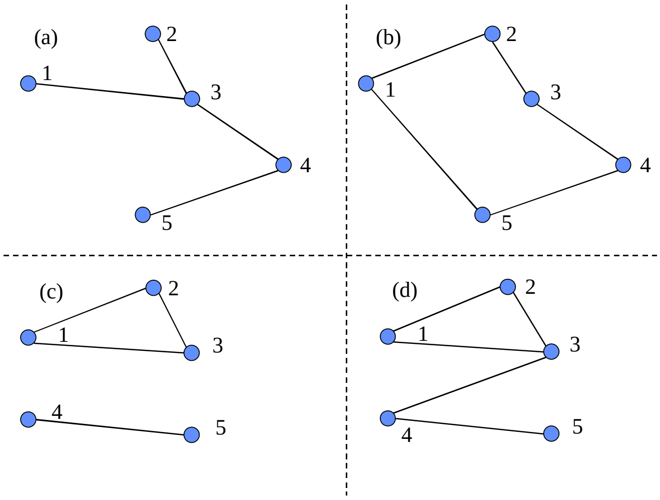

FIGURE 2. Vier Graphen

4.2. Generating partitions from graphs. According to the previous subsection, we need an equivalence relation. Here it is:

DEFINITION X.6 (connectedness as a relation). *Let $\mathcal{L}$ be a network on N. If players i and j (not necessarily $i \neq j$) are connected, we write $i \sim^{\mathcal{L}} j$, i.e., $\sim^{\mathcal{L}}$ is a relation on N.*

LEMMA X.3. *$\sim^{\mathcal{L}}$ defines an equivalence relation on N.*

EXERCISE X.7. *Show that $\sim^{\mathcal{L}}$ is an equivalence relation!*

The equivalence classes of the equivalence relation $\sim^{\mathcal{L}}$ form a partition:

DEFINITION X.7. *Let $\sim^{\mathcal{L}}$ be the equivalence relation given above. We note the resulting partition by $N/\mathcal{L}$. For any nonempty subset $S \subseteq N$, S is also partitioned (via $\sim^{\mathcal{L}(S)}$) and we define*

$$S/\mathcal{L} := S/\left(\mathcal{L}\left(S\right)\right).$$

EXERCISE X.8. *Determine the partitions of the player subset $\{1, 3, 4\}$ resulting from the four networks depicted in fig. 2.*

Do you see that a subset $K \subseteq N$ is internally connected iff $K/\mathcal{L} = \{K\}$ holds?

Assume any subset $S \subseteq N$ and consider two extreme networks, $\mathcal{L} = \emptyset$ and $\mathcal{L} = \mathcal{L}^{\text{voll}}$. We find

- that $S/\emptyset$ equals the atomic partition of S – every player is an island and
- that $S/\mathcal{L}^{\text{full}}$ equals the trivial partition $\{S\}$.

This observation can be generalized:

LEMMA X.4. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be networks on N such that $\mathcal{L}_1$ is a subnetwork of $\mathcal{L}_2$, $\mathcal{L}_1 \subseteq \mathcal{L}_2$. Then $S/\mathcal{L}_1$ is finer than $S/\mathcal{L}_2$ for every subset $S \subseteq N$.*

The proof of this lemma is not difficult. Consider any player $i \in S$. The component of $S/\mathcal{L}_1$ that contains i, $(S/\mathcal{L}_1)(i)$, consists of all the players from S that are connected to i. All these player also belong to $(S/\mathcal{L}_2)(i)$ because all the links contained in $\mathcal{L}_1$ are also contained in $\mathcal{L}_2$.

NOTATION X.1. *Given a network $\mathcal{L}$ on a player set N, the component containing player i is often written as C_i rather than $(N/\mathcal{L})(i)$.*

5. The Myerson game

5.1. Definition. The network value is due to Myerson (1977a). In builds on a coalition function $v \in \mathbb{V}_N$ and a network $\mathcal{L}$. We proceed in three steps:

- For all coalitions $S \subseteq N$,
  - find the partition $S/\mathcal{L}$ and
  - sum the worths $v(K)$ for all components K of that partition.

  These two steps define the network coalition function $v^{\mathcal{L}}$.
- Calculate the Shapley value $Sh\left(v^{\mathcal{L}}\right)$.

Accordingly, we first define the $\mathcal{L}$-game and then the network value:

DEFINITION X.8 (Myerson game). *Let $(v, \mathcal{L})$ be a network game. The Myerson game based on this network game is the coalition function $v^{\mathcal{L}}$ which is defined by*

$$v^{\mathcal{L}}(S) = \sum_{K \in S/\mathcal{L}} v(K).$$

Take an example found in Slikker & Nouweland (2001, S. 22), where we have $N = \{1,2,3\}$, the coalition function v given by

$$v(S) = \begin{cases} 0, & |S| \leq 1 \\ 60, & |S| = 2 \\ 72 & S = N \end{cases}$$

and $\mathcal{L} = \{12, 23\}$. While v is symmetrc, $v^{\mathcal{L}}$ is not. We obtain

$$v^{\mathcal{L}}(S) = \begin{cases} 0, & |S| \leq 1, S = \{1,3\} \\ 60, & S = \{1,2\}, S = \{2,3\} \\ 72 & S = N \end{cases}$$

EXERCISE X.9. *Given any coalition function $v \in \mathbb{V}_N$, determine the Myerson game $v^{\mathcal{L}}$ for $\mathcal{L} = \mathcal{L}^{full}$ and for $\mathcal{L} = \emptyset$.*

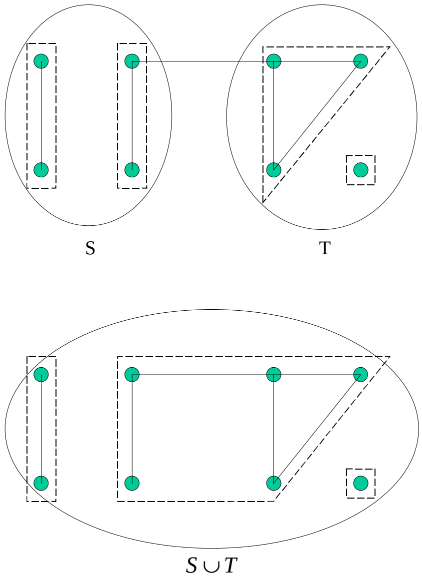

FIGURE 3. The upper partition is finer than the lower one.

Let us now look at the Myerson game for $N = \{1, 2, 3, 4\}$, the unanimity game $u_{\{1,3\}}$ and the network $\mathcal{L} = \{12, 23, 34\}$. The productive players 1 and 3 need player 2 in order to link up. Player 4 is of no help. Thus, we find

$$u^{\mathcal{L}}_{\{1,3\}}(K) = \begin{cases} 1, & K \supseteq \{1, 2, 3\} \\ 0, & \text{otherwise.} \end{cases}$$

and hence $u^{\mathcal{L}_2}_{\{1,3\}} = u_{\{1,2,3\}}$.

EXERCISE X.10. *Given* $N = \{1, 2, 3, 4\}$ *and the coalition function* $u_{\{1,3\}}$*, determine the Myerson game for* $\mathcal{L} = \{12, 23, 34, 41\}$.

5.2. Derived properties of Myerson games.

5.2.1. *Superadditivity.* Do Myerson games $v^{\mathcal{L}}$ inherit important properties from the basic game v? In particular, is $v^{\mathcal{L}}$ superadditive or convex or does $v^{\mathcal{L}}$ possess a nonempty core whenever v has these properties?

LEMMA X.5. *Let* $\mathcal{L}$ *be a network on* N. *If* $v \in \mathbb{V}_N$ *is superadditive, so is* $v^{\mathcal{L}}$.

For proof, consider two disjunct subsets S and T of N. $S/\mathcal{L}$ is a partition of S and $T/\mathcal{L}$ a partition of T. Since S and T are disjunct, $S/\mathcal{L} \cup T/\mathcal{L}$ is a partition of $S \cup T$. Consider, for example, fig. 3 where $S/\mathcal{L} \cup T/\mathcal{L}$ is the upper partition. It is finer than the lower partition, $(S \cup T)/\mathcal{L}$, because some links are cut in the upper part of the figure. Partition $(S \cup T)/\mathcal{L}$ has three components while partition $(S/\mathcal{L}) \cup (T/\mathcal{L})$ has four.

Continuing our proof, we find

$$\begin{aligned} v^{\mathcal{L}}(S\cup T) &= \sum_{C\in(S\cup T)/\mathcal{L}} v(C) \\ &\geq \sum_{C\in S/\mathcal{L}} v(C) + \sum_{C\in T/\mathcal{L}} v(C) \\ &= v^{\mathcal{L}}(R) + v^{\mathcal{L}}(S). \end{aligned}$$

5.2.2. *Convexity.* While superadditivity is transferred from v to $v^{\mathcal{L}}$, the same is not true for convexity. We follow Slikker & Nouweland (2001, p. 59) and consider the network game given by $N = \{1,2,3,4\}$, the "cycle" $L = \{12, 23, 34, 41\}$ and the coalition function v given by

$$v(S) = |S| - 1, S \neq \emptyset.$$

v is convex, as shown in chapter IV (see p. 63).

However, $v^{\mathcal{L}}$ is not convex. Note that the sets $\{1,2,3\}$, $\{1,3,4\}$ and $\{1,2,3,4\}$ are internally connected while $\{1,3\}$ is not. Therefore, we obtain

$$\begin{aligned} v^{\mathcal{L}}(\{1,2,3\}) &= v(\{1,2,3\}) = 2, \\ v^{\mathcal{L}}(\{1,3,4\}) &= v(\{1,3,4\}) = 2, \\ v^{\mathcal{L}}(\{1,2,3,4\}) &= v(\{1,2,3,4\}) = 3 \text{ und} \\ v^{\mathcal{L}}(\{1,3\}) &= v(\{1\}) + v(\{3\}) = 0 + 0 = 0. \end{aligned}$$

and player 2's marginal contributions to coalitions $\{1,3\}$ and $\{1,3,4\}$

$$\begin{aligned} MC_2^{\{1,3\}}\left(v^{\mathcal{L}}\right) &= v^{\mathcal{L}}(\{1,2,3\}) - v^{\mathcal{L}}(\{1,3\}) = 2 - 0 \\ &> 3 - 2 = v^{\mathcal{L}}(\{1,2,3,4\}) - v^{\mathcal{L}}(\{1,3,4\}) \\ &= MC_2^{\{1,3,4\}}\left(v^{\mathcal{L}}\right). \end{aligned}$$

This inequality contradicts convexity of $v^{\mathcal{L}}$.

6. The network value

6.1. Network games. We now define network games and the appropriate solution function:

Definition X.9 (network game). *For any player set N, every coalition function $v \in \mathbb{V}_N$ and any network $\mathcal{L} \in \mathfrak{L}_N$, $(v, \mathcal{L})$ is called a network game on N. The set of all network games on N is denoted by $\mathbb{V}_N^{net}$ and the set of all network games for all player sets N by $\mathbb{V}^{net}$.*

Definition X.10 (solution function for network games). *A function σ that attributes, for each network game $(v, \mathcal{L})$, a payoff to each of v's players,*

$$\sigma(v, \mathcal{L}) \in \mathbb{R}^{|N(v)|},$$

is called a solution function (on $\mathbb{V}^{net}$).

6.2. The Shapley value of the Myerson game. The sought-after network value is nothing but the Shapley value applied to the Myerson game:

DEFINITION X.11 (network value). *The network, or Myerson, value is the solution function My is given by*

$$My_i(v, \mathcal{L}) = Sh_i\left(v^{\mathcal{L}}\right), i \in N(v)$$

The network value is a generalization of the Shapley value:

LEMMA X.6. *We have $My\left(v, \mathcal{L}^{full}\right) = Sh(v)$.*

EXERCISE X.11. *Calculate the network payoffs for $N = \{1, 2, 3\}$, $\mathcal{L} = \{12, 23\}$ and the coalition functions*

- $u_{\{1,2\}}$ *and*
- $u_{\{1,3\}}$.

7. Properties of the network value

7.1. Summary of the properties. The network values has some notable properties.

THEOREM X.1 (properties of the communication value). *The network value obeys the component-decomposability axiom, the component-efficiency axiom, the superfluous-player axiom, the superfluous-link axiom, the additivity axiom and the balanced-contributions axiom.*

We consider all these properties in turn.

Networks induce partitions. Component decomposability and component efficiency means that we can restrict attention to these components. Payoffs do not depend on how the players outside are linked to each other (decomposability) and the payoff total for a component equals the component's worth (efficiency).

Superfluous players are null players not of the original game v but of the Myerson game $v^{\mathcal{L}}$. Of course, they do not affect the other players' payoffs. Similarly, superfluous links can be done away.

The fairness axiom is very close the the axiom of balanced contributions.

7.2. Components are islands. The reader is reminded of our notation $C_i = (N/\mathcal{L})(i)$ for networks $\mathcal{L}$ on N. Very close the AD value, the network value treats components as islands. This is obvious from both the component-decomposability axiom and the component-efficiency axiom:

DEFINITION X.12 (component-decomposability axiom). *A solution function σ (on $\mathbb{V}^{net}$) is said to obey component decomposability if*

$$\sigma_i(v, \mathcal{L}) = \sigma_i\left(v|_{C_i}, \mathcal{L}(C_i)\right)$$

holds for all network games $(v, \mathcal{L})$ and for all $i \in N$.

The network solution is component-decomposable. This means that the payoff for a player does not depend on how the graph $\mathcal{L}$ is structured outside player i's component. The payoff depends only on the coalition function restricted to C_i and on the network restricted to C_i.

Definition X.13 (component-efficiency axiom). *A solution function σ on $\mathbb{V}^{net}$ is said to obey the component-efficiency axiom if*

$$\sum_{i \in C_i} \sigma_i (v, \mathcal{L}) = v(C_i)$$

holds for all network games $(v, \mathcal{L})$ and for all components $C_i \in N/\mathcal{L}$.

We may conjecture the equality of the Myerson and the Aumann-Dreze value whenever both deal with the same partition, $\mathcal{P} = N/\mathcal{L}$. However, this is true only if all the links inside the components are present. In general, however, we have

$$\mu (v, \mathcal{L}) \neq \varphi^{AD} (v, N/\mathcal{L})$$

in general. Consider the player set $N = \{1, 2, 3, 4\}$, the network $\mathcal{L} = \{12, 23, 34, 41\}$ and the unanimity game $u_{\{1,3\}}$. sehen kann. According to exercise X.10 (p. 162) the Myerson game $u_{\{1,3\}}^{\mathcal{L}}$ is given by

$$u_{\{1,3\}}^{\mathcal{L}} (K) = \begin{cases} 1, & K \supseteq \{1, 2, 3\} \text{ or } K \supseteq \{1, 3, 4\} \\ 0, & \text{otherwise.} \end{cases}$$

You can confirm or believe the author that the Shapley payoffs are

$$\left(\frac{5}{12}, \frac{1}{12}, \frac{5}{12}, \frac{1}{12} \right).$$

Answer the following question and you see why the above inequality may well hold.

Exercise X.12. *Determine $N/\mathcal{L}$ and $\varphi^{AD} (v, N/\mathcal{L})$.*

7.3. Superfluous players and superfluous links. Superfluous players are the null players in Myerson games.

Definition X.14 (superfluous player). *Let $(v, \mathcal{L})$ be a network game. A player $i \in N$ is called superfluous (with respect to network game $(v, \mathcal{L})$) if*

$$v^{\mathcal{L}} (S) = v^{\mathcal{L}} (S \cup i)$$

holds for all $S \subseteq N$.

Obviously, a player is superfluous in $(v, \mathcal{L})$ if he is a null player in v. His payoff is zero and the other player are not affected if the links to this player are cut:

DEFINITION X.15 (superfluous-player axiom). *A solution function σ on $\mathbb{V}^{net}$ is said to obey the superfluous-player axiom if*

$$\sigma(v, \mathcal{L}) = \sigma(v, \mathcal{L} \backslash \mathcal{L}(i))$$

holds for all network games $(v, \mathcal{L})$ and for every superfluous player $i \in N$.

EXERCISE X.13. *Use the superfluous-player axiom, component decomposability and component efficiency to find the Myerson solution for $v = u_{\{1,2,3\}}^{\{1,2,3,4,5\}}$ and $\mathcal{L} = \{12, 23, 34, 45, 51, 13\}$ which is a more complicated game than the one offered on p. .*

Links, rather than players, may also be superfluous:

DEFINITION X.16 (superfluous link). *Let $(v, \mathcal{L})$ be a network game. A link $\ell \in \mathcal{L}$ is called superfluous (with respect to network game $(v, \mathcal{L})$) if*

$$v^{\mathcal{L}} = v^{\mathcal{L} \backslash \ell}$$

holds.

EXERCISE X.14. *Is there a superfluous link in the network game given by $v = u_{\{1,2\}}^{\{1,2,3\}}$ and $\mathcal{L} = \{12, 13\}$?*

DEFINITION X.17 (superfluous-link axiom). *A solution function σ on $\mathbb{V}^{net}$ is said to obey the superfluous-link axiom if*

$$\sigma(v, \mathcal{L}) = \sigma(v, \mathcal{L} \backslash \ell)$$

holds for all network games $(v, \mathcal{L})$ and for every superfluous links $\ell \in \mathcal{L}$.

7.4. Balanced contributions. In chapter VI we have seen that the threat to withdraw is a symmetrc property for the Shapley value (the axiom of balanced contributions). We have a similar result for networks:

DEFINITION X.18 (axiom of balanced contributions, one link). *A solution function σ on $\mathbb{V}^{net}$ is said to obey the axiom of balanced contributions if, for any two players $i, j \in N$,*

$$\sigma_i(v, \mathcal{L}) - \sigma_i(v, \mathcal{L} \backslash \{ij\}) = \sigma_j(v, \mathcal{L}) - \sigma_j(v, \mathcal{L} \backslash \{ij\})$$

holds for all network games $(v, \mathcal{L})$.

This means that two player i and j are affected equally by a dissolution of a direct link between them. The following axiom claims something similar, this time not for individual links but for all links entertained by the players:

DEFINITION X.19 (axiom of balanced contributions, all links). *A solution function σ on $\mathbb{V}^{net}$ is said to obey the axiom of balanced contributions if, for any two players $i, j \in N$,*

$$\sigma_i(v, \mathcal{L}) - \sigma_i(v, \mathcal{L} \backslash \mathcal{L}(j)) = \sigma_j(v, \mathcal{L}) - \sigma_j(v, \mathcal{L} \backslash \mathcal{L}(i))$$

holds for all network games $(v, \mathcal{L})$.

According to the second axiom, player i who cuts all his links harms player j as much as j can harm player i by removing all links $\mathcal{L}(j)$.

7.5. Axiomatization of the network value. Among the several known axiomatizations of the Myerson value, we like to highlight the two that make use of balanced contributions:

THEOREM X.2. *A solution concept σ on $\mathbb{V}^{net}$ fulfills the two axioms of*

- *component efficiency and*
- *balanced contributions (for one link or for all links) for all player sets $N \subseteq \mathbb{N}$,*

if and only if σ is the network value My.

8. Topics and literature

The main topics in this chapter are

- networks
- Myerson value

We introduce the following mathematical concepts and theorems:

- t

-

We recommend the textbook by Slikker & Nouweland (2001).

9. Solutions

Exercise X.1

Network $\mathcal{L}$ is $\mathcal{L} = \{\{1,2\},\{2,3\}\}$ or $\mathcal{L} = \{12, 23\}$. The subnetworks are $\mathcal{L}(1) = \{12\}$, $\mathcal{L}(2) = \mathcal{L}$ und $\mathcal{L}(4) = \emptyset$.

Exercise X.2

Have you found

- $\mathcal{L}(N) = \mathcal{L}$,
- $\mathcal{L}(\{1,2\}) = \begin{cases} \{12\}, & 12 \in \mathcal{L} \\ \emptyset, & 12 \notin \mathcal{L} \end{cases}$ and
- $\bigcup_{i \in N} \mathcal{L}(i) = \mathcal{L}$?

Exercise X.3

$\mathcal{L} \neq \emptyset$ implies the existence of another player $j \neq i$. If player i has no links to any other player, i is not connected (directly or indirectly) to j.

Exercise X.4

Two players i and j are directly connected if there is a $i - j$ trail containing only these two players. If such a trail does not exist but a trail containing these two players and also other players, the two players are indirectly connected.

Exercise X.5

Did you also obtain

property	is the father of	is of the same sex as
reflexive	no	yes
transitive	no	yes
symmetric	no	yes
asymmetric	yes	no
antisymmetric	no	no
complete	no	no

Exercise X.6

We have $[17] = [-23] = [1]$ and $[100] = [0]$. The relation "is of the same sex as" is an equivalence relation (see exercise X.5). The equivalent classes are "the set of all males" and "the set of all females".

Exercise X.7

We have to show three properties of $\sim^{\mathcal{L}}$:

- Reflexivity: For every player $i \in N$, $i \sim^{\mathcal{L}} i$ is immediate from the definition of connectedness.
- Symmetry: The existence of a trail from i to j implies the existence of a trail from j to i.
- Transitivity: Assume three player i, j and k from N such that i is connected to j and j is connected to k. If $i = j$ or $j = k$ holds, we are done. Otherwise, we have to construct a trail from i to k which we gain from merging the $i - j$ trail with the $j - k$ trail at j. (We refrain from giving a proper definition of merging.)

Exercise X.8

For the player subset $\{1, 3, 4\}$, we find the partitions

- (a) $\mathcal{P} = \{\{1, 3, 4\}\}$,
- (b) $\mathcal{P} = \{\{1\}, \{3, 4\}\}$,
- (c) $\mathcal{P} = \{\{1, 3\}, \{4\}\}$ and
- (d) $\mathcal{P} = \{\{1, 3, 4\}\}$.

Exercise X.9

We have $S/\mathcal{L}^{\text{full}} = \{S\}$ for every $S \subseteq N$ and therefore $v^{\mathcal{L}} = v$. For $\mathcal{L} = \emptyset$, $S/\mathcal{L}$ is the trivial partition of S and we obtain $v^{\mathcal{L}}(S) = \sum_{i \in S} v(\{i\})$.

Exercise X.10

Players 1 and 3 are not linked within $\{1, 3\}$, but are linked within both $\{1, 2, 3\}$ and $\{1, 3, 4\}$. Therefore, the Myerson game $u^{\mathcal{L}}_{\{1,3\}}$ is given by

$$\begin{aligned} u^{\mathcal{L}}_{\{1,3\}}(K) &= \begin{cases} u_{\{1,3\}}(K), & K \neq \{1, 3\} \\ 0, & K = \{1, 3\} \end{cases} \\ &= \begin{cases} 1, & K \supseteq \{1, 2, 3\} \text{ or } K \supseteq \{1, 3, 4\} \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Exercise X.11

We have $u^{\mathcal{L}}_{\{1,2\}} = u_{\{1,2\}}$ and $u^{\mathcal{L}}_{\{1,3\}} = u_{\{1,2,3\}}$ and hence

$$\begin{aligned} My\left(u_{\{1,2\}}, \mathcal{L}\right)) &= Sh\left(u_{\{1,2\}}\right) = \left(\frac{1}{2}, \frac{1}{2}, 0\right) \text{ and} \\ My\left(u_{\{1,3\}}, \mathcal{L}\right)) &= Sh\left(u_{\{1,2,3\}}\right) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right). \end{aligned}$$

Exercise X.12

$\mathcal{L}$ induces the trivial partition $N/\mathcal{L} = \{N\}$. Then the AD value equals the Shapley value and we find

$$\varphi^{AD}(v, N/\mathcal{L}) = \left(\frac{1}{2}, 0, \frac{1}{2}, 0\right).$$

Exercise X.14

Man kann vermuten, dass die Verbindung zwischen den beiden produktiven Spielern nicht überflüssig ist. Man sieht dies an

$$\begin{aligned} v^{\mathcal{L}}(\{1,2\}) &= v(\{1,2\}) = 1 \text{ und} \\ v^{\mathcal{L}\setminus\{12\}}(\{1,2\}) &= v(\{1\}) + v(\{2\}) = 0. \end{aligned}$$

Dagegen ist die Verbindung 13 überflüssig. Hierzu haben wir

$$v^{\{12,13\}}(S) = v^{\{12\}}(S) \text{ für alle } S \subseteq \{1,2,3\}$$

zu zeigen: Für alle Einerkoalitionen ist die Gleichung wegen Aufg. **??** richtig. Aus Symmetriegründen können wir uns bei den Zweierkoalitionen auf $\{1,2\}$ und $\{1,3\}$ beschränken:

$$\begin{aligned} v^{\{12,13\}}(\{1,2\}) &= 1 = v^{\{12\}}(\{1,2\}) \text{ und} \\ v^{\{12,13\}}(\{1,3\}) &= v(\{1,3\}) \\ &= 0 \\ &= v(\{1\}) + v(\{3\}) = v^{\{12\}}(\{1,3\}). \end{aligned}$$

Schließlich erhält man für die große Koalition

$$\begin{aligned} v^{\{12,13\}}(\{1,2,3\}) &= v(\{1,2,3\}) \\ &= 1 \\ &= v(\{1,2\}) + v(\{3\}) = v^{\{12\}}(\{1,2,3\}). \end{aligned}$$

10. Further exercises without solutions

Consider the coalition function v given by $N = \{1,2,3,4\}$ and

$$v(K) = \begin{cases} 0, & |K| \le 1 \\ 2, & K \in \{\{1,2\},\{1,3\},\{1,4\}\} \\ 3 & K \in \{\{2,3\},\{2,4\}\} \\ 5 & K \in \{\{3,4\},\{1,2,3\},\{1,2,4\}\} \\ 7, & K \in \{\{1,3,4\},\{2,3,4\},N\} \end{cases}$$

(1) Is v superadditive?
(2) Consider three networks $\mathcal{L}_a = \{12,14,34\}$, $\mathcal{L}_b = \{12,14,24,34\}$, $\mathcal{L}_c = \{12,13,24,34\}$. Determine the three Myerson games associated with these networks. Determine the Shapley values of these games.
(3) Comment!

CHAPTER XI

Permission and use values

1. Introduction

In this chapter, we introduce permission and use structures which can be formalized as a special kind of directed network. Both model "subordination", i.e., a superior-subordinate relationship. Permission means that a subordinate player cannot act without the permission of his superior. Using implies that the superior can automatically use the services supplied by his subordinates.

To consider a concrete example, assume a game v on $N = \{1, 2, 3\}$. Beginning with permission, let us assume that player 1 needs player 2's permission. We look at the six rank orders and determine the marginal contributions that take this permission structure (1 needs 2's permission) into account. For the rank order $(3, 1, 2)$, the marginal contributions are

- the standard one for player 3,
- no contribution for player 1 because player 2 is not present yet to give his permission, and
- the aggregate contribution $v(\{1, 2, 3\}) - v(\{3\})$ for player 2 because he brings to bear both player 1's and his own contribution.

We obtain the following table:

rank orders	MC_1	MC_2	MC_3
$(1, 2, 3)$	-	$v(\{1, 2\}) - v(\emptyset)$	$v(\{1, 2, 3\}) - v(\{1, 2\})$
$(1, 3, 2)$	-	$v(\{1, 2, 3\}) - v(\{3\})$	$v(\{3\}) - v(\emptyset)$
$(2, 1, 3)$	$v(\{1, 2\}) - v(\{2\})$	$v(\{2\}) - v(\emptyset)$	$v(\{1, 2, 3\}) - v(\{1, 2\})$
$(2, 3, 1)$	$v(\{1, 2, 3\}) - v(\{2, 3\})$	$v(\{2\}) - v(\emptyset)$	$v(\{2, 3\}) - v(\{2\})$
$(3, 1, 2)$	-	$v(\{1, 2, 3\}) - v(\{3\})$	$v(\{3\}) - v(\emptyset)$
$(3, 2, 1)$	$v(\{1, 2, 3\}) - v(\{2, 3\})$	$v(\{2, 3\}) - v(\{3\})$	$v(\{3\}) - v(\emptyset)$

For the special case of the gloves game $v = v_{\{1\},\{2,3\}}$, we find the permission payoffs $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$.

A different view is taken by the use-structure approach. If player 2 uses player 1, player 2 brings along 1 whenever he, player 2, enters the scene. For the rank order $(3, 2, 1)$, the marginal contributions are

- the standard one for player 3,

- the aggregate contribution $v(\{1,2,3\}) - v(\{3\})$ for player 2 because he uses both his own and also player 1's productivity, and
- no contribution for player 1.

We obtain the following table

rank orders	MC_1	MC_2	MC_3
$(1,2,3)$	$v(\{1\}) - v(\emptyset)$	$v(\{1,2\}) - v(\{1\})$	$v(\{1,2,3\}) - v(\{1,2\})$
$(1,3,2)$	$v(\{1\}) - v(\emptyset)$	$v(\{1,2,3\}) - v(\{1,3\})$	$v(\{1,3\}) - v(\{1\})$
$(2,1,3)$	-	$v(\{1,2\}) - v(\emptyset)$	$v(\{1,2,3\}) - v(\{1,2\})$
$(2,3,1)$	-	$v(\{1,2\}) - v(\emptyset)$	$v(\{1,2,3\}) - v(\{1,2\})$
$(3,1,2)$	$v(\{1,3\}) - v(\{3\})$	$v(\{1,2,3\}) - v(\{1,3\})$	$v(\{3\}) - v(\emptyset)$
$(3,2,1)$	-	$v(\{1,2,3\}) - v(\{3\})$	$v(\{3\}) - v(\emptyset)$

and the use payoffs $\left(\frac{1}{6}, \frac{2}{3}, \frac{1}{6}\right)$ for the gloves game $v_{\{1\},\{2,3\}}$.

Thus, the permission structure leads to a different value than the use structure although both can be represented by the same mathematical object (subordination structure) that we will introduce in section 2.

The permission-value part of this chapter draws heavily on Gilles, Owen & van den Brink (1992a), van den Brink & Gilles (1996b), and van den Brink (1999).

Both the permission and the use value are Shapley values of appropriately modified coalition functions (compare the procedure for the network value). Section 4 introduces autonomous coalitions needed for the definition of permission coalition functions while section 5 defines effective coalitions that help to define use coalition functions. Section 6 presents important axioms for permission and use values.

2. Subordination structures

"Needing permission" or "being used" is a relation on the set of players. If player 1 gives permission to, or uses, player 2, we write $2 \in \mathcal{S}(1)$, where the calligraphic letter $\mathcal{S}$ is reminiscent of subordinate.

Definition XI.1 (subordination structure). *Let N be a set (of players). A function $\mathcal{S} : N \to 2^N$ obeying $i \notin \mathcal{S}(i)$ for all $i \in N$ is called a subordination structure or a subordination relation. In case of $j \in \mathcal{S}(i)$ player i is called player j's superior and player j is called player i's subordinate. $\mathcal{S}^{-1}(j) = \{i \in N : j \in \mathcal{S}(i)\}$ is the set of player j's superiors. Depending on the interpretation, we sometimes call subordination structures permission structures or use structures.*

The set of all subordination structures on N is denoted by $\mathfrak{S}_N$. By $\mathfrak{S}$ we denote the set of subordination structures on any player set N. $\mathcal{S}_1$ is called a sub-subordination structure of $\mathcal{S}_2$ if $\mathcal{S}_1(i) \subseteq \mathcal{S}_2(i)$ for all players $i \in N$.

The full subordination structure $\mathcal{S}^{full}$ on N is defined by $\mathcal{S}^{full}(i) = N \backslash \{i\}$ for all $i \in N$. A subordination structure $\mathcal{S}^C$ gives rise to a clique $C \subseteq N$

if $\mathcal{S}^C$ *is defined by* $\mathcal{S}^C(i) = \begin{cases} C\backslash\{i\}, & i \in C \\ \emptyset, & i \notin C \end{cases}$. *The null subordination structure* $\mathcal{S}^\emptyset$ *is given by* $\mathcal{S}^\emptyset(i) = \emptyset$ *for all* $i \in N$.

EXERCISE XI.1. *Define the subordination structure* $\mathcal{S}$ *on* $N = \{1, 2, 3\}$ *where player* 1 *is the superior of players* 2 *and* 3 *while player* 3 *is player* 2*'s subordinate.*

In most applications, we want to exclude a subordination structure where

$$j \in \mathcal{S}(i) \text{ and } i \notin \mathcal{S}(j)$$

hold for some player $i, j \in N$:

DEFINITION XI.2 (asymmetric subordination structure). *A subordination structure* $\mathcal{S}$ *is called asymmetric, if*

$$j \in \mathcal{S}(i) \textit{ implies } i \notin \mathcal{S}(j)$$

for all players $i, j \in N$.

We summarize out notation for partitions, networks, and subordination structures:

partition $\mathcal{P}$	network $\mathcal{L}$	subordination structure $\mathcal{S}$
set of partitions $\mathfrak{P}$	set of networks $\mathfrak{L}$	set of sub. structures $\mathfrak{S}$
i's component $\mathcal{P}(i)$	i's links $\mathcal{L}(i)$	i's subordinates $\mathcal{S}(i)$

A subordination structure $\mathcal{S}$ tells the direct subordinates. Assume that 1 is the boss of 2 and 2 the boss of 3. Then 1 is an indirect boss of 3. The trails introduced in chapter X (p. 157) can be adapted to formalize this idea:

DEFINITION XI.3 (chain of command). *Let* $\mathcal{S}$ *be a subordination structure on* N. *The tuple* $T(i \to j) = \langle i = i_0, ..., j = i_k \rangle$ *is called a trail in* $\mathcal{S}$ *from* i *to* j *(a* $i - j$ *trail) if* $i_{\ell+1} \in \mathcal{S}(i_\ell)$ *holds for all* $\ell = 0, \ldots, k-1$. *The set of such trails is denoted by* $\mathbb{T}(i \to j)$.

The set of player i*'s direct or indirect subordinates is denoted by*

$$\hat{\mathcal{S}}(i) := \{j \in N\backslash\{i\} : \textit{a trail } T(i \to j) \textit{ exists}\}.$$

The set of player j*'s direct or indirect superiors is denoted by*

$$\hat{\mathcal{S}}^{-1}(j) := \{i \in N\backslash\{j\} : \textit{a trail } T(i \to j) \textit{ exists}\}.$$

The reader might note that an asymmetric subordination structure does not exclude $i \in \hat{\mathcal{S}}(i)$. Consider, for example, $N = \{1, 2, 3\}$ and the subordination structure $\mathcal{S}$ given by $\mathcal{S}(1) = \{2\}$, $\mathcal{S}(2) = \{3\}$, and $\mathcal{S}(3) = \{1\}$.

The definitions of $\mathcal{S}$, $\mathcal{S}^{-1}, \hat{\mathcal{S}}$, and $\hat{\mathcal{S}}^{-1}$ can be applied to coalitions rather than individual players in the obvious manner:

$$\begin{aligned}
\mathcal{S}(K) &: = \cup_{i\in K}\mathcal{S}(i),\\
\mathcal{S}^{-1}(K) &: = \cup_{i\in K}\mathcal{S}^{-1}(i),\\
\hat{\mathcal{S}}(K) &: = \cup_{i\in K}\hat{\mathcal{S}}(i),\\
\hat{\mathcal{S}}^{-1}(K) &: = \cup_{i\in K}\hat{\mathcal{S}}^{-1}(i).
\end{aligned}$$

Definition XI.4 (subordination game). *For any player set N, every coalition function $v \in \mathbb{V}_N$ and any subordination structure $\mathcal{S} \in \mathfrak{S}_N$, $(v, \mathcal{S})$ is called a subordination game. The set of all subordination games on N is denoted by $\mathbb{V}_N^{sub}$ and the set of all subordination games for all player sets N by $\mathbb{V}^{sub}$.*

3. Hierarchies

Before pursuing with the definition of permission games and use games, we have a look at a special class of subordination structures – at hierarchies. We distinguish between hierarchies and unique hierarchies:

Definition XI.5 (hierarchy). *A subordination structure $\mathcal{S} \in \mathfrak{S}_N$ is called a hierarchy on N if*

- *$\mathcal{S}$ is acyclic, i.e., if $i \notin \hat{\mathcal{S}}(i)$ holds, and*
- *$\mathcal{S}$ is connected, i.e., there exists a player $i_0 \in N$ with $\hat{\mathcal{S}}(i_0) = N\backslash\{i_0\}$.*

If, on top, $\left|\mathcal{S}^{-1}(j)\right| = 1$ for all $j \neq i_0$ holds, too, $\mathcal{S}$ is called a unique hierarchy.

Thus, we have two requirements:

- Nobody commands himself, directly or indirectly, which is a stronger requirement than just asymmetry. We exclude a subordination structure $\mathcal{S}$ defined by $\mathcal{S}(1) = \{2\}$, $\mathcal{S}(2) = \{3\}$, $\mathcal{S}(3) = \{1\}$.
- A big boss i_0 is the direct or indirect superior of all other players.

By these two requirements, the boss i_0 cannot have a superior. (Assume a player k with $i_0 \in \mathcal{S}(k)$. By connectedness, we have $k \in \hat{\mathcal{S}}(i_0)$ and hence $i_0 \in \hat{\mathcal{S}}(i_0)$, contradicting acyclicity.)

Exercise XI.2. *Look at the four subordination structures of fig. 1. Do they obey acyclicity and/or connectedness? Can you find a unique hierarchy?*

Definition XI.6 (hierarchy game). *A subordination game $(v, \mathcal{S})$ is called a hierarchy game (a unique-hierarchy game) if $\mathcal{S}$ is a hierarchy (a unique hierarchy). The set of all hierarchy games (unique-hierarchy games) on N is denoted by $\mathbb{V}_N^h$ $(\mathbb{V}_N^{uh})$.*

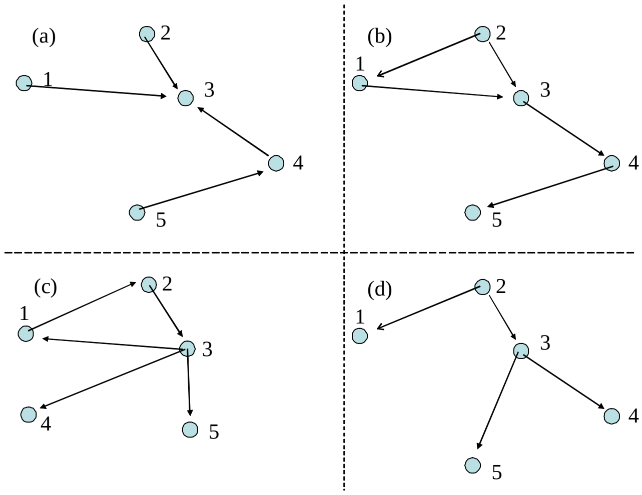

FIGURE 1. Hierarchies? Unique hierarchies?

DEFINITION XI.7 (domination). *Let* $(v, \mathcal{S})$ *be a hierarchy game with some player* i_0 *fulfilling* $\mathcal{S}(i_0) = N \backslash \{i_0\}$. *A player* $i \in N$ *dominates another player* $j \in N$, $j \neq i$, *if* i *is contained in every trail* $T(i_0, j)$. *By* $\bar{\mathcal{S}}(i)$ *we denote the set of all players that player* i *dominates.* $\bar{\mathcal{S}}^{-1}(j) := \left\{i \in N : j \in \bar{\mathcal{S}}(i)\right\}$ *is called the* j*'s set of dominating players.*

EXERCISE XI.3. *If* $\mathcal{S}$ *is a unique hierarchy, domination of* j *by* i *can be expressed by ... .*

For the axiomatizations, we need to delete a directed link from a hierarchy. Starting with a hierarchy $\mathcal{S}$ and considering a player j with at least two superiors ($\left|\mathcal{S}^{-1}(j)\right| \geq 2$), the deletion of the directed link between players h and j leads to the subordination structure $\mathcal{S}_{-(h,j)}$ which is defined by

$$\mathcal{S}_{-(h,j)}(i) = \begin{cases} \mathcal{S}(i) \setminus \{j\}, & i = h \\ \mathcal{S}(i), & i \neq h \end{cases}$$

Do you see that $\mathcal{S}_{-(h,j)}$ is a hierarchy if $\mathcal{S}$ is one? How about deleting links from unique hierarchies?

4. Autonomous coalitions and the permission game

In this chapter, we simultaneously introduce the permission and the use values. We indicated the definitions by way of marginal-contributions in the introduction. An alternative approach proceeds in two steps, similar to the network value. We first modify the given coalition function and then apply the Shapley value to that modified coalition function.

For the permission value, we need to define autonomous coalitions (see Gilles et al. 1992a, p. 281). An autonomous coalition contains all the superiors of players within that coalition and also the superiors of these superiors.

DEFINITION XI.8 (autonomous coalition). *Let $\mathcal{S}$ be a subordination structure on N. A coalition $K \subseteq N$ is called autonomous if $\hat{\mathcal{S}}^{-1}(K) \subseteq K$ holds.*

EXERCISE XI.4. *Consider the subordination structure $\mathcal{S}$ on $N = \{1, ..., 5\}$ given by*

$$\begin{aligned} \mathcal{S}(1) &= \{3\}, \\ \mathcal{S}(2) &= \emptyset, \\ \mathcal{S}(3) &= \{4\}, \\ \mathcal{S}(4) &= \{1\}, \\ \mathcal{S}(5) &= \{3\}. \end{aligned}$$

Find all the autonomous coalitions! How about the coalition $\{1, 3, 4\}$? How about the empty set?

The empty set and the grand coalition are always autonomous. Can you show that the union of two autonomous coalitions is autonomous? How about the intersection?

What worth should we accord to a coalition where some players have direct or indirect superiors outside? Gilles et al. (1992a, p. 280) distinguish between the conjunctive and the disjunctive approach. Under the first, we disregard players that have any direct or indirect superiors outside. Under the second, we disregard only those players where all direct or indirect superiors are outside.

In this chapter, we concentrate on the conjunctive approach:

DEFINITION XI.9 (autonomous subset). *Let $v \in \mathbb{V}_N$ be a coalition function, let $\mathcal{S} \in \mathfrak{S}_N$ be a subordination structure, and $K \subseteq N$ be a coalition. K's autonomous subset $aut(K)$ is defined by*

$$aut(K) := \bigcup_{\substack{A \subseteq K, \\ A \text{ autonomous}}} A.$$

Thus, a coalition's autonomous subset (called sovereign part by Gilles et al. 1992a, p. 281) is its largest autonomous subset.

DEFINITION XI.10 (permission game). *Let $(v, \mathcal{S})$ be a subordination game. The permission game based on this subordination game is the coalition function $v^{\mathcal{S}}$ which is defined by*

$$v^{\mathcal{S}}(K) = v(aut(K)).$$

EXERCISE XI.5. *Let K be an autonomous coalition under the subordination structure $\mathcal{S}$. Determine $v^{\mathcal{S}}(K)$!*

EXERCISE XI.6. *Determine the permission games* $u_{\{1,2\}}^{\mathcal{S}_a}$ *and* $u_{\{1,2\}}^{\mathcal{S}_b}$ *for* $N = \{1,2,3\}$ *and*

$$\begin{aligned} \mathcal{S}_a(1) &= \{2\}, \mathcal{S}_a(2) = \{3\}, \mathcal{S}_a(3) = \emptyset \text{ and} \\ \mathcal{S}_b(1) &= \{2\}, \mathcal{S}_b(2) = \emptyset, \mathcal{S}_b(3) = \{1\}. \end{aligned}$$

LEMMA XI.1. *Let* v *and* w *be coalition functions on* N. *The permission game* $(v+w)^{\mathcal{S}}$ *equals the sum of the permission games* $v^{\mathcal{S}} + w^{\mathcal{S}}$.

The proof is not difficult and follows from

$$\begin{aligned} (v+w)^{\mathcal{S}}(K) &= (v+w)(aut(K)) \text{ (definition permission game)} \\ &= v(aut(K)) + w(aut(K)) \text{ (vector sum)} \\ &= v^{\mathcal{S}}(K) + w^{\mathcal{S}}(K) \text{ (definition permission game).} \end{aligned}$$

LEMMA XI.2. *Let* $\mathcal{S}$ *be a subordination structure. If* v *is a monotonic coalition function, so is the permission game* $v^{\mathcal{S}}$.

Consider two coalitions E and F with $E \subseteq F$ for a proof. Because of

$$aut(E) = \bigcup_{\substack{A \subseteq E, \\ A \text{ autonomous}}} A \subseteq \bigcup_{\substack{A \subseteq F, \\ A \text{ autonomous}}} A = aut(F)$$

we have $v^{\mathcal{S}}(E) = v(aut(E)) \leq v(aut(F)) = v^{\mathcal{S}}(F)$.

DEFINITION XI.11 (solution function for subordination games). *A function* σ *that attributes, for each subordination game* $(v,\mathcal{S})$, *a payoff to each of* v*'s players,*

$$\sigma(v,\mathcal{S}) \in \mathbb{R}^{|N(v)|},$$

is called a solution function (on $\mathbb{V}^{sub}$*).*

DEFINITION XI.12 (permission value). *The permission value is the solution function* Per *given by*

$$Per_i(v,\mathcal{S}) = Sh_i\left(v^{\mathcal{S}}\right), i \in N(v)$$

where $v^{\mathcal{S}}$ *is the permission game based on* $\mathcal{S}$.

LEMMA XI.3. *We have* $Per(v,\mathcal{S}) = Sh(v)$ *for the null subordination structure* $\mathcal{S}$ *given by* $\mathcal{S}(i) = \emptyset$ *for all* $i \in N$.

EXERCISE XI.7. *Calculate the permission payoffs for* $N = \{1,2,3\}$ *and the subordination structure* $\mathcal{S}$ *given by* $\mathcal{S}(1) = \{2\}$, $\mathcal{S}(2) = \emptyset, \mathcal{S}(3) = \{1\}$ *and the coalition functions*

- $u_{\{1,2\}}$ *and*
- $u_{\{1,3\}}$.

Clique-players obtain the same permission payoff:

LEMMA XI.4. *Let* $C \subseteq N$ *be a clique and let* $\mathcal{S}^C$ *be the associated subordination structure. Then, we have* $Per_i\left(v,\mathcal{S}^C\right) = Per_j\left(v,\mathcal{S}^C\right)$ *for all* $i,j \in C$.

5. Effective coalitions and the use game

Paralleling the autonomous coalitions and the permission game, we now introduce the concept of effective coalitions and the use game. Effective coalitions are those that contain all the players that may be used by the players in the game.

DEFINITION XI.13 (effective coalition). *Let $\mathcal{S}$ be a subordination structure on N. A coalition $K \subseteq N$ is called effective if $\mathcal{S}(K) \subseteq K$ holds.*

Do you see that a coalition K is effective if and only if $\hat{\mathcal{S}}(K) \subseteq K$ holds?

EXERCISE XI.8. *Consider the subordination structure $\mathcal{S}$ on $N = \{1, ..., 5\}$ given by*

$$\begin{aligned} \mathcal{S}(1) &= \{3\}, \\ \mathcal{S}(2) &= \emptyset, \\ \mathcal{S}(3) &= \{4\}, \\ \mathcal{S}(4) &= \{1\}, \\ \mathcal{S}(5) &= \{3\}. \end{aligned}$$

Find all the effective coalitions! How about the coalition $\{1, 3, 4\}$? How about the empty set?

The empty set and the grand coalition are always autonomous.

- Can you show that the union of two autonomous coalitions is autonomous.
- How about the intersection?

The worth to be attributed to a coalition seems obvious: for a coalition K just form the union of K with $\hat{\mathcal{S}}(K)$:

DEFINITION XI.14 (effective superset). *Let $v \in \mathbb{V}_N$ be a coalition function, let $\mathcal{S} \in \mathfrak{S}_N$ be a subordination structure, and $K \subseteq$ be a coalition. K's effective superset $eff(K)$ is defined by*

$$eff(K) := K \cup \hat{\mathcal{S}}(K).$$

Thus, a coalition's effective superset ist its smallest effective superset.

DEFINITION XI.15 (use game). *Let $(v, \mathcal{S})$ be a subordination game. The use game based on this subordination game is the coalition function $v^{\mathcal{S}}$ which is defined by*

$$v^{\mathcal{S}}(K) = v(eff(K)).$$

EXERCISE XI.9. *Determine the use games $u_{\{1,2\}}^{\mathcal{S}_a}$ and $u_{\{1,2\}}^{\mathcal{S}_b}$ for $N = \{1, 2, 3\}$ and*

$$\begin{aligned} \mathcal{S}_a(1) &= \{2\}, \mathcal{S}_a(2) = \{3\}, \mathcal{S}_a(3) = \emptyset \text{ and} \\ \mathcal{S}_b(1) &= \{3\}, \mathcal{S}_b(2) = \emptyset, \mathcal{S}_b(3) = \{1\}. \end{aligned}$$

LEMMA XI.5. *Let v and w be coalition functions on N. The use game $(v+w)^{\mathcal{S}}$ equals the sum of the use games $v^{\mathcal{S}}+w^{\mathcal{S}}$.*

EXERCISE XI.10. *Can you prove the above lemma?*

LEMMA XI.6. *Let $\mathcal{S}$ be a subordination structure. If v is a monotonic coalition function, so is the use game $v^{\mathcal{S}}$.*

EXERCISE XI.11. *Show that monotonicity of v is passed on to $v^{\mathcal{S}}$.*

DEFINITION XI.16 (use value). *The use value is the solution function Use given by*

$$Use_i(v,\mathcal{S}) = Sh_i\left(v^{\mathcal{S}}\right), i \in N(v)$$

where $v^{\mathcal{S}}$ is the use game based on $\mathcal{S}$.

LEMMA XI.7. *We have $Use(v,\mathcal{S}) = Sh(v)$ for the (null) subordination structure $\mathcal{S}$ given by $\mathcal{S}(i)=\emptyset$ for all $i \in N$.*

EXERCISE XI.12. *Calculate the use payoffs for $N=\{1,2,3\}$ and the subordination structure $\mathcal{S}$ given by $\mathcal{S}(1)=\{2\}$, $\mathcal{S}(2)=\emptyset, \mathcal{S}(3)=\{1\}$ and the coalition functions*

- *$u_{\{1,2\}}$ and*
- *$u_{\{1,3\}}$.*

LEMMA XI.8. *Let $C \subseteq N$ be a clique and let $\mathcal{S}^C$ be the associated subordination structure. Then, we have $Use_i\left(v,\mathcal{S}^C\right) = Use_j\left(v,\mathcal{S}^C\right)$ for all $i,j \in C$.*

The grand coalition as a special clique implies the full subordination structure. Lemmata XI.4 and XI.8 imply equal payoffs for all players:

COROLLARY XI.1. *For the full subordination structure $\mathcal{S}^{full}: N \to 2^N$ defined by $\mathcal{S}^{full}(i) = N\backslash\{i\}$ for all $i \in N$, we have*

$$Use\left(v,\mathcal{S}^{full}\right) = Per\left(v,\mathcal{S}^{full}\right) = \left(\frac{v(N)}{n},...,\frac{v(N)}{n}\right)$$

for all coalition functions $v \in \mathbb{V}_N$.

6. Important axioms for permission and use values

Solution functions σ on $\mathbb{V}^{\text{sub}}$ might obey one or several of the following axioms. The subordination structures may be hierarchies (as in van den Brink 2003), but need not be.

DEFINITION XI.17 (additivity axiom). *A solution function σ on $\mathbb{V}_N^h$ is said to obey the additivity axiom if we have*

$$\sigma(v+w,\mathcal{S}) = \sigma(v,\mathcal{S}) + \sigma(w,\mathcal{S})$$

for any two coalition functions $v,w \in \mathbb{V}$ with $N(v)=N(w)$ and any subordination structure $\mathcal{S} \in \mathfrak{S}_{N(v)}$.

EXERCISE XI.13. *Does the additivity axiom hold for the permission value and/or the use value? Hint: Consult the lemmata XI.1 and XI.5.*

DEFINITION XI.18 (null-player axiom). *A solution function σ on $\mathbb{V}_N^{sub}$ is said to obey the null-player axiom if we have*

$$\sigma_i(v, \mathcal{S}) = 0$$

for all subordination games $(v, \mathcal{L})$ and for every null player $i \in N$.

The null-player axiom does not hold for the permission value (consider the permission payoffs for the unanimity game $u_{\{1,2\}}$, exercise XI.7). Neither does the use value fulfill the null-player axiom (exercise XI.12).

DEFINITION XI.19 (inessential player). *Let $(v, \mathcal{S})$ be a subordination game. A player $i \in N$ is called inessential (with respect to $(v, \mathcal{S})$) if*

$$v(K) = v(K \cup \{j\})$$

holds for all $K \subseteq N$ and for all $j \in \{i\} \cup \hat{\mathcal{S}}(i)$.

DEFINITION XI.20 (inessential-player axiom). *A solution function σ on $\mathbb{V}_N^{sub}$ is said to obey the inessential-player axiom if*

$$\sigma_i(v, \mathcal{S}) = 0$$

holds for all subordination games $(v, \mathcal{S})$ and for every inessential player $i \in N$.

An inessential player $i \in N$ with respect to $(v, \mathcal{S})$ is a null player with respect to $v^{\mathcal{S}}$ for both the permission-structure and the use-structure interpretation. We begin with the permission value. Let $K \subseteq N$ be any coalition that does not contain i. The set $\Delta K := aut(K \cup \{i\}) \backslash aut(K)$ contains

- player i if i does not have any superiors outside and
- some players from K for whom i is a superior.

Thus, we find $\Delta K \backslash \{i\} = K \cap \hat{\mathcal{S}}(i)$ and

$$\begin{aligned}
& v^{\mathcal{S}}(K \cup \{i\}) - v^{\mathcal{S}}(K) \\
= \;& v(aut(K \cup \{i\})) - v(aut(K)) \\
= \;& \sum_{j \in \Delta K} MC_j^{K_j}(v) = \sum_{j \in \{i\} \cup \left(K \cap \hat{\mathcal{S}}(i)\right)} MC_j^{K_j}(v)
\end{aligned}$$

with suitably chosen $K_j \subseteq N$. Since i is inessential, all these marginal contributions are zero so that i is indeed a null palyer with respect to $v^{\mathcal{S}}$.

EXERCISE XI.14. *Show that the use value obeys the inessential-player axiom.*

A player is necessary if the worth of any coalition that he does not belong to is zero. Within in the framework of simple games, such a player would be addressed as a veto player.

DEFINITION XI.21 (necessary player). *Let $(v, \mathcal{S})$ be a subordination game. A player $i \in N$ is called necessary (with respect to $(v, \mathcal{S})$) if*

$$v(K) = 0$$

holds for all $K \subseteq N \backslash \{i\}$.

DEFINITION XI.22 (necessary-player axiom). *A solution function σ on $\mathbb{V}_N^{sub}$ is said to obey the necessary-player axiom if*

$$\sigma_i(v, \mathcal{S}) \geq \sigma_j(v, \mathcal{S})$$

holds for every monotionic coalition function v and for every necessary player $i \in N$.

According to van den Brink & Gilles (1996b, p. 129), the permission value fulfills the necessary player axiom. Indeed, a necessary player with respect to v is also a necessary player with respect to $v^{\mathcal{S}}$. If a necessary player i is not contained in some coalition K, it is also not contained in $aut(K) \subseteq K$ and we have $v^{\mathcal{S}}(K) = v(aut(K)) = 0$. We now compare player i's marginal contributions with those of any other player j:

- For a coalition $E \subseteq N \backslash \{i, j\}$ we have $v^{\mathcal{S}}(E \cup \{j\}) = v^{\mathcal{S}}(E) = 0$ because i is contained in neither E nor $E \cup \{j\}$. By the monotionicity of v and hence of $v^{\mathcal{S}}$ (see lemma XI.2), we have $v^{\mathcal{S}}(E \cup \{i\}) \geq v^{\mathcal{S}}(E)$. Therefore, $MC_i^E \geq MC_j^E = 0$ for all those coalitions E that host neither i nor j.
- The previous point implies $v^{\mathcal{S}}(E \cup \{i\}) \geq v^{\mathcal{S}}(E \cup \{j\})$ and hence

$$\begin{aligned} MC_j^{E \cup \{i\}}\left(v^{\mathcal{S}}\right) &= v^{\mathcal{S}}(E \cup \{i\} \cup \{j\}) - v^{\mathcal{S}}(E \cup \{i\}) \\ &\leq v^{\mathcal{S}}(E \cup \{j\} \cup \{i\}) - v^{\mathcal{S}}(E \cup \{j\}) \\ &= MC_j^{E \cup \{i\}}\left(v^{\mathcal{S}}\right). \end{aligned}$$

Since the Shapley value is the average of the marginal contributions with respect to all rank orders, the permission value fulfills the superior-player axiom and hence the dominant-player axiom.

However, the use value does not fulfill the necessary-player axiom. The reason is that the necessary player i (with respect to v!) need to be necessary with respect to $v^{\mathcal{S}}$. Consider $N = \{1, 2, 3\}$, the unanimity game $u_{\{2,3\}}$ and the subordination structure $\mathcal{S}$ given by $\mathcal{S}(1) = \{2\}$ and $\mathcal{S}(2) = 3$. The productive player 3 is a necessary player (as is player 2). But his payoff is zero which you can see by a rank-order argument. If player 3 is first, the productive player 2 is still missing so that player 3's marginal contribution is 0. If players 1 or 2 are first, both their marginal contributions are 1. Therefore, we find the use payoffs $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$ with $Use_3\left(u_{\{2,3\}}, \mathcal{S}\right) < Use_1\left(u_{\{2,3\}}, \mathcal{S}\right)$.

Definition XI.23 (efficiency axiom). *A solution function* σ *on* $\mathbb{V}_N^{sub}$ *is said to obey the efficiency axiom if*

$$\sum_{i\in N}\sigma_i(v,\mathcal{S})=v(N)$$

holds for all subordination games $(v,\mathcal{S})$.

Exercise XI.15. *Does the efficiency axiom hold for the permission value and/or the use value?*

Definition XI.24 (dominant player). *A solution function* σ *on* $\mathbb{V}_N^h$ *(!) is said to obey the dominant-player axiom if we have*

$$\sigma_i(v,\mathcal{S})\geq\sigma_j(v,\mathcal{S})$$

for every monotionic coalition function v *whenever player* $i\in N$ *dominates player* $j\neq i$ *(see definition XI.7, p. 175).*

Definition XI.25 (superior player). *A solution function* σ *on* $\mathbb{V}_N^{sub}$ *is said to obey the superior-player axiom if we have*

$$\sigma_i(v,\mathcal{S})\geq\sigma_j(v,\mathcal{S})$$

for every monotionic coalition function v *and for every two players* $i,j\in N$ *with* $j\in\mathcal{S}(i)$.

The superior-player axiom is stronger than the dominant-player axiom because if i dominates j, i is also j's superior. That is, if we can show that a value fulfills the superior-player axiom, it also fulfills the dominant-player axiom.

We show that both axioms are fulfilled by both the permission and the use value. Beginning with the permission structure $\mathcal{S}$, consider players i and j with $j\in\mathcal{S}(i)$. We show that i's marginal contributions are higher than j's ones.

- For a coalition $E\subseteq N\backslash\{i,j\}$ we have $v^{\mathcal{S}}(E\cup\{j\})=v^{\mathcal{S}}(E)$ because j does not have i's permission. By the monotionicity of v and hence of $v^{\mathcal{S}}$ (see lemma XI.2), we have $v^{\mathcal{S}}(E\cup\{i\})\geq v^{\mathcal{S}}(E)$. Therefore, $MC_i^E\geq MC_j^E=0$ for all those coalitions E that host neither i nor j.
- The previous point implies $v^{\mathcal{S}}(E\cup\{i\})\geq v^{\mathcal{S}}(E\cup\{j\})$ and hence

$$\begin{aligned}MC_j^{E\cup\{i\}}\left(v^{\mathcal{S}}\right) &= v^{\mathcal{S}}(E\cup\{i\}\cup\{j\})-v^{\mathcal{S}}(E\cup\{i\})\\ &\leq v^{\mathcal{S}}(E\cup\{j\}\cup\{i\})-v^{\mathcal{S}}(E\cup\{j\})\\ &= MC_j^{E\cup\{i\}}\left(v^{\mathcal{S}}\right).\end{aligned}$$

Since the Shapley value is the average of the marginal contributions with respect to all rank orders, the permission value fulfills the superior-player axiom and hence the dominant-player axiom.

EXERCISE XI.16. *Show that the use value also fulfills the superior-player axiom (and hence the dominant-player axiom).*

The last axiom to consider is a fairness axiom in the spirit of balanced contributions known from chapters VI and X. Consider a player j with at least two superiors h and g. If h ceases to be j's superior, j and g are equally affected.

DEFINITION XI.26 (balanced contributions). *A solution function σ on $\mathbb{V}_N^h$ (!) is said to obey the balanced-contribution axiom if, for all players $h, j, g \in N$ with $h \neq g$ and $j \in S(g) \cap S(h)$, we have*

$$\sigma_j(v, \mathcal{S}) - \sigma_j\left(v, \mathcal{S}_{-(h,j)}\right) = \sigma_i(v, \mathcal{S}) - \sigma_i\left(v, \mathcal{S}_{-(h,j)}\right) \text{ for all } i \in \{g\} \cup \bar{\mathcal{S}}^{-1}(g)$$

holds for all subordination games $(v, \mathcal{S})$.

Note that the equality does not only apply to g himself but also to all players that dominate g (the players from $\bar{\mathcal{S}}^{-1}(g)$).

We present two different axiomatizations. The first refers to hierarchies:

THEOREM XI.1 (axiomatization of the permission value). *The permission value on hierarchies is axiomatized by the additivity axiom, the inessential-player axiom, the necessary-player axiom, the efficiency axiom, the dominant-player axiom, and the balanced-contribution axiom.*

The second axiomatization refers to any subordination structures:

THEOREM XI.2 (axiomatization of the permission value). *The permission value on subordination structures is axiomatized by the additivity axiom, the inessential-player axiom, the necessary-player axiom, the efficiency axiom, and the superior-player axiom.*

7. Topics and literature

The main topics in this chapter are

- outside-option values
- Casajus value
- Wiese value
- component efficiency
- splitting axiom

We introduce the following mathematical concepts and theorems:

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We recommend.

8. Solutions

Exercise XI.1

The subordination structure $\mathcal{S}$ on $\{1,2,3\}$ is given by $\mathcal{S}(1) = \{2,3\}$, $\mathcal{S}(2) = \{3\}$, and $\mathcal{S}(3) = \emptyset$.

Exercise XI.2

Graph (a) obeys acyclicity, but violates connectedness. It is not a hierarchy.

Graph (b) depicts a hierarchy with player 2 as the boss, but not a unique one – player 3 has two superiors.

Graph (c) obeys connectedness where players 1, 2 and 3 are all big bosses. The graph violates acyclicity. It is not a hierarchy.

Graph (d) is a unique hierarchy.

Exercise XI.3

If $\mathcal{S}$ is a unique hierarchy, domination of j by i can be expressed by $j \in \hat{\mathcal{S}}(i)$.

Exercise XI.4

The autonomous coalitions are

$$\emptyset, \{2\}, \{5\}, \{2,5\}, \{1,3,4,5\}, N.$$

Exercise XI.5

Since K is autonomous, we have

$$\begin{aligned} aut(K) &= \bigcup_{\substack{A \subseteq K, \\ A \text{ autonomous}}} A \\ &= K \end{aligned}$$

and hence $v^{\mathcal{S}}(K) = v(aut(K)) = v(K)$.

Exercise XI.6

We find $u_{\{1,2\}}^{\mathcal{S}_a} = u_{\{1,2\}}$ and $u_{\{1,2\}}^{\mathcal{S}_b} = u_N$.

Exercise XI.7

We have $u_{\{1,2\}}^{\mathcal{S}} = u_N$ and hence $Per\,(v, \mathcal{S}) = Sh\,(u_N) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. By $u_{\{1,3\}}^{\mathcal{S}} = u_{\{1,3\}}$ we obtain $Per\,(v, \mathcal{S}) = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$.

Exercise XI.8

The effective coalitions are

$$\emptyset, \{2\}, \{1,3,4\}, \{1,2,3,4\}, \{1,3,4,5\}, \{1,2,3,4,5\}$$

Exercise XI.9

We find $u_{\{1,2\}}^{\mathcal{S}_a} = u_{\{1\}}$ and

$$u_{\{1,2\}}^{\mathcal{S}_b}(K) = \begin{cases} 1, & 1 \in K \text{ or } 3 \in K \\ 0, & \text{otherwise} \end{cases}$$

Exercise XI.10

Just copy the proof of lemma XI.1 (p. 177). You find

$$\begin{aligned} (v+w)^{\mathcal{S}}(K) &= (v+w)\,(eff\,(K)) \text{ (definition use game)} \\ &= v\,(eff\,(K)) + w\,(eff\,(K)) \text{ (vector sum)} \\ &= v^{\mathcal{S}}(K) + w^{\mathcal{S}}(K) \text{ (definition use game).} \end{aligned}$$

Exercise XI.11

Assume two coalitions E and F fulfilling $E \subseteq F \subseteq N$. First of all, $E \subseteq F$ implies

$$eff\,(E) = E \cup \hat{\mathcal{S}}(E) \subseteq F \cup \hat{\mathcal{S}}(F) = eff\,(F).$$

By the monotionicity of v, we find $v^{\mathcal{S}}(E) = v\,(eff\,(E)) \leq v\,(eff\,(F)) = v^{\mathcal{S}}(F)$.

Exercise XI.12

We find

$$u_{\{1,2\}}^{\mathcal{S}}(K) = \begin{cases} 1, & 1 \in K \text{ or } 3 \in K \\ 0, & \text{otherwise} \end{cases}$$

and hence $Use\left(u_{\{1,2\}}\right) = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$. By $u_{\{1,3\}}^{\mathcal{S}} = u_{\{3\}}$ the use payoffs are $(0,0,1)$ for the second unanimity game.

Exercise XI.13

The Shapley value is additive so that we find

$$\begin{aligned} & Sh\left((v+w)^{\mathcal{S}}\right) \\ =\ & Sh\left(v^{\mathcal{S}} + w^{\mathcal{S}}\right) \\ =\ & Sh\left(v^{\mathcal{S}}\right) + Sh\left(w^{\mathcal{S}}\right) \end{aligned}$$

for the permission value Per as well as for the use value Use.

Exercise XI.14

Let $K \subseteq N$ be any coalition that does not contain i. The set $\Delta K := eff\,(K \cup \{i\}) \backslash eff\,(K)$ contains

- player i if i does not have any superiors inside K and
- some players from outside K for whom i but not a player from K is a superior.

Thus, we find $\Delta K\backslash\{i\} = \hat{\mathcal{S}}(i)\backslash\hat{\mathcal{S}}(K)$

$$\begin{aligned} & v^{\mathcal{S}}(K\cup\{i\}) - v^{\mathcal{S}}(K) \\ = \; & v(eff(K\cup\{i\})) - v(eff(K)) \\ = \; & \sum_{j\in\Delta K} MC_j^{K_j}(v) = \sum_{j\in\{i\}\cup\hat{\mathcal{S}}(i)\backslash\hat{\mathcal{S}}(K)} MC_j^{K_j}(v) \end{aligned}$$

with suitably chosen $K_j \subseteq N$. Since i is inessential, this sum of marginal contributions is zero so that i is indeed a null palyer with respect to $v^{\mathcal{S}}$.

Exercise XI.15

Yes, the additivity axiom holds for both values because it holds for the Shapley value.

Exercise XI.16

The proof is very similar to the one for the permission value: For a coalition $E \subseteq N\backslash\{i,j\}$ we have $v^{\mathcal{S}}(E\cup\{j\}) \leq v^{\mathcal{S}}(E\cup\{i\})$ because $v^{\mathcal{S}}$ is monotonic (see lemma XI.6) and $eff(E\cup\{j\})$ is a subset of $eff(E\cup\{i\})$ by $j \in \mathcal{S}(i)$. Therefore, we find

$$\begin{aligned} MC_j^E\left(v^{\mathcal{S}}\right) &= v^{\mathcal{S}}(E\cup\{j\}) - v^{\mathcal{S}}(E) \\ &\leq v^{\mathcal{S}}(E\cup\{i\}) - v^{\mathcal{S}}(E) \\ &= MC_i^E\left(v^{\mathcal{S}}\right). \end{aligned}$$

The rest of the proof is just a copy of the proof for the permission value.

9. Further exercises without solutions

CHAPTER XII

Hierarchies, wages, and allocation

1. Introduction

In this chapter (which is the joint output by André Casajus, Tobias Hiller and Harald Wiese), we suggest a wage scheme that accounts for the hierarchical structure of an enterprise. Besides results on how the hierarchy affects wage differentials between levels of the hierarchy, we deal with the allocation of employees to the different levels.

- Die Position innerhalb der Hierarchie eines Unternehmens spielt eine entscheidende Rolle bei der Bestimmung der Entlohnung eines Mitarbeiters. Explizit wird dies insbesondere bei Beschäftigten im Strukturvertrieb deutlich.
- Im kürzlich erschienenen Aufsatz von van den Brink (2008) wird die hierarchische Struktur eines Unternehmens mit Hilfe von Konzepten der kooperativen Spieltheorie abgebildet. In seinem Ansatz sind dabei die Beziehungen zwischen den Vorgesetzten und ihren direkten Mitarbeitern über alle Beschäftigten gleich stark.
- Der vorliegende Aufsatz nutzt ebenfalls die kooperative Spieltheorie, um hierarchische Strukturen von Unternehmen abzubilden. Er lässt dabei verschiedene Intensitäten der Beziehungen zwischen Vorgesetzten und Mitarbeitern zu. Diese Intensität gibt dabei Auskunft darüber, in welchem Ausmaß der Vorgesetzte am Erfolg bzw. Misserfolg seines Mitarbeiters partizipiert.
- Für diese reichhaltigere hierarchische Struktur eines Unternehmens wird ein Entlohnungsschema vorgestellt, das die Standardergebnisse der Literatur repliziert.
- Damit wird ein Bezugsrahmen begründet, der zum einen die Ableitung weiterer theoretischer Ergebnisse erlaubt und zudem empirische Untersuchungen zum Themengebiet „Hierarchie und Entlohnung“ anleiten kann.

2. Einleitung

Die Lohnstruktur innerhalb von Unternehmen ist eines der Interessensgebiete personalwirtschaftlicher Untersuchungen. Eine wichtige Rolle bei der Bestimmung der Löhne in Unternehmen spielt die Hierarchie des Unternehmens. Hierarchien im Sinne dieses Aufsatzes sind dabei zunächst von Einkommensleitern (engl.: job ladder) abzugrenzen. Hierarchien sind dadurch gekennzeichnet, dass durch sie eine klare Vorgesetzten-Mitarbeiter-Struktur geschaffen wird. Jedem Mitarbeiter, bis auf jenem an der Spitze des Unternehmens, kann genau ein direkter Vorgesetzter zugeordnet werden (Radner 1992, Meagher 2001). Dieser Vorgesetzte kontrolliert seine direkten Mitarbeiter und kann ihnen Anweisungen erteilen. Bei Einkommensleitern hingegen dienen verschiedene Mitarbeiterebenen ausschließlich der Lohndifferenzierung, d.h. den Mitarbeitern einer Ebene sind keine Mitarbeiter der nächstniedrigeren Ebene zugewiesen (Lazear & Rosen 1981, Carmichael 1983, Prendergast 1993).

Explizit sind die Effekte einer Hierarchie auf Entlohnung und Allokation von Mitarbeitern im Strukturvertrieb (auch: network marketing, multi-level marketing), wie er beispielsweise bei Amway, der Deutschen Vermögensberatung, der HMI-Organisation oder AWD Anwendung findet (Frehrking & Schöffski 1994), erkennbar. Zum einen hängt die Entlohnung der Mitarbeiter von der eigenen Produktivität (Verkaufsleistung) ab. Zum anderen profitieren sie von der Produktivität der Mitarbeiter in der bei ihnen beginnenden Subhierarchie.

Anhand der Festlegung des prozentualen Bonus' bei Amway wird dies deutlich. Amway liefert Produkte an so genannte Geschäftspartner. Diese Produkte können zum einen dem Selbstverbrauch dienen, zum anderen jedoch mit einer Handelsspanne von 30% weiterverkauft werden. Diese beiden Aktivitäten beschreiben die produktive Leistung der Geschäftspartner. Daneben werben Geschäftspartner weitere Geschäftspartner, wodurch sich eine Hierarchie aufbaut. Der Bonus steigt in Stufen und hängt von der Umsatzsumme ab, die der Geschäftspartner selbst (eigene Produktivität) und alle von ihm direkt oder indirekt geworbenen Geschäftspartner (Produktivität der Partner in der Subhierarchie) erwirtschaften. Daher ist der prozentuale Bonus eines höher positionierten Geschäftspartners mindestens so hoch wie der Bonus der ihm unterstellten Partner. Seinen beispielhaft drei Partner – A, B und C – betrachtet, wobei B und C von A geworben wurden. Sind B und C fleißige Käufer für den Eigenbedarf bzw. Verkäufer, erhalten sie beispielsweise die Bonusstufen 3% (für B) bzw. 6% (für C). Der dem Partner A zugerechnete Umsatz erlaubt daher mindestens Bonusstufe 6% für ihn, so dass er eine 6%ige (oder mehr, je nach seinem eigenen Umsatz) Bonuszahlung auf den Umsatz seiner Gruppe erhält, von der die Bonuszahlungen für B und C abgezogen werden. Die höhere Hierarchiestufe bedeutet für ihn, dass er auch mit einer geringen persönlichen Produktivität in den Genuss einer höheren Bonusstufe kommt.

Der Aufsatz lässt sich von dieser Wirkungsweise einer Hierarchie auf die Entlohnung der Mitarbeiter inspirieren und basiert daher auf zwei Grundannahmen, **H** und **P**. Annahme **P** besagt, dass die Entlohnung eines Beschäftigten eine Funktion der Produktivität ist. Dabei ist nicht nur die Grenzproduktivität relevant, also der zusätzliche Output des Beschäftigten beim aktuellen Beschäftigungsstand. Vielmehr ist auch die zusätzliche Produktivität in Bezug auf kleinere Teilgruppen der Mitarbeiter (marginale Beiträge genannt) relevant für die Bezahlung. Dies ist plausibel, da die Produktivität in Bezug auf andere Gruppen als die gesamten Beschäftigten Aussagen darüber trifft, welche alternativen Verdienstmöglichkeiten ein Mitarbeiter besitzt. Allerdings ist empirische Evidenz für diese Annahme schwer beschaffbar, da die marginalen Beiträge (in Bezug auf alle Beschäftigten oder in Bezug auf Teilgruppen) kaum messbar sein dürften.

Die zweite Annahme, Annahme **H**, erkennt die am Beispiel Amway skizzierte Tatsache an, dass neben der Produktivität die hierarchische Stellung des Mitarbeiters Auswirkung auf seine Entlohnung hat. Dabei geht es nicht vordergründig darum, dass eine hohe Produktivität mit einer hohen Hierarchiestufe belohnt wird. Dies kann der Fall sein (siehe auch den Literaturüberblick) und später wird im Aufsatz eine Beziehung in gerade dieser Hinsicht hergestellt. Zunächst jedoch soll argumentiert werden, dass bei einer gegebenen Hierarchie die oberen Hierarchiestufen von den Leistungen der unteren Hierarchiestufen profitieren oder für die Fehler unterer Hierarchiestufen „geradestehen“ müssen. Genauer wird angenommen, dass ein Vorgesetzter von den ihm direkt untergebenen Mitarbeitern profitiert. Da diese jedoch ihrerseits an den Leistungen der wiederum niedrigeren Hierarchiestufen partizipieren, ergibt sich zudem eine indirekte, aber mit zunehmendem Hierarchieabstand weniger wichtige Partizipation.

Natürlich existiert bei der modelltheoretischen Umsetzung der Annahmen **P** und **H** einiger Spielraum. Im Aufsatz werden die diesbezüglichen konkreteren Annahmen (Unterannahmen von **P** und **H**) in zweierlei Weise explizit. Zunächst wird eine Formel präsentiert, die die Auszahlungen aller Beschäftigten in Abhängigkeit von den Produktivitäten und in Abhängigkeit von der Stellung in einer Hierarchie wiedergibt. Die Rechtfertigung des Entlohnungsschemas erfolgt anschließend durch ein System von einfachen und plausiblen Eigenschaften bzw. Axiomen, die ebenfalls eindeutig zu diesen Auszahlungen führen. Mit Hilfe eines Axiomensystems kann zudem für ein Entlohnungsschema leicht überprüft werden, ob es bestimmte geforderte Kriterien erfüllt. Dies ist dann der Fall, wenn die Kriterien direkt zum Axiomensystem gehören oder aus diesen Axiomen ableitbar sind. Die Methode der Axiomatisierung erleichtert ferner den Vergleich verschiedener Entlohnungsschemata, da genau feststellbar ist, in welchen Eigenschaften sich diese unterscheiden.

Die beiden Annahmen **P** und **H** werden mit Hilfe der kooperativen Spieltheorie umgesetzt. Dazu wird die ökonomische Situation mit Hilfe der Koalitionsfunktion modelliert. Diese ordnet jeder Koalition, d.h. jeder

Teilmenge der Spielermenge, im Folgenden die Menge aller Beschäftigten, einen Wert zu, der die Leistungsfähigkeit dieser Koalition widerspiegelt. Aus diesen Werten generiert nun die Shapley-Lösung (Shapley 1953b), das bekannteste Lösungskonzept der kooperativen Spieltheorie, Auszahlungen als arithmetisches Mittel der marginalen Beiträge. Die Shapley-Lösung ist somit eine konkrete Umsetzung von Annahme **P**.

Zur Formalisierung von Annahme **H**, muss die hierarchische Struktur in die Bestimmung der Mitarbeiter-Auszahlungen einbezogen werden. Dieser Aufsatz nutzt dazu, ähnlich wie van den Brink (2008), gerichtete Graphen bzw. eine Kontrollstruktur (permission structure). Sind zwei Mitarbeiter in einem solchen Graphen durch einen Pfeil verbunden, so ist derjenige, auf den der Pfeil zeigt, der Vorgesetzte des anderen.

Die Modellierung von van den Brink (2008) besitzt dabei die Eigenschaft, dass die einzelnen Dominanzbeziehungen, d.h. der Einfluss des Vorgesetzten auf seine direkten Mitarbeiter, gleich stark sind. Allerdings ist es eine recht plausible Vermutung, dass diese Beziehungen zwischen einem Vorgesetzten und seinen direkten Mitarbeitern in Unternehmen unterschiedlich stark ausgeprägt sein können. An anderer Stelle in der Literatur wurde diese Idee unterschiedlich starker gerichteter Verbindungen zwischen Spielern beispielsweise bei der Bestimmung der Macht von Spielern in Netzwerken durch van den Brink & Gilles (2000) und Herings, van der Laan & Talman (2005) aufgegriffen.

Diese unterschiedliche Stärke spielt in diesem Aufsatz eine zentrale Rolle und ergänzt somit die in van den Brink (2008) entwickelte Abbildung von Unternehmenshierarchien in die Welt der kooperativen Spieltheorie. Die Hierarchie wird durch einen gewichteten gerichteten Graphen bzw. eine gewichtete Kontrollstruktur modelliert. Die Idee des in diesem Aufsatz entwickelten Lösungskonzepts für diese reichhaltigere Struktur lässt sich dann wie folgt formulieren. Auf einer ersten Stufe erhalten die Mitarbeiter zunächst auf Grundlage der Koalitionsfunktion ihre Auszahlungen gemäß dem Shapley-Lösungskonzept. Anschließend wirkt die Hierarchie umverteilend. Von der erhaltenden Auszahlung muss jeder Mitarbeiter, außer jener an der Spitze des Unternehmens, einen gewissen Anteil an seinen Vorgesetzten abführen. Dies korrespondiert damit, dass Vorgesetzte zu einem Teil am Erfolg bzw. Misserfolg ihrer Mitarbeiter partizipieren, in diesem Fall mit dem oben erwähnten Anteil. Die Höhe dieses Anteils kann beispielsweise von der Führungspersönlichkeit bzw. dem Führungsstil des Vorgesetzten, aber auch von der durch die Personalverantwortlichen festgelegten Stärke des vertikalen Hierarchieunterschieds zwischen beiden Mitarbeitern abhängen. Das so skizzierte Lösungskonzept kann als Entlohnungsschema, das die Hierarchie des Unternehmens berücksichtigt, interpretiert werden.

Im Rahmen dieser Wirkungsweise von Hierarchien auf die Entlohnung kann u.a. gezeigt werden, dass in Unternehmen, die in einem bestimmten Sinne symmetrisch sind, Mitarbeiter in höheren Hierarchieebenen besser entlohnt werden als jene in niedrigeren. Zudem kann eine Erklärung dafür geliefert werden, dass produktivere Mitarbeiter in höheren Hierarchieebenen des Unternehmens anzutreffen sind – die Führungskraft an der Spitze des Unternehmens erhält den größten relativen Anteil des Erfolgs von den Mitarbeitern, die in der ihr direkt unterstellten Hierarchieebene tätig sind. Entscheidet diese Führungskraft über die Allokation der Mitarbeiter, so wird sie die produktivsten dieser Ebene zuordnen.

Die Stoßrichtung des Aufsatzes ist in der Hauptsache theoretisch und methodisch. Wir stellen uns dem Problem, positive Aussagen über die Wirkungen von Hierarchien auf die Entlohnung und Allokation von Mitarbeitern zu treffen. Hierzu wird das Shapley-Lösungskonzept (Annahme **P**) mit einem gewichteten gerichteten Graphen (Annahme **H**) angereichert. Alternative Modellierungen der Wirkungsweise von Hierarchien sind möglich. Der Ansatz öffnet sich der Kritik insofern, als sich diese direkt an den noch vorzustellenden Unterannahmen von **P** und **H** festmachen lässt. Das präsentierte Entlohnungsschema bietet damit der empirischen und theoretischen Hierarchieforschung einen neuen Ansatzpunkt. Effizienz- oder Anreizprobleme lassen sich, zumindest in der jetzigen Form, nicht mit dem präsentierten Entlohnungsschema behandeln. Ursache dafür ist, dass in Übereinstimmung mit dem Effizienz-Axiom der Shapley-Lösung vorausgesetzt wird, dass alle Hierarchien zum gleichen Gesamtoutput führen.

Der Aufsatz ist folgendermaßen gegliedert. Der nächste Abschnitt bietet einen Literaturüberblick, in dem zunächst verschiedene in der Literatur diskutierte Ursachen für das Entstehen von Hierarchien dargestellt werden. Anschließend folgt die Zusammenfassung der Ergebnisse bisheriger Modelle zur Analyse der Auswirkungen der Hierarchie eines Unternehmens auf dessen Lohnstruktur. Dabei wird u.a. auf den schon erwähnten Aufsatz von van den Brink (2008) näher eingegangen. Abschließend werden weitere bisherige Ansätze der kooperativen Spieltheorie kurz dargestellt, die sich ebenfalls der Modellierung von Hierarchien bzw. Einkommensleitern widmen. Im Abschnitt **??** werden die grundlegenden Begriffe und Definitionen der kooperativen Spieltheorie eingeführt. Dabei wird u.a. die hierarchische Struktur eines Unternehmens in die Sprache der kooperativen Spieltheorie übersetzt. In Abschnitt 4 schließt sich die Einführung und Axiomatisierung des Lösungskonzepts/Entlohnungsschemas an. Personalwirtschaftliche Implikationen hinsichtlich der vertikalen Lohndifferenzen zwischen den Hierarchieebenen werden in Abschnitt 5 und hinsichtlich der Allokation der Mitarbeiter auf die Ebenen in Abschnitt 6 gezogen. Abschnitt 7 schließt den Aufsatz mit einer Zusammenfassung.

3. Literaturüberblick

In der Literatur werden verschiedene Ursachen für das Entstehen von Hierarchien in Unternehmen angeführt. Zum einen soll durch sie für die Mitarbeiter ein Anreiz gesetzt werden, ihren Arbeitseinsatz zu steigern, da sie damit die Wahrscheinlichkeit erhöhen, auf eine Hierarchiestufe mit einer besseren Entlohnung zu gelangen (Mirrlees 1976, Calvo & Wellisz 1979). Bessere Informationsverarbeitung ist ein weiteres Motiv zur Schaffung von Hierarchien. Zum einen können die durch ein Unternehmen zu bewältigenden Probleme durch eine Hierarchie in viele kleinere zerlegt werden, deren Lösungen anschließend wieder zusammengesetzt werden. Zum anderen fallen die benötigten Informationen zur Lösung der Probleme häufig dezentral an. Infolgedessen kann, durch eine dezentrale Erfassung und anschließende Koordination der Informationen gemäß der hierarchischen Struktur, der Informationsverarbeitungsprozess beschleunigt werden. Allerdings steigen mit der Einführung zusätzlicher Positionen innerhalb der Hierarchie auch die Kosten der Informationsverarbeitung (Keren & Levhari 1979, Geanakoplos & Milgrom 1991, Radner 1992, Radner 1993, Prat 1997, Borland & Eichberger 1998, Meagher 2003). Ein weiterer und hier letztgenannter Grund für das Bestehen von Hierarchien sind die Kontrollaktivitäten der Vorgesetzten gegenüber den Mitarbeitern. Sie können mit einer gewissen Wahrscheinlichkeit den Arbeitseinsatz oder den Output der Mitarbeiter feststellen. Je mehr Hierarchiestufen existieren bzw. je kleiner die Kontrollspanne, desto höher ist die Wahrscheinlichkeit, den wahren Arbeitseinsatz festzustellen, und umso mehr agieren die Mitarbeiter im Sinne des Unternehmens. Allerdings, und darin besteht der trade-off, steigen auch die Kosten der Überwachung (Stiglitz 1975, Calvo & Wellisz 1979, Rosen 1982, Qian 1994).

In den Modellen von Calvo & Wellisz (1979), Rosen (1982), Waldman (1984) und Qian (1994) spielen die Fähigkeiten der Mitarbeiter eine entscheidende Rolle bei ihrer Allokation innerhalb der Hierarchie. Diese Fähigkeit wird dabei in diesen Aufsätzen, bis auf Waldman (1984), als deren Überwachungs- bzw. Kontrolltalent modelliert. Ein Resultat aller Aufsätze ist die Allokation der fähigsten Mitarbeiter an die Spitze der Hierarchie, da die dort getroffenen Entscheidungen (bzw. durchgeführten Kontrollaktivitäten) im Vergleich zu Entscheidungen auf niedrigeren Hierarchiestufen einen größeren Einfluss auf den Unternehmensgewinn ausüben. Auf Grund dieses höheren Einflusses werden die dort beschäftigten Mitarbeiter höher entlohnt als Mitarbeiter in niedrigeren Ebenen. Zudem zeigen die Autoren in ihren Modellen, dass der erwähnte Multiplikatoreffekt der höheren Hierarchieebenen dazu führt, dass die Löhne stärker mit der Hierarchieebene steigen, als die Fähigkeiten bzw. Produktivität der Mitarbeiter. Waldman

(1984) und Qian (1994) deduzieren aus ihren Modellen ferner, dass die Mitarbeiter an der Spitze von großen Unternehmen, gemessen an der Zahl der Beschäftigten, ein höheres Entgelt erhalten als diejenigen bei kleinen Unternehmen, da der genannte Multiplikatoreffekt bei diesen großen Unternehmen stärker wirkt.

Der Aufsatz von van den Brink (2008) nutzt, wie bereits erwähnt, die kooperative Spieltheorie, um die Auswirkungen der Unternehmenshierarchie auf vertikale Lohndifferenzen im Unternehmen zu analysieren. Die Grundidee seiner Arbeit ist dabei, dass ein Mitarbeiter die Zustimmung aller seiner Vorgesetzten benötigt, um eine Entscheidung treffen zu können bzw. um produktiv zu sein (conjunctive approach). Einer Koalition wird demzufolge der Wert zugeordnet, den ihre autonome Teilmenge, d.h. die Menge der Beschäftigten, deren Vorgesetzte ebenfalls in der betrachteten Koalition enthalten sind, erzielt. Anschließend wird auf Grundlage dieser restringierten Koalitionsfunktion die Shapley-Auszahlung der Mitarbeiter ermittelt. Ein erster Beitrag zu diesem Ansatz, in dem die Grundidee des Lösungskonzepts skizziert wird, stammt von Gilles, Owen & van den Brink (1992b). Die Axiomatisierung des Konzepts erfolgt durch van den Brink & Gilles (1996a).

Eine Prämisse von van den Brink (2008) ist dabei, dass ausschließlich die Mitarbeiter ohne Unterstellte (Arbeiter) zur Wertschöpfung beitragen. Aus seinem Modell kann er die folgenden Ergebnisse ableiten. Liegt dem Spiel eine monotone Koalitionsfunktion zugrunde, d.h. kein Arbeiter senkt durch seinen Beitritt den Wert einer Gruppe, so erhält ein Vorgesetzter eine Entlohnung, die mindestens so hoch ist, wie die seines bestbezahlten direkten Mitarbeiters. Ist die Koalitionsfunktion zudem konvex, d.h. der Beitrag eines Arbeiters zu einer Gruppe steigt bei deren Erweiterung um andere Arbeiter, ist die obere Lohngrenze eines Vorgesetzten die Lohnsumme seiner direkten Mitarbeiter.

Neben diesem Ansatz entwickelt van den Brink (1997) ein Konzept, bei dem ein Mitarbeiter nur die Zustimmung aller Mitarbeiter einer Befehlskette zwischen ihm und dem Vorgesetzten an der Spitze des Unternehmens benötigt (disjunctive approach). Für den Fall, dass jeder Mitarbeiter, bis auf jener an der Spitze des Unternehmens, nur einen direkten Vorgesetzten besitzt, fällt dieser Ansatz mit dem erstgenannten zusammen. Allerdings weist auch dieser Ansatz den Kritikpunkt auf, dass die Dominanzbeziehungen zwischen den Mitarbeitern gleich stark sind.

Neben gerichteten Graphen können in der kooperativen Spieltheorie auch ungerichtete Graphen für die Bestimmung der Auszahlung der Spieler Berücksichtigung finden. Die durch einen solchen Graphen verbundenen Spieler sind symmetrisch in dieser Verbindung (Myerson 1977b), so dass mit diesen Graphen keine Vorgesetzten-Mitarbeiter-Struktur bzw. Hierarchie abgebildet werden kann.

Ein weiterer Ansatz der kooperativen Spieltheorie zur Analyse von hierarchischen Strukturen bzw. Einkommensleitern in Unternehmen stammt von Kalai & Samet (1987a) und nutzt ein Gewichtungssystem mit Ebenen. Die zugewiesenen Gewichte bzw. Ebenen lassen sich als Verantwortung der Mitarbeiter interpretieren (Owen 1968). Allerdings erlaubt es das Lösungskonzept nicht, einem Mitarbeiter in einer höheren Ebene direkte Beschäftigte in der nächstniedrigeren zuzuordnen, so dass im Rahmen dieses Ansatzes keine Hierarchien betrachtet werden können. Zudem ist die vertikale Abstufung zwischen den Beschäftigten, die in unserem Konzept eine zentrale Stellschraube ist, durch die Ebenenstruktur fest vorgegeben und nur das Ausmaß der horizontalen Abstufung zwischen den Mitarbeitern kann durch die Wahl der Gewichte feingesteuert werden.

3.1. Die Hierarchie eines Unternehmens. Dem Aufsatz von van den Brink (2008) folgend und in Übereinstimmung mit der in Abschnitt 2 präsentierten Definition einer Unternehmenshierarchie wird die Hierarchie eines Unternehmens durch eine Funktion $\mathcal{S}: N \to 2^N$ abgebildet, die jedem Arbeitnehmer $i \in N$ seine Untergebenen bzw. direkten Mitarbeiter zuordnet. $\mathcal{S}$ kann dabei als ein gerichteter Graph aufgefasst werden (Bollobás 2002). $\mathcal{S}(i)$ bezeichnet die Menge der direkten Mitarbeiter von i. Dabei gilt $i \notin \mathcal{S}(i)$. Die Mitarbeiter, die in der Menge $\mathcal{S}^{-1}(i) = \{j \in N : i \in \mathcal{S}(j)\}$ enthalten sind, werden als is direkte Vorgesetzte angespochen. Ein Pfad T in N von i zu j ist eine Folge von Mitarbeitern $T(i,j) = \langle r_0, r_1, \ldots, r_{k-1}, r_k \rangle$ mit $i = r_0$, $j = r_k$ und $r_{\ell+1} \in \mathcal{S}(r_\ell)$ für alle $\ell = 0, \ldots, k-1$. Der Pfad kann als „Befehlskette“ zwischen den Mitarbeitern i und j interpretiert werden, wobei Mitarbeiter i ein direkter oder indirekter Vorgesetzter von Mitarbeiter j ist. Der Pfad $\langle i_0, i_0 \rangle$ wird als trivialer Pfad bezeichnet. Die Menge aller Mitarbeiter, die einem Arbeitnehmer i direkt oder indirekt unterstehen, wird mit $\hat{\mathcal{S}}(i) := \{j \in N \backslash \{i\} : \text{es existiert ein Pfad von } i \text{ zu } j\}$ bezeichnet. Entsprechend wird die Menge von is direkten und indirekten Vorgesetzten mit $\hat{\mathcal{S}}^{-1}(i) := \{j \in N \backslash \{i\} : \text{es existiert ein Pfad von } j \text{ zu } i\}$ angesprochen.

Wie in der Literatur üblich, wird für eine Unternehmenshierarchie eine Baumstruktur angenommen (Radner 1992, Meagher 2001), d.h. es gibt genau einen Mitarbeiter i_0, der keinen Vorgesetzten besitzt, und jeder der anderen Mitarbeiter hat genau einen direkten Vorgesetzten und kann dabei nicht sein eigener indirekter Vorgesetzter sein. Formal notiert bedeutet dies that S is a unique hierarchy:

- es existiert ein Arbeitnehmer $i_0 \in N$, so dass $\mathcal{S}^{-1}(i_0) = \emptyset$ und $\hat{\mathcal{S}}(i_0) = N \backslash \{i_0\}$ gilt,
- für jeden Mitarbeiter $i \in N \backslash \{i_0\}$ ist $\left|\mathcal{S}^{-1}(i)\right| = 1$ erfüllt und
- es gilt $i \notin \hat{\mathcal{S}}(i)$ für alle $i \in N$.

Neben der Unternehmenshierarchie $\mathcal{S}$ werden durch das in diesem Aufsatz vorgestellte Entlohnungsschema auch verschiedene Stärken der Beziehung zwischen einem Vorgesetzten und dessen direkten Mitarbeiter berücksichtigt. Der Vektor $w : N \to \mathbb{R}$ ordnet jedem Mitarbeiter i ein Gewicht w_i, $0 \leq w_i \leq 1$, zu, das über die Stärke der Partizipation des Vorgesetzten an den Erfolgen bzw. Misserfolgen seiner Mitarbeiter Auskunft gibt. Ein Mitarbeiter i muss dabei den Anteil w_i seines (Miss-)Erfolgs an den Vorgesetzten abtreten. Für i_0 gilt $w_{i_0} = 0$. Die w_i können z.B. durch den Führungsstil des Vorgesetzten geprägt sein oder von den Personalverantwortlichen des Unternehmens bewusst gesetzt werden, um die Eigenständigkeit von Mitarbeitern zu steuern oder die Lohnstruktur des Unternehmens zu bestimmen.

Weist ein Gewichtsvektor allen Mitarbeitern, außer i_0, das gleiche Gewicht $\bar{w}$ zu, d.h. $w_i = w_j = \bar{w}$ für alle $i, j \in N \backslash \{i_0\}$, so wird der entsprechende Vektor ebenfalls mit $\bar{w}$ angesprochen. Für spätere Beweise ist es notwendig den Gewichtsvektor $w[K]$ für alle $K \subseteq N$ zu definieren. Dieser weist allen Beschäftigten $i \in K$ das Gewicht null zu, $w[K]_i = 0$. Alle übrigen Mitarbeiter $j \in N \backslash K$ erhalten ihr ursprüngliches Gewicht zugeordnet, $w[K]_j = w_j$. Beispielsweise ist der Vektor $w[\{i\}]$ gegeben durch $(w_1, \ldots, w_{i-1}, 0, w_{i+1}, \ldots, w_n)$. Vereinfachend wird häufig $w[i, j, \ldots]$ anstelle von $w[\{i, j, \ldots\}]$ notiert.

Definition XII.1. *Die hierarchische Struktur eines Unternehmens wird durch die Hierarchie $\mathcal{S}$ sowie den Gewichtsvektor w dargestellt. Ein hierarchisches Spiel ist das Tupel $(N, v, \mathcal{S}, w)$.*

Das folgende Beispiel soll die Notation verdeutlichen und wird später bei der Berechnung der H-Auszahlungen nochmals aufgegriffen.

Example XII.1. *Ein Unternehmen beschäftigt fünf Mitarbeiter, $N = \{1, 2, 3, 4, 5\}$. Die hierarchische Struktur ist gegeben durch $\mathcal{S}(3) = \mathcal{S}(4) = \mathcal{S}(5) = \emptyset$, $\mathcal{S}(2) = \{3, 4\}$, $\mathcal{S}(1) = \{2, 5\}$ sowie den Gewichtsvektor $w = (w_1, \ldots, w_5) = \left(0, \frac{1}{4}, \frac{2}{3}, \frac{1}{2}, \frac{1}{4}\right)$ (siehe Abbildung* **??***). In diesem Unternehmen bestimmt sich beispielsweise die Menge aller (direkten und indirekten) Vorgesetzten von Mitarbeiter 3 als $\hat{\mathcal{S}}^{-1}(3) = \{1, 2\}$ und der Pfad zwischen Mitarbeiter 1 und Mitarbeiter 3 ist gegeben durch: $T(1 \to 3) = \langle 1, 2, 3 \rangle$.*

4. Das H-Lösungskonzept

4.1. Definition. Die grundlegende Überlegung für das in diesem Abschnitt vorgestellte Lösungskonzept für hierarchische Spiele bzw. das Entlohnungsschema für ein Unternehmen mit hierarchischer Struktur umfasst zwei Elemente. Für die Erstellung des Unternehmensergebnisses arbeiten alle Mitarbeiter (symmetrisch) zusammen (im Sinne von Alchian & Demsetz 1972), d.h. die hierarchische Struktur hat auf die Zusammenarbeit der Mitarbeiter keinen Einfluss. Das so erwirtschaftete Ergebnis wird zunächst gemäß dem

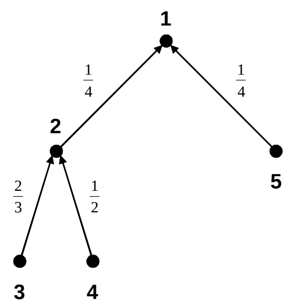

FIGURE 1. Example of a hierarchical structure

Shapley-Lösungskonzept und somit leistungsbezogen auf Basis der individuellen marginalen Beiträge auf die Beschäftigten verteilt. Im Anschluss wirkt die hierarchische Struktur umverteilend von unteren Ebenen zu oberen. Ein Mitarbeiter i auf der untersten Ebene, $\mathcal{S}(i) = \emptyset$, muss von seiner Shapley-Auszahlung den Anteil w_i an seinen direkten Vorgesetzten j, $\mathcal{S}^{-1}(i) = \{j\}$, abführen. Dieser wiederum muss von seinen insgesamt erhaltenen (Brutto-)Zahlungen, seine Shapley-Auszahlung und die Zahlungen seiner direkten Mitarbeiter an ihn, den Anteil w_j an seinen direkten Vorgesetzten weiterreichen etc.

Alle Mitarbeiter j, für die $j \in \hat{\mathcal{S}}^{-1}(g)$ gilt bzw. die im Pfad $T(i_0 \to g)$ enthalten sind, erhalten einen Anteil der Shapley-Auszahlung des Mitarbeiters g. Ein beliebiger Beschäftigter $i \in N$ erhält von der Shapley-Auszahlung des Mitarbeiters g den Anteil:

$$f_i(\mathcal{S}, w, g) = \begin{cases} [1 - w_i] \prod\limits_{\substack{l \in \hat{\mathcal{S}}(i), \\ l \in T(i_0 \to g)}} w_l, & i \in T(i_0 \to g), \\ 0, & \text{sonst.} \end{cases} \tag{XII.1}$$

Zwei Spezialfälle können für den Anteil $f_i(\mathcal{S}, w, g)$ betrachtet werden. Gilt beispielsweise $g = i$, so existiert kein $l \in T(i_0 \to g)$, der zugleich Mitarbeiter von i ist. In diesem Fall muss der Mitarbeiter den Anteil w_i seiner Shapley-Auszahlung an seinen direkten Vorgesetzten abführen, der Anteil $1 - w_i$ verbleibt bei ihm. Gilt hingegen $i = i_0$, so muss i nichts nach oben abführen und er erhält den Anteil $\prod_{l \in \mathcal{S}(i), l \in T(i_0 \to g)} w_l$. Sollte der Mitarbeiter i nicht im Pfad $T(i_0 \to g)$ enthalten sein, so erhält er den Anteil null von gs Shapley-Auszahlung (letzte Formelzeile).

Mit Hilfe dieser Anteile kann jetzt die Auszahlung $H_i(N, v, \mathcal{S}, w)$, die ein Mitarbeiter $i \in N$ auf Grund des Entlohnungsschemas erwarten kann, bestimmt werden als:

$$H_i\left(N,v,\mathcal{S},w\right)=\sum_{j=1}^{n} f_i\left(\mathcal{S},w,j\right)\cdot \mathrm{Sh}_j\left(N,v\right). \tag{XII.2}$$

Zusammen mit Formelzeile XII.1 kann die Beziehung zwischen der Netto- und der Brutto-Entlohnung eines Mitarbeiters i betrachtet werden:

$$H_i\left(N,v,\mathcal{S},w\right)=\left(1-w_i\right)\cdot H_i\left(N,v,\mathcal{S},w\left[i\right]\right). \tag{XII.3}$$

Die Brutto-Entlohnung $H_i\left(N,v,\mathcal{S},w\left[i\right]\right)$ ist jener Betrag, den i erhalten würde, wenn er nichts an seinen direkten Vorgesetzten abführen muss. Wird dieser Betrag mit $\left(1-w_i\right)$ multipliziert, so ergibt sich die Netto-Auszahlung, d.h. die H-Auszahlung des Beschäftigten. Die Brutto-Auszahlung des Mitarbeiters i kann dabei aus den Brutto-Auszahlungen seiner direkten Mitarbeiter bestimmt werden über:

$$H_i\left(N,v,\mathcal{S},w\left[i\right]\right)=\mathrm{Sh}_i\left(N,v\right)+\sum_{j\in\mathcal{S}(i)} w_j\cdot H_j\left(N,v,\mathcal{S},w\left[j\right]\right). \tag{XII.4}$$

Remark XII.1. *Die Gleichungen XII.3 und XII.4 definieren H_i induktiv, da $H_i\left(N,v,\mathcal{S},w\left[i\right]\right)=\mathrm{Sh}_i\left(N,v\right)$ falls $\mathcal{S}\left(i\right)=\emptyset$.*

Example XII.2. *Sei zusätzlich zur Hierarchie $\mathcal{S}$ und dem Gewichtsvektor w aus Beispiel XII.1 die Leistungsfähigkeit der einzelnen Koalitionen $K\subseteq N$ durch folgende Koalitionsfunktion gegeben:*

$$v\left(K\right)=\begin{cases} 0, & \left|K\right|\leq 1\\ 10, & \left|K\right|=2\\ 20, & \left|K\right|=3\\ 40, & \left|K\right|=4\\ 60, & K=N.\end{cases}$$

Die Mitarbeiter sind im Spiel $\left(N,v\right)$ symmetrisch, so dass für ihre Shapley-Auszahlungen $\mathrm{Sh}_i\left(N,v\right)=12$, $i=1,\ldots,5$, resultiert. Die H-Auszahlungen ergeben sich als:

$$\begin{aligned} H_1\left(N,v,\mathcal{S},w\right) &= \underbrace{12}_{\mathrm{Sh}_1(N,v)}+\frac{1}{4}\cdot\underbrace{26}_{H_2(N,v,\mathcal{S},w[2])}+\frac{1}{4}\cdot\underbrace{12}_{H_5(N,v,\mathcal{S},w[5])}=21,5\\ H_2\left(N,v,\mathcal{S},w\right) &= \underbrace{\left(1-\frac{1}{4}\right)}_{1-w_2}\cdot\underbrace{\left(\underbrace{12}_{\mathrm{Sh}_2(N,v)}+\frac{2}{3}\cdot\underbrace{12}_{H_3(N,v,\mathcal{S},w[3])}+\frac{1}{2}\cdot\underbrace{12}_{H_4(N,v,\mathcal{S},w[4])}\right)}_{H_2(N,v,\mathcal{S},w[2])}\\ &= 19,5\end{aligned}$$

$$
\begin{aligned}
H_3(N,v,\mathcal{S},w) &= \underbrace{\left(1-\frac{2}{3}\right)}_{1-w_3}\cdot \underbrace{12}_{\mathrm{Sh}_3(N,v)} = 4\\
H_4(N,v,\mathcal{S},w) &= \underbrace{\left(1-\frac{1}{2}\right)}_{1-w_4}\cdot \underbrace{12}_{\mathrm{Sh}_4(N,v)} = 6\\
H_5(N,v,\mathcal{S},w) &= \underbrace{\left(1-\frac{1}{4}\right)}_{1-w_5}\cdot \underbrace{12}_{\mathrm{Sh}_5(N,v)} = 9.
\end{aligned}
$$

Die Wirkung der hierarchischen Struktur zeigt sich beispielsweise beim Vergleich der Auszahlungen $H_3(N,v,\mathcal{S},w)$ und $H_4(N,v,\mathcal{S},w)$. Beide Mitarbeiter sind im Produktionsprozess symmetrisch. Auf Grund der unterschiedlichen Stärke der Verbindung zu ihren gemeinsamen direkten Vorgesetzten, erhält Mitarbeiter 4 eine höhere H-Auszahlung als Mitarbeiter 3. Ein Blick auf die Auszahlungen der Mitarbeiter 2 und 5 gibt weiteren Aufschluss über die Auswirkungen der hierarchischen Struktur. Beide Beschäftigte sind im Produktionsprozess symmetrisch und müssen den gleichen Anteil ihrer Brutto-Auszahlungen an ihren Vorgesetzten abführen. Allerdings erhält Mitarbeiter 2 einen Teil der Brutto-Auszahlungen seiner direkten Mitarbeiter, so dass seine H-Auszahlung über der von Mitarbeiter 5 liegt.

4.2. Axiomatisierung. Die zur Charakterisierung des H-Entlohnungsschemas verwendeten Axiome lassen sich in zwei Gruppen einteilen. Die erste Gruppe stellt dabei sicher, dass, wenn alle Gewichte null sind, für die H-Auszahlungen der Mitarbeiter deren Shapley-Auszahlungen resultieren. Die zweite Gruppe der Axiome bewirkt dann die gewünschte Umverteilungswirkung der hierarchischen Struktur.

Die erste Axiomengruppe (Annahme **P**) beinhaltet eine Abwandlung der vier „klassischen" Eigenschaften zur Charakterisierung des Shapley-Lösungskonzepts. Die erste wird als **Effizienz-Axiom** bzw. **Budget-Neutralität** bezeichnet. Ein Entlohnungsschema φ, das diese Eigenschaft erfüllt, schüttet nur jene Lohnsumme an alle Mitarbeiter aus, die von diesen auch erwirtschaftet wurde, d.h. $\sum_{i\in N}\varphi_i(N,v,\mathcal{S},w) = v(N)$. Das zweite Axiom ist das **Additivitäts-Axiom**. Es besagt, dass $\varphi_i(N,v+v',\mathcal{S},w) = \varphi_i(N,v,\mathcal{S},w) + \varphi_i(N,v',\mathcal{S},w)$ für zwei Koalitionsfunktionen v und v' für alle Mitarbeiter $i\in N$ gilt. Für die Auszahlung der Mitarbeiter soll es demnach keine Rolle spielen, ob die Auszahlungen der Spiele $(N,v,\mathcal{S},w)$ und $(N,v',\mathcal{S},w)$ addiert werden oder zunächst die Koalitionsfunktionen addiert werden und anschließend für das so gewonnene Spiel $(N,v+v',\mathcal{S},w)$ die Auszahlung bestimmt wird. Das dritte Axiom, das **schwache Nullspieler-Axiom**, verlangt von einem Entlohnungsschema φ, dass einem Mitarbeiter i, der zu keiner Koalition etwas beiträgt, d.h. $v(K) = v(K\cup\{i\})$ für alle $K\subseteq$

N, beim Gewichtsvektor $w[N]$ (alle Gewichte sind auf null gesetzt) eine Auszahlung null zugewiesen wird, $\varphi_i(N, v, \mathcal{S}, w[N]) = 0$. Das letzte Axiom dieser Axiomengruppe ist das **schwache Symmetrie-Axiom**. Sind zwei Mitarbeiter i und j aus N im Spiel (N, v) symmetrisch, d.h. es gilt $v(K \cup \{j\}) = v(K \cup \{i\})$ für alle $K \subseteq N \backslash \{i, j\}$, so sollen beide gleich entlohnt werden, wenn der Gewichtsvektor $w[N]$ lautet, $\varphi_i(N, v, \mathcal{S}, w[N]) = \varphi_j(N, v, \mathcal{S}, w[N])$.

Die zweite Axiomengruppe (Annahme **H**) beinhaltet ebenfalls vier Axiome. Das erste wird **Brutto-Netto-Axiom** genannt und ist bereits in Gleichung XII.3 notiert und erläutert. Das zweite Axiom wird als **Abspaltungs-Axiom** bezeichnet. Wird das Gewicht einer Verbindung zweier Mitarbeiter auf null gesetzt, so verliert der Vorgesetzte dadurch brutto jenen Betrag, den der direkte Mitarbeiter bei seiner Netto-Auszahlung hinzubekommt. Wenn i der direkte Vorgesetzte von j ist, $j \in \mathcal{S}(i)$, so verlangt das Axiom demnach, dass

$$\varphi_i(N, v, \mathcal{S}, w[i]) - \varphi_i(N, v, \mathcal{S}, w[i, j]) = \varphi_j(N, v, \mathcal{S}, w[j]) - \varphi_j(N, v, \mathcal{S}, w)$$

erfüllt ist. Das dritte zur Axiomatisierung des H-Entlohnungsschemas verwendete Axiom ist das **Isolations-Axiom**. Wird bei einem Mitarbeiter sowohl das Gewicht der Verbindung zu seinem direkten Vorgesetzten, als auch die Gewichte zu seinen direkten Mitarbeitern auf null gesetzt, so erhält dieser seine Shapley-Auszahlung, d.h. jene Auszahlung die resultiert, wenn alle Gewichte des hierarchischen Spiels auf null gesetzt sind,

$$\varphi_i(N, v, \mathcal{S}, w[\{i\} \cup \mathcal{S}(i)]) = \varphi_i(N, v, \mathcal{S}, w[N]).$$

Das letzte zur Axiomatisierung benötigte Axiom ist das **Unabhängigkeits-Axiom**. Nimmt ein Mitarbeiter $i \in N$ an zwei hierarchischen Spielen $(N, v, \mathcal{S}, w)$ und $(N, v, \mathcal{S}, w')$ teil, deren Gewichtsvektoren w und w' für i und seine ihm unterstellten Mitarbeiter identisch (aber nicht notwendig für andere) sind, d.h. $w_j = w'_j$, wenn $j \in \hat{\mathcal{S}}(i) \cup \{i\}$, so erhält dieser Spieler in beiden Spielen die gleiche Entlohnung zugewiesen, $\varphi_i(N, v, \mathcal{S}, w) = \varphi_i(N, v, \mathcal{S}, w')$.

THEOREM XII.1. *Das H-Entlohnungsschema ist das einzige Lösungskonzept, welches das Effizienz-Axiom, das Additivitäts-Axiom, das schwache Nullspieler-Axiom, das schwache Symmetrie-Axiom, das Brutto-Netto-Axiom, das Abspaltungs-Axiom, das Isolations-Axiom und das Unabhängigkeits-Axiom erfüllt.*

Der Beweis dieses Theorems ist in Anhang A dargestellt.

5. Personalwirtschaftliche Implikationen: Entlohnung

In diesem Abschnitt werden die sich für ein Unternehmen mit hierarchischer Struktur $(\mathcal{S}, w)$ ergebenden Implikationen hinsichtlich der Entlohnung der Mitarbeiter gezogen, wenn die Löhne gemäß dem H-Entlohnungsschema festgelegt werden.

DEFINITION XII.2. *Ein Mitarbeiter wird als* ***unwesentlich*** *bezeichnet, wenn sowohl er als auch seine direkten und indirekten Mitarbeiter zu allen Koalitionen den marginalen Beitrag null leisten. Formal notiert heißt dies, dass $i \in N$ ein unwesentlicher Mitarbeiter in $(N, v, \mathcal{S}, w)$ ist, wenn $v(K) = v(K \backslash \{j\})$ mit $j \in \hat{\mathcal{S}}(i) \cup \{i\}$ für jede Koalition $K \subseteq N$ erfüllt ist.*

LEMMA XII.1. *Ein unwesentlicher Mitarbeiter erhält vom H-Entlohnungsschema eine Auszahlung null zugewiesen.*

Diese Eigenschaft folgt direkt aus den Gleichungen XII.1 und XII.2. Sie ist plausibel, da sowohl durch i als auch seine Abteilung kein Beitrag zum Ergebnis des Unternehmens erfolgt. Somit erhalten alle betroffenen Mitarbeiter zunächst die Shapley-Auszahlung null zugewiesen. Die anschließende Umverteilung von unten nach oben gemäß der Hierarchie $\mathcal{S}$ und dem Gewichtsvektor w bleibt folgenlos.

DEFINITION XII.3. *Ein* ***einflussloser unproduktiver*** *Mitarbeiter i ist dadurch gekennzeichnet, dass er zu allen Koalitionen den marginalen Beitrag null leistet und zudem von seinen direkten Mitarbeitern jeweils den Anteil null ihrer Brutto-Auszahlungen erhält. In formaler Schreibweise bedeutet dies, dass $v(K) = v(K \backslash \{i\})$ für jede Koalition $K \subseteq N$ und zugleich $w_j = 0$ für alle $j \in \mathcal{S}(i)$ erfüllt ist.*

LEMMA XII.2. *Einem* ***einflusslosen unproduktiven*** *Mitarbeiter wird vom H-Entlohnungsschema ebenfalls die Auszahlung null zugewiesen.*

Auch diese Eigenschaft ist plausibel und folgt direkt aus den Gleichungen XII.1 und XII.2.

Aus diesen beiden Eigenschaften kann zudem auf die Entlohnung eines Mitarbeiters i geschlossen werden, der zu allen Koalitionen den marginalen Beitrag null beiträgt und der über keine ihm folgenden Mitarbeiter verfügt, d.h. $v(K) = v(K \backslash \{i\})$ ist für jede Koalition $K \subseteq N$ erfüllt und zugleich gilt $\mathcal{S}(i) = \emptyset$.

LEMMA XII.3. *Einem unproduktiven Mitarbeiter, dem keine weiteren direkten Mitarbeiter zugewiesen werden, wird eine Auszahlung null zugeordnet, d.h. $H_i(N, v, \mathcal{S}, w) = 0$.*

Das H-Entlohnungsschema sorgt nicht automatisch dafür, dass ein Vorgesetzter eine höhere Entlohnung erhält als seine direkten Mitarbeiter. Dies mag man als einen Vorteil unseres Lösungskonzepts werten. Einem Mitarbeiter mit sehr seltenen und wichtigen Fähigkeiten ist bisweilen ein so hohes Gehalt zu zahlen, dass er mehr bekommt als einige seiner Vorgesetzten. Ein theoretischer Grenzfall, in dem der Vorgesetzte in jedem Fall weniger erhält, ist leicht zu konstruieren. Ist dieser Vorgesetzte beispielsweise ein einflussloser unproduktiver Mitarbeiter und seine direkten Mitarbeiter erzielen positive Auszahlungen für das Spiel (N, v), so erhalten sie eine höhere H-Auszahlung als ihr direkter Vorgesetzter. Allerdings können die Gewichte so gewählt werden, dass das Entgelt des Vorgesetzten jenes der direkten Mitarbeiter übersteigt. Ist beispielsweise ein Mitarbeiter i ein einflussloser unproduktiver Mitarbeiter und seine direkten Mitarbeiter $j \in \mathcal{S}(i)$ erhalten im Spiel (N, v) positive Shapley-Auszahlungen, so führen z.B. die Gewichte $w_j > 0,5$ und $w_i = 0$ zu einem höheren Lohn für i, $H_i(N, v, \mathcal{S}, w) > H_j(N, v, \mathcal{S}, w)$.

Theorem XII.2. *Wird in einem Unternehmen für alle Beziehungen zwischen Vorgesetzten und Mitarbeitern $i \in N \setminus \{i_0\}$ ein einheitliches Gewicht $\bar{w}$ verwendet,* $0 < \bar{w} < 1$, *und alle Mitarbeiter erzielen eine positive Shapley-Auszahlung,* $\mathrm{Sh}_i(N, v) > 0$, *dann existiert ein $\bar{w}$ so, dass für alle Mitarbeiter $i, j \in N$ mit $j \in \mathcal{S}(i)$ $H_j(N, v, \mathcal{S}, \bar{w}) < H_i(N, v, \mathcal{S}, \bar{w})$ erfüllt ist.*

Der Beweis dieser Aussage findet sich in Anhang B. Für den Fall, dass die Gewichte von den Personalverantwortlichen des Unternehmens festgelegt werden, zeigt sich hier die Bedeutung dieser Entscheidung und ihrer Auswirkung auf die vertikale Lohnstruktur des Unternehmens.

Für die folgende Aussage zur vertikalen Lohnstruktur eines Unternehmens ist zunächst die Definition von Hierarchieleveln bzw. -ebenen sowie von symmetrischen Unternehmen notwendig. Die Definition von Hierarchieebenen ist dabei an Gilles et al. (1992b) angelehnt:

Definition XII.4. *Die Hierarchie $\mathcal{S}$ bestimmt eine Partition bzw. Leveleinteilung $\mathfrak{L} = (L_0, .., L_M)$ der Spielermenge N mit*

- $L_0 = \{i_0\}$ *und*
- $L_k = \left\{ i \in N \setminus \bigcup_{u=0}^{k-1} L_u \,\middle|\, \mathcal{S}^{-1}(i) \subseteq L_{k-1} \right\}$, $1 \leq k \leq M$, $L_M \neq \emptyset$ *und* $L_{M+1} = \emptyset$.

Das Level L_M ist die niedrigste Hierarchieebene des Unternehmens. Bei dieser Definition von Hierarchieebenen ist der Abstand zum Vorgesetzten i_0 entscheidend für die Zuordnung zu einem Level, sie kann daher als top-down-Hierarchie bezeichnet werden. Eine andere, hier nicht verwendete Definition für Hierarchieebenen, bestimmt die Levelzugehörigkeit an Hand des Abstands zu den Beschäftigten ohne direkte Mitarbeiter (bottom-up-Hierarchie) (Gilles et al. 1992b). Im Beispiel XII.1 führt die top-down-Leveldefinition zur Leveleinteilung $L_0 = \{1\}$, $L_1 = \{2, 5\}$ und $L_2 = \{3, 4\}$. Die Mitarbeiter 2 und 5 besitzen den gleichen Abstand zu Mitarbeiter 1, so dass sie der gleichen Ebene zugeordnet werden. In der bottom-up-Leveldefinition würde sich folgende Leveleinteilung ergeben: $L_0 = \{1\}$, $L_1 = \{2\}$ und

$L_2 = \{3, 4, 5\}$; die Mitarbeiter 3, 4 und 5 besitzen keine direkten Mitarbeiter und sind deshalb der niedrigsten Ebene zugeordnet.

DEFINITION XII.5. *Ein Unternehmen wird als symmetrisch bezeichnet, wenn*

- $\mathrm{Sh}_i(N, v) =: \overline{\mathrm{Sh}}(N, v)$ *für alle* $i \in N$ *gilt,*
- $w_i = \bar{w}$, $0 < \bar{w} < 1$, *für alle* $i \in N \backslash \{i_0\}$ *erfüllt ist sowie*
- $|\mathcal{S}(i)| = s \geq 1$ *für alle* $i \in N \backslash L_M$ *eingehalten wird.*

Dabei bezeichnet $|\mathcal{S}(i)|$ die Anzahl der direkten Mitarbeiter von i bzw. seine Kontrollspanne. Diese ist in einem symmetrischen Unternehmen für jeden Mitarbeiter, außer jenen auf dem niedrigsten Hierarchielevel, gleich groß. Damit fallen beide Leveldefinitionen zusammen. Zudem müssen alle Beschäftigten, bis auf i_0, den gleichen Anteil ihrer Brutto-Auszahlungen an ihren direkten Vorgesetzten abführen. Bei der Erstellung des Unternehmensergebnisses sind die Mitarbeiter ferner symmetrisch. Es kann dann gezeigt werden:

THEOREM XII.3. *In einem symmetrischen Unternehmen mit monotoner Koalitionsfunktion* v *und* $v(N) > 0$ *erhalten die Mitarbeiter in höheren Hierarchieebenen eine bessere Entlohnung als jene in niedrigeren. Formal kann demnach gezeigt werden, dass* $H_i(N, v, \mathcal{S}, w) \geq H_j(N, v, \mathcal{S}, w)$ *für alle* $i \in L_k$, $j \in L_{k+1}$, $0 \leq k \leq M - 1$, *erfüllt ist.*

Der Beweis dieses Theorems findet sich wiederum im Anhang B. Für symmetrische Unternehmen mit monotoner Koalitionsfunktion kann somit das Standardergebnis der Literatur, dass die Mitarbeiter auf höheren Ebenen besser entlohnt werden als jene auf niedrigeren, auch mit dem H-Entlohnungsschema repliziert werden.

6. Personalwirtschaftliche Implikationen: Allokation

Bisher wurde im Aufsatz eine feste Zuordnung der Mitarbeiter auf die durch die Hierarchie geschaffenen Positionen angenommen. Im Folgenden wird diese Annahme aufgehoben und die Allokation der Mitarbeiter auf die verschiedenen Hierarchieebenen thematisiert. Hierfür muss zunächst eine abstrakte hierarchische Struktur eingeführt werden, die ausschließlich die Hierarchie und ihre Gewichte ohne die Zuordnung der Mitarbeiter zu den Positionen beinhaltet. Mit P wird dabei die Menge aller Positionen bezeichnet. Die Funktion T legt die Beziehung zwischen den Positionen fest, $T : P \to 2^P$. Dies geschieht in Analogie zur Funktion $\mathcal{S}$, die die Beziehung zwischen den Mitarbeitern festlegt. Die Funktion T erfüllt dabei die gleichen Anforderungen wie die Funktion $\mathcal{S}$ (siehe Abschnitt 3.1). Es gilt also beispielsweise, dass zu jeder Position, bis auf jene an der Spitze des Unternehmens, o, genau eine vorgesetzte Position existiert, $\left|T^{-1}(x)\right| = 1$ für alle $x \in P\backslash\{o\}$. Die Gewichte sind ebenfalls an die Positionen gebunden. Der Vektor mit den positionsbezogenen Gewichten wird mit m bezeichnet und ordnet jeder Position den relativen Anteil zu, den ein Mitarbeiter auf dieser Position an den direkten Vorgesetzten abgeben muss, $m : P \rightarrow [0, 1]$. Dabei gilt in Analogie zum Gewichtsvektor w, dass das der Position o zugeordnete Gewicht null beträgt, $m_o = 0$.

Definition XII.6. *Eine abstrakte hierarchische Struktur wird durch das Tupel* (P, T, m) *beschrieben.*

Neben der abstrakten hierarchischen Struktur existiert ein kooperatives Spiel (N, v). Die Verbindung zwischen beiden stellt die Besetzungsfunktion β her. Diese bijektive Funktion ordnet die Mitarbeiter den einzelnen Positionen zu, $\beta : N \rightarrow P$. Die Menge aller Zuordnungen wird mit $B(T, N)$ bezeichnet. Zu jedem β gibt es die Hierarchie $\mathcal{S}^\beta$ und den Gewichtsvektor w^β, die wie folgt bestimmt werden. Für die Menge der direkten Beschäftigten von Mitarbeiter $i \in N$ unter der Zuordnung β ergibt sich $\mathcal{S}^\beta(i) := \beta^{-1}(T(\beta(i)))$. Dabei bezeichnet $\beta(i)$ zunächst die Position von i unter β. Mit $T(\beta(i))$ sind dann die Positionen angesprochen, die der Position $\beta(i)$ direkt unterstellt sind. Der Ausdruck $\beta^{-1}(T(\beta(i)))$ liefert schließlich die Menge der Mitarbeiter, die diese Positionen einnehmen. Analog kann die durch β festgelegte Menge der direkten und indirekten Mitarbeiter sowie die der direkten und indirekten Vorgesetzten von Mitarbeiter i definiert werden.

Das Gewicht eines Mitarbeiters $i \in N$ wird ermittelt über $w_i^\beta := m(\beta(i))$. Dabei bezeichnet $\beta(i)$ wiederum die Position von i unter β und $m(\beta(i))$ somit das Gewicht der von i eingenommenen Position.

Auch für die abstrakte hierarchische Struktur lässt sich in Analogie zur Definition XII.4 eine Leveleinteilung definieren.

Definition XII.7. *Die Funktion* T *bestimmt eine Partition bzw. Leveleinteilung* $\mathfrak{L}^P = \left(L_0^P, .., L_M^P\right)$ *der Positionen* P *mit*

- $L_0^P = \{o\}$ *und*
- $L_k^P = \left\{ x \in P\backslash\bigcup_{u=0}^{k-1} L_u^P \middle| T^{-1}(x) \subseteq L_{k-1}^P \right\}$, $1 \leq k \leq M$, $L_M^P \neq \emptyset$ *und* $L_{M+1}^P = \emptyset$.

Aus der Leveldefinition für die Positionen und der Zuordnung β kann wiederum die Leveleinteilung der Mitarbeiter gewonnen werden. Es gilt dabei $L_k^\beta := \beta^{-1}\left(L_k^P\right)$.

Für die Aussage bezüglich der Allokation der Beschäftigten auf die einzelnen Positionen der Unternehmenshierarchie ist zudem eine weitere Symmetrie-Definition erforderlich.

DEFINITION XII.8. *Eine abstrakte hierarchische Struktur eines Unternehmens wird als symmetrisch hinsichtlich ihrer Gewichte bezeichnet, wenn für alle* $x \in L_k^P$*, mit* $k = 0, \ldots, M$*,* $m_x =: m_{L_k^P}$ *erfüllt ist.*

Die Anforderungen sind somit geringer als bei der Definition symmetrischer Unternehmen, bei denen zudem Symmetrie im Produktionsprozess, gleiche Kontrollspannen sowie über alle Level konstante Gewichte $\bar{w}$ gefordert wurden.

THEOREM XII.4. *Sei ein Unternehmen mit symmetrischer abstrakter hierarchischer Struktur* (P, T, m)*, mit* $0 < m_x < 1$ *für alle* $x \in P \backslash \{o\}$*, sowie dem Tupel* (N, v)*, mit* $|N| = |P|$*, gegeben. Mitarbeiter* i_0*, welcher bereits der Position* o *zugeordnet wurde,* $\beta^{-1}(o) = i_0$*, entscheidet über die weitere Ausgestaltung der Funktion* β*. Er wählt dabei jene Zuordnung* β_{opt}*, die seine Auszahlung maximiert:*

$$\beta_{opt} \in \underset{\beta \in B(T,N), \beta(o) = i_0}{\arg\max} H_{i_0}\left(N, v, w^\beta, \mathcal{S}^\beta\right).$$

Resultat ist eine Zuordnung der Mitarbeiter auf die verschiedenen Positionen, so dass aus $i \in L_k^\beta$ *und* $j \in L_l^\beta$*, mit* $1 \le k \le l \le M$*,* $\mathrm{Sh}_i(N, v) \ge \mathrm{Sh}_j(N, v)$ *folgt. Mitarbeiter* i_0 *wird somit die produktivsten Mitarbeiter, gemessen an ihren Shapley-Auszahlungen, in die ihm direkt unterstellte Ebene zuordnen.*

Der Beweis ist im Anhang B gegeben. Aus diesem Theorem folgt unmittelbar das nächste Korollar.

COROLLARY XII.1. *Hat Mitarbeiter* i_0 *die übrigen Beschäftigten entsprechend ihrer Produktivität den einzelnen Ebenen zugeordnet, so hat kein Mitarbeiter aus* $N \backslash \{i_0\}$ *einen Anreiz, in dem bei ihm beginnenden Teil der Hierarchie die Zuordnung der Mitarbeiter zu den einzelnen Positionen abzuändern.*

Auch unser Ansatz kann somit, ähnlich wie Calvo & Wellisz (1979), Rosen (1982), Waldman (1984) und Qian (1994), eine Erklärung dafür liefern, dass produktive Mitarbeiter in höhere Hierarchieebenen befördert werden. Die Koalitionsfunktion, und somit auch das Gesamtergebnis des Unternehmens, $v(N)$, bleibt durch diese Allokation allerdings unverändert.

7. Zusammenfassung

Im vorliegenden Aufsatz wurde mit Hilfe der kooperativen Spieltheorie die hierarchische Struktur eines Unternehmens durch die Hierarchie $\mathcal{S}$ und den Gewichtsvektor w abgebildet. Zusammen mit dem Wissen um die Spielermenge N und die Koalitionsfunktion v kann damit jedem Mitarbeiter des Unternehmens eine Auszahlung bzw. Entlohnung zugeordnet werden. Diese berücksichtigt zum einen die Leistung der Mitarbeiter, da in die Berechnung der Entlohnung deren jeweilige Shapley-Auszahlung einfließt. Zum anderen findet die Hierarchie mit ihrer umverteilenden Wirkung bei der Festlegung der Mitarbeiter-Entlohnung Berücksichtigung. Damit wird die Idee aufgegriffen, dass Vorgesetzte am (Miss-)Erfolg ihrer Mitarbeiter partizipieren.

Es ist dabei nicht ohne weitere Annahmen sichergestellt, dass Vorgesetzte eine höhere Entlohnung erhalten als ihre direkten Mitarbeiter. Es kann jedoch gezeigt werden, dass es für jedes Unternehmen, in dem kein Mitarbeiter unproduktiv ist, ein einheitliches $\bar{w}$, $0 < \bar{w} < 1$, gibt, so dass die Mitarbeiter in höheren Hierarchieebenen besser entlohnt werden als jene in niedrigeren. Damit wird die Bedeutung der personalpolitischen Entscheidung über den Gewichtsvektor w für die vertikalen Lohndifferenzen des Unternehmens hervorgehoben. Für einen Spezialfall, den des symmetrischen Unternehmens, kann gezeigt werden, dass jeder Vorgesetzte unabhängig vom gewählten Gewichtsvektor $\bar{w}$, mit $0 < \bar{w} < 1$, eine höhere Entlohnung erhält als die ihm zugeordneten Mitarbeiter. Das Resultat der besseren Entlohnung auf höheren Hierarchiestufen ist zugleich ein Ergebnis in den Modellen von Calvo & Wellisz (1979), Rosen (1982), Waldman (1984) und Qian (1994), das somit für den Spezialfall symmetrischer Unternehmen Bestätigung findet.

Des Weiteren kann der hier vorgestellte Ansatz eine Erklärung dafür liefern, dass produktivere Mitarbeiter höheren Hierarchieebenen zugeordnet werden. Diese Allokation liegt im Interesse des Mitarbeiters an der Spitze des Unternehmens, i_0. Ist dieser für die Zuordnung der übrigen Mitarbeiter verantwortlich, so wird er die produktivsten von ihnen der ihm direkt unterstellten Hierarchieebene zuordnen. Auch hiermit werden die Ergebnisse der Modelle von Calvo & Wellisz (1979), Rosen (1982), Waldman (1984) und Qian (1994) repliziert.

In Bezug zum Aufsatz von van den Brink (2008) (in der Folge mit vdB abgekürzt) kann festgestellt werden, dass kein Faktor α so existiert, dass durch Multiplikation mit diesem die H-Auszahlungen aus den entsprechenden vdB-Auszahlungen gewonnen werden können. Beispielhaft kann zur Verdeutlichung ein Unternehmen mit drei Beschäftigten, $N = \{1, 2, 3\}$, der Hierarchie $\mathcal{S}$ mit $\mathcal{S}(1) = \{2, 3\}$ und $\mathcal{S}(2) = \mathcal{S}(3) = \emptyset$, dem Gewichtsvektor w und der Koalitionsfunktion v betrachtet werden. Für den Fall, dass die marginalen Beiträge gemäß dem Shapley-Lösungskonzept gewichtet werden, erhält Mitarbeiter 2 die vdB-Auszahlung $\theta_2^{\text{Sh}}(N, v, \mathcal{S}) = \frac{1}{3} \cdot \frac{1}{2} \cdot [v(\{1, 2\}) - v(\{1\})] + \frac{1}{3} \cdot [v(N) - v(\{1, 3\})]$. Eine andere Gewichtung der marginalen Beiträge, wie sie bei van den Brink (2008) möglich ist, ändert nichts am Umstand, dass der Wert $v(\{2\})$ nicht in die Berechnung der vdB-Auszahlung von Mitarbeiter 2 einfließt. Die H-Auszahlung für diesen Mitarbeiter bestimmt sich dagegen als

$$\begin{aligned} H_2(N, v, \mathcal{S}, w) = (1 - w_2) \cdot \\ &\left(\frac{1}{3} \cdot v(\{2\}) + \frac{1}{6} [v(\{1,2\}) - v(\{1\})] \right. \\ &\left. + \frac{1}{6} [v(\{2,3\}) - v(\{3\})] + \frac{1}{3} [v(N) - v(\{1,3\})] \right). \end{aligned}$$

Wird nun die Koalitionsfunktion v geeignet verändert, in dem beispielsweise $v(\{2\})$ variiert wird, so verändert sich die H-Auszahlung des Mitarbeiters 2, seine vdB-Auszahlung bleibt jedoch konstant, d.h. es gibt keinen konstanten Faktor α, mit dem vor und nach der Variation von $v(\{2\})$ durch Multiplikation aus der vdB-Auszahlung die H-Auszahlung von Mitarbeiter 2 bestimmt werden kann.

Ein Ergebnis des vdB-Aufsatzes, in dem ausschließlich die Mitarbeiter auf der untersten Ebene gemäß bottom-up-Hierarchie produktiv sind, lautet, dass $\theta_i(N, v, \mathcal{S}) \geq \theta_j(N, v, \mathcal{S})$ für alle $j \in \mathcal{S}(i)$ gilt, sofern dem Spiel eine monotone Koalitionsfunktion zu Grunde liegt. Hierbei wird θ_i verwendet, um anzudeuten, dass die Gewichtung der marginalen Beiträge nicht unbedingt der im Shapley-Lösungskonzept folgen muss. In dieser allgemeinen Form gilt dies für den hier entwickelten Ansatz nicht, wie die Theoreme XII.2 und XII.3 zeigen. Allerdings lassen sich auch für den Spezialfall, dass ausschließlich die Beschäftigten des untersten Hierarchielevels produktiv sind, Gewichte finden, die sicherstellen, dass Mitarbeiter in höheren Ebenen besser entlohnt werden als jene in niedrigeren. Ist die dem Spiel zu Grunde liegende Koalitionsfunktion zudem konvex, so resultiert im vdB-Aufsatz eine obere Lohngrenze eines Vorgesetzten in Höhe der Lohnsumme seiner direkten Mitarbeiter. Eine solch allgemeine Aussage lässt sich für das H-Entlohnungsschema nicht treffen.

Für zukünftige Forschungsarbeit ergeben sich aus dem in diesem Aufsatz vorgestellten Ansatz zwei Ausgangspunkte. Zum einen wurden bisher die Auswirkungen der Hierarchie auf den Produktionsprozess nicht berücksichtigt; alle Mitarbeiter arbeiten symmetrisch zusammen. Allerdings ist die Koordination der Mitarbeiter eine Funktion von Führungskräften, die in einer Weiterentwicklung des H-Entlohnungsschemas berücksichtigt werden sollte. Zum anderen wurde das vom Unternehmen bzw. einzelnen Koalitionen erwirtschaftete Ergebnis bisher nicht von der Hierarchie und dem Gewichtsvektor beeinflusst. Somit ist es bisher nicht möglich, für Unternehmen, die entsprechend dem H-Entlohnungsschema die Löhne der Mitarbeiter festlegen, eine optimale das Ergebnis maximierende Hierarchie zu bestimmen.

Anhang A

Axiomatisierung beinhaltet zweierlei. Zum einen haben wir zu zeigen, dass das H-Entlohnungsschema die in Satz XII.1 genannten Axiome erfüllt. Zum anderen müssen wir beweisen, dass jedes Entlohnungsschema, das diesen Axiomen gehorcht, bereits das H-Entlohnungsschema ist.

Wir beginnen mit der Überprüfung des Effizienz-Axioms

$$\sum_{i=1}^{n} H_i (N, v, \mathcal{S}, w) = \sum_{i=1}^{n} \sum_{j=1}^{n} f_i (\mathcal{S}, w, j) \cdot \mathrm{Sh}_j (N, v) . \tag{XII.5}$$

Es ist zu zeigen, dass $\sum_{i=1}^{n} f_i (\mathcal{S}, w, j) = 1$ für alle $j \in N$ gilt. Für den Fall, dass $w_j = 0$ ist, folgt dies direkt aus Formelzeile XII.1; $f_i (\mathcal{S}, w, j)$ ist genau dann eins, wenn $i = j$ gilt. Für i_0 folgt demnach sofort $\sum_{i=1}^{n} f_i (\mathcal{S}, w, i_0) = 1$, da ausschließlich Mitarbeiter i_0, auf Grund von $w_0 = 0$, den Anteil eins erhält, alle anderen Beschäftigten erhalten den Anteil null. Für $i \in L_1$ resultiert für die Summe der Anteile an is Shapley-Auszahlung (siehe Gl. XII.1)

$$\begin{aligned} & \sum_{g \in T(i_0, i)} [1 - w_g] \prod_{l \in \hat{\mathcal{S}}(g),\ l \in T(i_0, i)} w_l \\ = \quad & [1 - w_0] \cdot w_i + [1 - w_i] = 1. \end{aligned}$$

Es folgt nun der Beweis mit Hilfe der Induktion. Angenommen $\sum_{i=1}^{n} f_i (\mathcal{S}, w, j)$ $= 1$ gilt für alle $j \in L_d$, $d \geq 1$. Es wird jetzt ein r aus dem nächstniedrigeren Level L_{d+1} zusätzlich in den Pfad aufgenommen, dessen direkter Vorgesetzter in L_d der Beschäftige t ist, d.h. $t = \mathcal{S}^{-1} (r)$. Es resultiert:

$$\begin{aligned} & \sum_{i=1}^{n} f_i (\mathcal{S}, w, r) = \sum_{i \in T(i_0, r)} f_i (\mathcal{S}, w, r) \\ = \quad & \sum_{i \in T(i_0, r)} [1 - w_i] \prod_{l \in \hat{\mathcal{S}}(i),\ l \in T(i_0, r)} w_l \\ = \quad & \sum_{i \in T(i_0, j)} [1 - w_i] \prod_{l \in \hat{\mathcal{S}}(i),\ l \in T(i_0, r)} w_l + [1 - w_r] \prod_{l \in \hat{\mathcal{S}}(r),\ l \in T(i_0, r)} w_l \\ = \quad & \sum_{i \in T(i_0, j)} w_r \cdot [1 - w_i] \prod_{l \in \hat{\mathcal{S}}(i),\ l \in T(i_0, j)} w_l + [1 - w_r] \prod_{l \in \hat{\mathcal{S}}(r),\ l \in T(i_0, r)} w_l \\ = \quad & w_r \cdot \sum_{i=1}^{n} f_i (\mathcal{S}, w, j) + [1 - w_r] \prod_{l \in \hat{\mathcal{S}}(r),\ l \in T(i_0, r)} w_l \\ = \quad & w_r \cdot 1 + [1 - w_r] = 1. \end{aligned}$$

Dies schließt den Beweis, dass $\sum_{i=1}^{n} f_i (\mathcal{S}, w, j) = 1$ gilt. Formelzeile XII.5 kann jetzt als

$$\sum_{i=1}^{n} H_i (N, v, \mathcal{S}, w) = \sum_{j=1}^{n} 1 \cdot \mathrm{Sh}_j (N, v) .$$

notiert werden. Da das Shapley-Lösungskonzept Effizienz erfüllt, $\sum_{j=1}^{n} \mathrm{Sh}_j (N, v) = v(N)$, weist auch das H-Entlohnungsschema diese Eigenschaft auf.

Die Additivitätseigenschaft überträgt sich ebenfalls vom Shapley-Lösungskonzept auf das H-Entlohnungsschema, wie bereits aus Gleichung XII.2 ersichtlich ist. Auf Grund der Tatsache, dass das Shapley-Lösungskonzept dem Symmetrie-Axiom und dem Nullspieler-Axiom genügt und $H_i (N, v, \mathcal{S}, w[N]) = \mathrm{Sh}_i (N, v)$ für einen beliebigen Mitarbeiter $i \in N$ gilt, sind durch das H-Entlohnungsschema das schwache Symmetrie-Axiom und das schwache Nullspieler-Axiom erfüllt.

Die Erfüllung des Unabhängigkeits-Axioms folgt direkt aus den Gleichungen XII.1 und XII.2. Die Formelzeile XII.3 zeigt, dass H dem Brutto-Netto-Axiom ebenfalls gerecht wird. Aus der Gleichung XII.4 folgt

$$H_i (N, v, \mathcal{S}, w[\{i\} \cup \mathcal{S}(i)]) = \mathrm{Sh}_i (N, v) = H_i (N, v, \mathcal{S}, w[N]),$$

d.h. das Isolations-Axiom. Unter Nutzung von Gleichung XII.4 resultiert zudem

$$\begin{aligned} H_i (N, v, \mathcal{S}, w[i]) - H_i (N, v, \mathcal{S}, w[i,j]) &= w_j \cdot H_j (N, v, \mathcal{S}, w[j]) \\ &= H_j (N, v, \mathcal{S}, w[j]) - H_j (N, v, \mathcal{S}, w) \end{aligned}$$

für alle $j \in \mathcal{S}(i)$, wobei bei der Umformung zur zweiten Zeile Gleichung XII.3 genutzt wurde, d.h. das H-Entlohnungsschema erfüllt das Abspaltungs-Axiom. Somit ist bewiesen, dass alle genannten Axiome erfüllt werden.

Der zweite Teil der Axiomatisierung verlangt, dass bei Anwendung der Axiome auf hierarchische Spiele $(N, v, \mathcal{S}, w)$ zur Ermittlung der Mitarbeiter-Auszahlungen genau jene Entlohnung resultiert, die das H-Entlohnungsschema vorsieht.

Angenommen ein Entlohnungsschema φ erfüllt das Effizienz-Axiom, das Additivitäts-Axiom, das schwache Nullspieler-Axiom, das schwache Symmetrie-Axiom, das Brutto-Netto-Axiom, das Abspaltungs-Axiom, das Isolations-Axiom und das Unabhängigkeits-Axiom. Mit den ersten vier Axiomen wird für den Gewichtsvektor $w[N]$ die Shapley-Auszahlung axiomatisiert, d.h. $\varphi_i (N, v, \mathcal{S}, w[N]) = \mathrm{Sh}_i (N, v)$. Für die Brutto-Auszahlung eines Mitarbeiters i auf der niedrigsten Hierarchieebene, $\mathcal{S}(i) = \emptyset$, folgt dann

$$\begin{aligned} \varphi_i (N, v, \mathcal{S}, w[i]) &= \varphi_i (N, v, \mathcal{S}, w[\{i\} \cup \mathcal{S}(i)]) \\ &\overset{\textbf{Isolation}}{=} \varphi_i (N, v, \mathcal{S}, w[N]) = \mathrm{Sh}_i (N, v), \end{aligned}$$

und damit für seine Netto-Auszahlung

$$\varphi_i (N, v, \mathcal{S}, w) \overset{\textbf{Brutto-Netto}}{=} (1 - w_i) \cdot \mathrm{Sh}_i (N, v).$$

Sei $\mathcal{S}(i) = \{j_1, j_2, \ldots, j_s\}$, so resultiert

$$\begin{aligned}
& \varphi_i(N, v, \mathcal{S}, w[i]) - \varphi_i(N, v, \mathcal{S}, w[\{i\} \cup \mathcal{S}(i)]) \\
= & \sum_{\ell=1}^{s} \varphi_i(N, v, \mathcal{S}, w[\{i\} \cup \{j_1, j_2, \ldots, j_{\ell-1}\}]) \\
& -\varphi_i(N, v, \mathcal{S}, w[\{i\} \cup \{j_1, j_2, \ldots, j_\ell\}]) \\
& \overset{\textbf{Abspaltung}}{=} \sum_{\ell=1}^{s} \varphi_{j_\ell}(N, v, \mathcal{S}, w[\{j_1, j_2, \ldots, j_\ell\}]) \\
& -\varphi_{j_\ell}(N, v, \mathcal{S}, w[\{j_1, j_2, \ldots, j_{\ell-1}\}]) \\
& \overset{\textbf{Unabhängigkeit}}{=} \sum_{\ell=1}^{s} \varphi_{j_\ell}(N, v, \mathcal{S}, w[j_\ell]) - \varphi_{j_\ell}(N, v, \mathcal{S}, w) \\
& \overset{\textbf{Brutto-Netto}}{=} \sum_{j \in \mathcal{S}(i)} w_j \cdot \varphi_j(N, v, \mathcal{S}, w[j]).
\end{aligned}$$

Es wird die Differenz ermittelt, zwischen der Brutto-Auszahlung von Mitarbeiter i und seiner Auszahlung, wenn zudem die Stärke der Verbindungen zu seinen direkten Mitarbeitern auf null gesetzt sind. Dabei können die Gewichte mit einem Schlag oder einzeln auf null reduziert werden (Übergang Zeile 1 zu 2). Den Betrag, den i brutto verliert, erhalten seine direkten Mitarbeiter jeweils netto hinzu (Zeile 2 zu 3). Dabei ist für einen einzelnen direkten Mitarbeiter nur der „Moment“ relevant, wenn die eigene Verbindung zum Vorgesetzten angetastet wird (Zeile 3 zu 4). Beim Übergang zur letzten Zeile wird dann mit Hilfe des Brutto-Netto-Axioms nochmals leicht umformuliert.

Da

$$\varphi_i(N, v, \mathcal{S}, w[\{i\} \cup \mathcal{S}(i)]) \overset{\textbf{Isolation}}{=} \varphi_i(N, v, \mathcal{S}, w[N]) = \mathrm{Sh}_i(N, v)$$

gilt, resultiert für die Brutto-Auszahlung eines Mitarbeiters i

$$\varphi_i(N, v, \mathcal{S}, w[i]) = \mathrm{Sh}_i(N, v) + \sum_{j \in \mathcal{S}(i)} w_j \cdot \varphi_j(N, v, \mathcal{S}, w[j]).$$

Zusammen mit dem Brutto-Netto-Axiom ist H somit induktiv bestimmt, d.h. φ ist eindeutig.

Anhang B

Beweis zu Theorem XII.2

Auf Grund der positiven Shapley-Auszahlungen und des Intervalls für das einheitliche Gewicht $\bar{w}$, $0 < \bar{w} < 1$, für alle $i \in N \backslash \{i_0\}$, resultiert $H_j(N, v, \mathcal{S}, \bar{w}) > 0$ für alle $j \in N$. Für den Mitarbeiter i_0 resultiert zudem auf Grund der Gleichungen XII.3 und XII.4

$$H_{i_0}(N, v, \mathcal{S}, \bar{w}) = \mathrm{Sh}_{i_0}(N, v) + \sum_{j \in \mathcal{S}(i_0)} \frac{\bar{w}}{1-\bar{w}} H_j(N, v, \mathcal{S}, \bar{w}).$$

Da $\mathrm{Sh}_{i_0}(N,v) > 0$ gilt, resultiert $H_{i_0}(N,v,\mathcal{S},\bar{w}) > H_j(N,v,\mathcal{S},\bar{w})$ bereits für den Fall, dass i_0 nur einen direkten Mitarbeiter besitzt, $j \in \mathcal{S}(i_0)$, sobald $\bar{w} \geq \frac{1}{2}$ erfüllt ist.

Für alle Mitarbeiter $i \in N\backslash\{i_0\}$, mit $\mathcal{S}(i) \neq \emptyset$, folgt

$$H_i(N,v,\mathcal{S},\bar{w}[i]) = \mathrm{Sh}_i(N,v) + \sum_{j\in\mathcal{S}(i)} \bar{w} \cdot H_j(N,v,\mathcal{S},\bar{w}[j]) \tag{XII.6}$$

aus den Gleichungen XII.2 und XII.4. Es wird jetzt gezeigt, dass es für jeden Mitarbeiter $k \in \mathcal{S}(i)$ ein $w_k < 1$ so gibt, dass $H_i(N,v,\mathcal{S},\bar{w}[i]) > H_k(N,v,\mathcal{S},\bar{w}[k])$ für alle $\bar{w} \geq w_k$ erfüllt ist. Angenommen, dies gilt nicht. Dann gibt es eine Folge von Gewichten $(\bar{w}_n)_{n\in\mathbb{N}}$, $\bar{w}_n < 1$, $\bar{w}_n \to 1$ die zu

$$H_k(N,v,\mathcal{S},\bar{w}_n[k]) \geq H_i(N,v,\mathcal{S},\bar{w}_n[i]) \tag{XII.7}$$

führt. Wird diese Gleichung mit Gleichung XII.6 zusammengefügt so resultiert

$$H_k(N,v,\mathcal{S},\bar{w}_n[k]) \geq \mathrm{Sh}_i(N,v) + \sum_{j\in\mathcal{S}(i)} \bar{w}_n \cdot H_j(N,v,\mathcal{S},\bar{w}_n[j])$$

und damit

$$(1-\bar{w}_n)\cdot H_k(N,v,\mathcal{S},\bar{w}_n[k]) \geq \mathrm{Sh}_i(N,v) + \sum_{j\in\mathcal{S}(i)\backslash\{k\}} \bar{w}_n \cdot H_j(N,v,\mathcal{S},\bar{w}_n[j]).$$

Während die linke Seite der Gleichung für $n \to \infty$ gegen null konvergiert ($H_k(N,v,\mathcal{S},\bar{w}_n[k])$ ist auf Grund von Gleichung XII.2 endlich), konvergiert die rechte Gleichungsseite gegen eine echt positive Zahl, d.h. ein Widerspruch konnte gezeigt werden. Wird nun $w^* := \max\left(\frac{1}{2}, \max_{i\in N\backslash\{i_0\},k\in\mathcal{S}(i)} w_k\right)$ gesetzt für alle Mitarbeiter $i \in N\backslash\{i_0\}$ und $j \in \mathcal{S}(i)$, so ergibt sich

$$\begin{gathered} H_i(N,v,\mathcal{S},\bar{w}) = (1-\bar{w})\cdot H_i(N,v,\mathcal{S},\bar{w}[i]) > \\ (1-\bar{w})\cdot H_j(N,v,\mathcal{S},\bar{w}[j]) = H_j(N,v,\mathcal{S},\bar{w}) \end{gathered}$$

für alle $w^* < \bar{w} < 1$. Zusammen mit dem Ergebnis zu Mitarbeiter i_0 schließt dies den Beweis.

Beweis zu Theorem XII.3

Es wird allgemein ein Unternehmen mit M Ebenen angenommen, das die geforderten Eigenschaften hinsichtlich der hierarchischen Struktur und der Koalitionsfunktion erfüllt. In drei Schritten wird gezeigt,

- dass die Mitarbeiter der untersten Ebene weniger verdienen als jene in der zweitniedrigsten,
- dass bei zwei beliebigen Ebenen unterhalb der höchsten, die Mitarbeiter in der höheren besser entlohnt werden als diejenigen in der niedrigeren und schließlich
- dass die (der) Mitarbeiter in höchsten Ebene eine bessere Entlohnung erhält als ein beliebiger Mitarbeiter in der zweithöchsten Ebene.

Die Auszahlung für je einen Mitarbeiter auf der niedrigsten Ebene L_M und der zweitniedrigsten L_{M-1} bestimmt sich durch:

$$\begin{aligned} H_{\bar{M}}(N, v, \mathcal{S}, \bar{w}) &= (1-\bar{w}) \cdot \overline{\mathrm{Sh}} \\ H_{\overline{M-1}}(N, v, \mathcal{S}, \bar{w}) &= (1-\bar{w}) \cdot \left[\overline{\mathrm{Sh}} + s \cdot \bar{w} \cdot \overline{\mathrm{Sh}}\right]. \end{aligned}$$

Dabei bezeichnet $H_{\bar{M}}(N, v, \mathcal{S}, \bar{w})$ die Entlohnung, die die Mitarbeiter im Level L_M erhalten, $H_{\bar{M}}(N, v, \mathcal{S}, \bar{w}) := H_i(N, v, \mathcal{S}, \bar{w})$ für $i \in L_M$. Es folgt unmittelbar:

$$H_{\overline{M-1}}(N, v, \mathcal{S}, \bar{w}) - H_{\bar{M}}(N, v, \mathcal{S}, \bar{w}) = (1-\bar{w}) \cdot s \cdot \bar{w} \cdot \overline{\mathrm{Sh}} > 0.$$

Für den zweiten Teil des Beweises werden ausschließlich die Brutto-Auszahlungen der Mitarbeiter verglichen. Da ein einheitliches $\bar{w}$ und somit auch $(1-\bar{w})$ für alle Mitarbeiter existiert, können die Ergebnisse auf die Netto-Entlohnungen übertragen werden. Der Beweis erfolgt durch Induktion. Für alle Ebenen $L_M, L_{M-1}, \ldots, L_t, L_r$ wurde bereits gezeigt, dass $H_{\bar{r}}(N, v, \mathcal{S}, \bar{w}[\bar{r}]) > H_{\bar{t}}(N, v, \mathcal{S}, \bar{w}[\bar{t}])$ gilt. Dabei ist $H_{\bar{r}}(N, v, \mathcal{S}, \bar{w}[\bar{r}])$ definiert durch $H_{\bar{r}}(N, v, \mathcal{S}, \bar{w}[\bar{r}]) := H_i(N, v, \mathcal{S}, \bar{w}[i])$ für $i \in L_r$. Wird nun eine Ebene L_s mit $s = r-1$, $s \neq 0$, betrachtet, so folgt für die Brutto-Auszahlung der Mitarbeiter in L_s

$$H_{\bar{s}}(N, v, \mathcal{S}, \bar{w}[\bar{s}]) = \overline{\mathrm{Sh}} + s \cdot \bar{w} \cdot H_{\bar{r}}(N, v, \mathcal{S}, \bar{w}[\bar{r}]).$$

Der Vergleich mit der Brutto-Auszahlung der Ebene L_r ergibt:

$$\begin{aligned} & \overline{\mathrm{Sh}} + s \cdot \bar{w} \cdot H_{\bar{r}}(N, v, \mathcal{S}, \bar{w}[\bar{r}]) - H_{\bar{r}}(N, v, \mathcal{S}, \bar{w}[\bar{r}]) \\ = \; & s \cdot \bar{w} \cdot H_{\bar{r}}(N, v, \mathcal{S}, \bar{w}[\bar{r}]) + \overline{\mathrm{Sh}} - H_{\bar{r}}(N, v, \mathcal{S}, \bar{w}[\bar{r}]) \\ = \; & s \cdot \bar{w} \cdot H_{\bar{r}}(N, v, \mathcal{S}, \bar{w}[\bar{r}]) - s \cdot \bar{w} \cdot H_{\bar{t}}(N, v, \mathcal{S}, \bar{w}[\bar{t}]) \\ = \; & s \cdot \bar{w} \cdot \underbrace{\left(H_{\bar{r}}(N, v, \mathcal{S}, \bar{w}[\bar{r}]) - H_{\bar{t}}(N, v, \mathcal{S}, \bar{w}[\bar{t}])\right)}_{>0} > 0, \end{aligned}$$

womit die Induktion geschlossen ist.

Es folgt nun im letzten Schritt der Vergleich der Entlohnung für den Mitarbeiter auf der höchsten Ebene, L_0, mit derjenigen auf der nächstniedrigeren. Aus dem zweiten Teil des Beweises folgt, dass $H_i(N, v, \mathcal{S}, \bar{w}) > H_j(N, v, \mathcal{S}, \bar{w})$ für alle $i \in L_k$, $j \in L_{k+1}$, $1 \leq k \leq M-1$, erfüllt ist. Die Netto-Auszahlung für i_0 bestimmt sich durch:

$$H_{\bar{0}}(N, v, \mathcal{S}, \bar{w}) = \overline{\mathrm{Sh}} + s \cdot \bar{w} \cdot H_{\bar{1}}(N, v, \mathcal{S}, \bar{w}[\bar{1}]).$$

Diese Netto-Entlohnung wird jetzt mit der Brutto-Entlohnung eines Mitarbeiters auf Ebene L_1 verglichen. Sollte bereits die Netto-Entlohnung des Mitarbeiters i_0 höher liegen als der Brutto-Lohn seiner direkten Mitarbeiter in Ebene L_1, so gilt dies auch für deren Netto-Löhne, da diese durch Multiplikation mit dem Faktor $(1-\bar{w}) < 1$ aus deren Brutto-Entlohnung gewonnen werden:

$$
\begin{aligned}
& H_{\bar{0}}(N,v,\mathcal{S},\bar{w}) - H_{\bar{1}}(N,v,\mathcal{S},\bar{w}[\bar{1}]) \\
= \; & \overline{\mathrm{Sh}} + s\cdot\bar{w}\cdot H_{\bar{1}}(N,v,\mathcal{S},\bar{w}[\bar{1}]) - H_{\bar{1}}(N,v,\mathcal{S},\bar{w}[\bar{1}]) \\
= \; & s\cdot\bar{w}\cdot H_{\bar{1}}(N,v,\mathcal{S},\bar{w}[\bar{1}]) - s\cdot\bar{w}\cdot H_{\bar{2}}(N,v,\mathcal{S},\bar{w}[\bar{2}]) \\
= \; & s\cdot\bar{w}\cdot \underbrace{\left(H_{\bar{1}}(N,v,\mathcal{S},\bar{w}[\bar{1}]) - H_{\bar{2}}(N,v,\mathcal{S},\bar{w}[\bar{2}])\right)}_{>0} > 0
\end{aligned}
$$

und damit: $H_{\bar{0}}(N,v,\mathcal{S},\bar{w}) > H_{\bar{1}}(N,v,\mathcal{S},\bar{w}[\bar{1}]) > H_{\bar{1}}(N,v,\mathcal{S},\bar{w})$.

Beweis zu Theorem XII.4

Der Beweis erfolgt über einen Widerspruch. Würde i_0 eine Zuordnung $\check{\beta} \in B(T,N)$ der übrigen Mitarbeiter so wählen, dass die Mitarbeiter $i,j \in N$, mit $\mathrm{Sh}_i(N,v) > \mathrm{Sh}_j(N,v)$, den Leveln wie folgt zugewiesen werden: $i \in L_k^{\check{\beta}}$ und $j \in L_{k-1}^{\check{\beta}}$, mit $2 \leq k \leq M$, so gibt es ein $\hat{\beta} \in B(T,N)$, das die Entlohnung des Mitarbeiters i_0 steigert. Es ist klar, dass hier die Betrachtung der Ebenen k und $k-1$ anstelle allgemein von Ebenen k und l, $l < k$ ausreicht. Diese Zuordnung $\hat{\beta}$ legt $\hat{\beta}(j) = \check{\beta}(i)$ und $\hat{\beta}(i) = \check{\beta}(j)$ fest, d.h. die Mitarbeiter i und j tauschen ihre Positionen. Alle übrigen Mitarbeiter $l \in N\backslash\{i,j\}$ werden unter beiden Zuordnungen den gleichen Positionen zugewiesen, d.h. $\hat{\beta}^{-1}(l) = \check{\beta}^{-1}(l)$.

Es wird nun die Differenz der Auszahlungen des Mitarbeiters i_0 unter beiden Zuordnungen bestimmt und gezeigt, dass die Auszahlung unter $\hat{\beta}$ höher ist als unter $\check{\beta}$

$$
\begin{aligned}
& H_{i_0}\left(N,v,\mathcal{S}^{\check{\beta}},w^{\check{\beta}}\right) - H_{i_0}\left(N,v,\mathcal{S}^{\hat{\beta}},w^{\hat{\beta}}\right) \\
= \; & f_{i_0}\left(\mathcal{S}^{\check{\beta}},w^{\check{\beta}},i\right)\cdot \mathrm{Sh}_i(N,v) + f_{i_0}\left(\mathcal{S}^{\check{\beta}},w^{\check{\beta}},j\right)\cdot \mathrm{Sh}_j(N,v) \\
& -\left[f_{i_0}\left(\mathcal{S}^{\hat{\beta}},w^{\hat{\beta}},i\right)\cdot \mathrm{Sh}_i(N,v) + f_{i_0}\left(\mathcal{S}^{\hat{\beta}},w^{\hat{\beta}},j\right)\cdot \mathrm{Sh}_j(N,v)\right] \\
= \; & f_{i_0}\left(\mathcal{S}^{\check{\beta}},w^{\check{\beta}},i\right)\cdot \mathrm{Sh}_i(N,v) + f_{i_0}\left(\mathcal{S}^{\check{\beta}},w^{\check{\beta}},j\right)\cdot \mathrm{Sh}_j(N,v) \\
& -\left[f_{i_0}\left(\mathcal{S}^{\check{\beta}},w^{\check{\beta}},j\right)\cdot \mathrm{Sh}_i(N,v) + f_{i_0}\left(\mathcal{S}^{\check{\beta}},w^{\check{\beta}},i\right)\cdot \mathrm{Sh}_j(N,v)\right] \\
= \; & \underbrace{\left[\mathrm{Sh}_i(N,v) - \mathrm{Sh}_j(N,v)\right]}_{>0}\cdot \underbrace{\left[f_{i_0}\left(\mathcal{S}^{\check{\beta}},w^{\check{\beta}},i\right) - f_{i_0}\left(\mathcal{S}^{\check{\beta}},w^{\check{\beta}},j\right)\right]}_{<0} < 0.
\end{aligned}
$$

Der Umformung von der ersten zur zweiten Formelzeile liegt zu Grunde, dass die Zahlungen, die i_0 von allen anderen Mitarbeitern $l \in N\backslash\{i,j\}$ erhält, unverändert sind, da diese in ihren Positionen verbleiben (siehe Gl. XII.1) und zudem die Shapley-Auszahlungen aller Mitarbeiter nicht verändert werden. Somit ist es ausreichend, den gesamten absoluten Betrag, den i_0 von i und j unter beiden Zuordnungen erhält, zu vergleichen. Beim Übergang zu nächsten Formelzeile wird berücksichtigt, dass gemäß der Gleichung XII.1, der Leveldefinition XII.7 sowie der Annahme über die abstrakte Hierarchie (siehe

Definition XII.8) gilt $f_{i_0}\left(\mathcal{S}^{\hat{\beta}}, w^{\hat{\beta}}, i\right) = f_{i_0}\left(\mathcal{S}^{\check{\beta}}, w^{\check{\beta}}, j\right)$, $f_{i_0}\left(\mathcal{S}^{\hat{\beta}}, w^{\hat{\beta}}, j\right) = f_{i_0}\left(\mathcal{S}^{\check{\beta}}, w^{\check{\beta}}, i\right)$. In der letzten Formelzeile wird dann annahmegemäß $\mathrm{Sh}_i\left(N, v\right) > \mathrm{Sh}_j\left(N, v\right)$ verwendet. Zudem folgt aus Gleichung XII.1, der Leveldefinition XII.7 sowie der Annahme über die abstrakte Hierarchie (siehe Definition XII.8) $f_{i_0}\left(\mathcal{S}^{\check{\beta}}, w^{\check{\beta}}, i\right) = f_{i_0}\left(\mathcal{S}^{\check{\beta}}, w^{\check{\beta}}, j\right) \cdot w_i^{\check{\beta}} < f_{i_0}\left(\mathcal{S}^{\check{\beta}}, w^{\check{\beta}}, j\right)$.

Mit Hilfe dieser Überlegungen wurde gezeigt, dass für zwei benachbarte Ebenen gilt, dass i_0 eine Zuordnung wählt, die den produktiveren Mitarbeiter der höheren Ebenen zuordnet. Damit folgt schließlich, dass aus $i \in L_k^{\beta}$ und $j \in L_l^{\beta}$, mit $1 \leq k \leq \cdots \leq l \leq M$, $\mathrm{Sh}_i\left(N, v\right) \geq \mathrm{Sh}_j\left(N, v\right)$ resultiert.

Extensions and exogenous payoffs

Part E

Extensions and exogenous payoffs

Methodogically, we approach two new ideas. First of all, we allow for the possibility that players in a coalition do not contribute their maximum. In chapter XIII, we have players who work part-time in one firm and the rest of their time in another. Chapter XIV considers shirking civil servants who also do not work as much as they could possibly do. Both cases necessitate an extension of the coalition function that we introduce in the following chapter XIII. In terms of application, this chapter gives a cooperative answer to the question of how to determine the boundaries of the firm.

The second new idea refers to exogenous payments. We present two different values for exogenous payments which come together with interesting applications. Chapter XIV deals with the boundaries of the civil service in an economy while chapter XV considers a real-estate market and the realtor's pricing policy.

CHAPTER XIII

Firms and markets

1. Introduction

One of the central problems for economic theory concerns the boundaries of the firms: What kind of economic activity is conducted through markets and what kind is conducted through firms? The relevant literature goes back to Ronald Coase (1937) (Nobel prize winner 1991) and has been pursued by Oliver Williamson (1975) (Nobel prize winner 2009) and many others. In this paper, we will try to apply cooperative game theory to the boundaries-of-the-firm problem.

The basic idea of this paper is to model firms by way of an employment relation between players. Every player has an endowment of 100% of his time. He may choose to give away part of his time to other players. He may then be termed a worker while the other player becomes an employer. A player can be both a worker (who spends part of his time in another player's firm) and an employer (who uses other players' time in his own firm). We will formally introduce the employment relation in section 3.1. In section 3.2, we will work the employment relation into coalition functions. Here, one can proceed via Owen's multilinear extension defined in Owen (1972) or via Lovasz' minimum extension defined in Lovasz (1983). We will argue that for our purposes, the minimum extension makes more sense.

It should have become obvious that we will make use of cooperative game theory from the viewpoint of positive economics. It seems to us that Pareto efficiency (as embodied in most solution concepts of cooperative game theory) is troublesome from a positive perspective. After all, the microeconomics of markets and firms teaches us a number of reasons why efficiency might fail. Most of these reasons have to do with informational deficits of one sort or another.

We will distinguish two types of inefficiencies, market inefficiency and organizational inefficiency. We first turn to market inefficiency. In markets, gains from trade may remain unexploited because the economic agents do not know each other or do not trust each other. Principal-agent problems of the hidden-information variety (lemons) contribute to this inefficiency, the seminal papers being Akerlof (1970) and Spence (1973). Another source of inefficiency is uncertainty about the reservation prices. If the prospective seller's reservation price is lower than the prospective buyer's willingness to pay, the bargain may still fail because the seller takes the risk to demand a relatively high ask price, and/or the buyer takes the risk to offer a relatively small bid price as shown by Myerson & Satterthwaite (1983).

A special case of market inefficiency concerns the bargaining between workers and employers. These bargaining processes define the employment relation. For reasons of tractability, we will disregard this special type of market inefficiency.

Within our cooperative-game-theory model, market inefficiencies (apart from wage bargaining) are reflected by partitions on the player set. We let chance decide which partition will form. For the components within the stochastic partitions, we assume component efficiency (as do Aumann and Drèze) which will in general violate overall efficiency. In contrast, the Shapley value assumes the effective formation of the grand coalition (all players together).

We model two kinds of organizational inefficiency. First, an entrepreneur who employs at least one worker has fixed costs of setting up the appropriate organization. Second, organizational inefficiencies concern the principal-agent problems of hidden actions (Holmstrom (1979) and Milgrom & Roberts (1992)) and the team-production problems (Holmstrom (1982)). Here, we simply assume that the time put to productive use in a firm is less than the time given to the firm. I.e. if a worker is employed for five hours, he is productive only four hours, say. We will address this inefficiency by "team-production inefficiency". Outside game theory, organizational theory provides another justification for these costs. Crozier & Friedberg (1980, 47) address the "difficulties men must overcome in order to form and maintain" an organization. Our fixed costs refer to the forming problem and the team-production costs to the maintaining aspect.

This paper is not the first one to use cooperative game theory in order to theorize about the boundaries of the firm. Hart & Moore (1990) propose a model where players may or may not own crucial factors of production. Players with ownership rights are called employers, players without, employees. Ownership rights feed into the incentives of players to undertake specific human-capital investments. Hart and Moore's paper can be seen as a generalization of Williamson's analysis, a generalization of considerable scope as any reader of this beautiful paper will realize.

For Hart and Moore, an employer is defined by his ownership of assets. Ownership of assets confers control over players not owning assets (employees). Intimately related to Hart and Moore's approach is the work by Rajan & Zingales (1998). These authors argue that access to resources rather than allocation of ownership rights are crucial to our understanding of firms and crucial to the incentives to specialize. Another important, but more distant work using the control of access is Rajan & Zingales (2001). While we find that these approaches yield important insights, we venture into a more traditional direction and choose to model the employment relationship directly. This approach, we argue, is closer to the intuitive understanding of what employment entails.

Hart & Moore (1990, p. 1150) suggest that some nonhuman assets are

> "an important ingredient of any theory of the firm. The reason is that in the absence of any nonhuman assets, it is unclear what authority or control means. Authority over what? Control over what? Surely integration does not give a boss direct control over workers' human capital, in the absence of slavery."

While we think that Hart and Moore's approach (using assets to define employment) does make sense, their above argument goes to far. It seems natural to us that bosses do indeed control their employees while the employees may have some leeway not to do as told (team-production inefficiency, in our model). Still, employees are not slaves. After all, players can decide to terminate an employment relationship. The way we model employment is close to the view expressed by Batt (1929, p. 6) and endorsed by Coase (1937):

> "The master must have the right to control the servant's work ... . It is this right of control, of being entitled to tell the servant when to work (within the hours of service) ... and what work to do and how to do it (within the terms of such service) which is the dominant characteristic in this relation and marks off the servant from an independent contractor ..."

While our agents are not slaves, we still do not model wage formation. Instead, we assume employment relations that maximize welfare. In fact, this is precisely our method to endogenize employment relations. This amounts to assuming that wage bargaining does not lead to labour market inefficiencies. In this respect, our model is similar to that by Hart and Moore. They do not address the problem of how agents bargain over assets but look for control structures that maximize welfare, given that control structures influence specific investments.

Fig. 1 presents a comparison of Hart and Moore's model with ours. While their model combines cooperative and noncooperative game theory, we stay within the confines of cooperative game theory.

We now turn to a specific example. Assume a baker (B) and a chocolate maker (C). In isolation, they produce and sell bread and chocolate, respectively. Together, they produce and sell chocolate bread. The consumers like

Hart and Moore, 1990

sequence:	control structure	investment by agents	payoff
solution:	welfare maximization	Nash equilibrium	Shapley

this paper

sequence:	employment relation
solution:	welfare maximization

FIGURE 1. Comparing Hart and Moore's model with ours

chocolate bread a lot, hence the coalition function v is given by

$$\begin{aligned} v(B) &= 80, \\ v(C) &= 40, \\ v(B,C) &= 200. \end{aligned}$$

We would certainly like answers to the following questions:

- Will the two agents produce separately and buy or sell chocolate or bread on the market?
- If a firm turns out to be optimal, will the baker employ the chocolate maker or vice versa?
- Can an economic situation be imagined where both agents found firms, i.e. where the baker employs the chocolate maker and the chocolate maker employs the baker?

The paper is organized as follows. The next section presents coalition functions and the two most famous extensions of coalition functions. We will give an intuitive explanation for the Lovasz extension and argue that it is better suited than the Owen extension for the purposes of this paper. Section 3 presents a formal account of the employment relation and defines a coalition function on the basis of this relation. It also defines welfare in case of market and organizational inefficiencies. In section 4, the model is then applied to any superadditive two-player game. Section 4 concludes the paper.

2. Extensions of coalition functions

Taking up the baker and chocolate maker example from the introduction, we are interested in defining the worth the baker and the chocolate maker can produce if they work together (in a firm, say). We assume that the baker spends $\frac{1}{2}$ of his time and the chocolate maker $\frac{1}{3}$ of his time. $\left(\frac{1}{2}, \frac{1}{3}\right)$ is an example of a part-time coalition. Formally, a part-time coalition is a function

$$N \to [0,1]\,.$$

Player $i \in N$ gives part of his time s_i, $0 \leq s_i \leq 1$, to some coalition. In case of $s \in \{0,1\}^N$, we identify s with

$$\mathbf{K}\,(s) := \{i \in N : s_i = 1\}\,.$$

Definition XIII.1. *An extension of a coalition function v on N is a function*

$$v^{ext} : \mathbb{R}_+^N \to \mathbb{R}$$

obeying

$$v^{ext}\,(s) = v\,(\mathbf{K}\,(s))\,, s \in \{0,1\}^N\,.$$

Note that while part-time coalitions are vectors with components between 0 and 1, we have $\mathbb{R}_+^n := \{x \in \mathbb{R}^n : x \geq 0\}$ (rather than $[0,1]^n$) as the extension's domain (this is the usual way to define extensions). (By $x \geq 0$, we mean $x_i \geq 0$ for all $i = 1, ..., n.$)

Following Owen (1972), the so-called multilinear extension (MLE) is defined by

$$v^{MLE}\,(s) := \sum_{T \in 2^N \backslash \{\emptyset\}} d^v\,(T) \cdot \prod_{i \in T} s_i \tag{XIII.1}$$

while the Lovasz extension is given by

$$v^{\ell}\,(s) := \sum_{T \in 2^N \backslash \{\emptyset\}} d^v\,(T) \cdot \min_{i \in T} s_i. \tag{XIII.2}$$

In particular, we have

$$u_T^{MLE}\,(s) := \prod_{i \in T} s_i, T \subseteq N, T \neq \emptyset$$

and

$$u_T^{\ell}\,(s) := \min_{i \in T} s_i, T \subseteq N, T \neq \emptyset$$

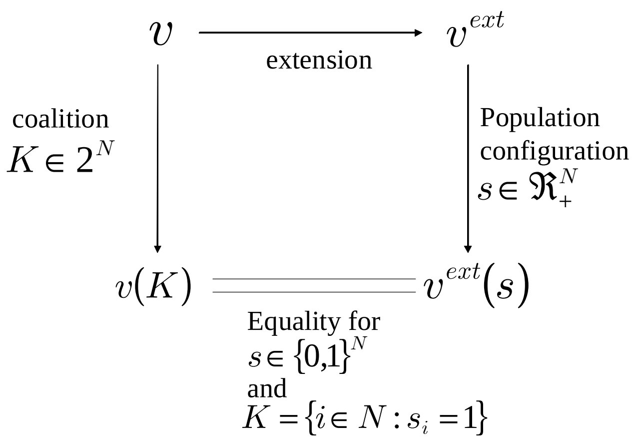

FIGURE 2. A coalition function and its extension

EXAMPLE XIII.1. *Assuming $s_2 \leq s_3 \leq s_4$ without loss of generality and using equation **??** and example **??**, the apex game's Lovasz extension is*

$$\begin{aligned} h^{\ell}(s) &= -\min_{i\in\{1,2,3\}} s_i - \min_{i\in\{1,2,4\}} s_i - \min_{i\in\{1,3,4\}} s_i + \min_{i\in\{2,3,4\}} s_i \\ &\quad + \min_{i\in\{1,2\}} s_i + \min_{i\in\{1,3\}} s_i + \min_{i\in\{1,4\}} s_i \\ &= \begin{cases} -3s_1 + s_2 + 3s_1, & s_1 \leq s_2 \leq s_3 \leq s_4 \\ -2s_2 - s_1 + s_2 + s_2 + 2s_1, & s_2 \leq s_1 \leq s_3 \leq s_4 \\ -2s_2 - s_3 + s_2 + s_2 + s_3 + s_1, & s_2 \leq s_3 \leq s_1 \leq s_4 \\ 2s_2 - s_3 + s_2 + s_2 + s_3 + s_4, & s_2 \leq s_3 \leq s_4 \leq s_1 \end{cases} \\ &= \begin{cases} s_2, & s_1 \leq s_2 \leq s_3 \leq s_4 \\ s_1, & s_2 \leq s_1 \leq s_3 \leq s_4 \\ s_1, & s_2 \leq s_3 \leq s_1 \leq s_4 \\ s_4, & s_2 \leq s_3 \leq s_4 \leq s_1 \end{cases} \end{aligned}$$

This result also makes intuitive sense. In case of $s_1 \leq s_2 \leq s_3 \leq s_4$ (first line) the maximal value $\min_{i\in\{2,3,4\}} s_i$ is achieved if the three small players cooperate. Player 1 cooperates with players 3 or 4 in the second line and with player 4 in the third line and fourth line. In each case, the scarce player's size defines the worth.

Raus Anfang??

We will now introduce an alternative characterization of the Lovasz (or minimum) extension. Let s be a vector from $\mathbb{R}_+^n$ and let $x = (x_1, ..., x_k) \in \mathbb{R}_+^k$, $k \leq n$, be a strict ordering of s such that

$$\{x_j : j = 1, ..., k\} = \{s_i : i = 1, ..., n\}$$

and

$$0 \leq x_1 < x_2 < ... < x_k.$$

Also, let $x_0 := 0$. Of course, we have strict inequality $k < n$ if there are players $i, j \in N$, $i \neq j$, such that $s_i = s_j$. We now define

$$K_j := \{i \in N : s_i \geq x_j\}, j = 1, ..., k$$

and

$$f_s(K) := \begin{cases} x_j - x_{j-1}, & K = K_j \text{ for one } j \in \{1, ..., k\} \\ 0, & \text{otherwise} \end{cases}$$

f_s attributes $x_j - x_{j-1}$ to a coalition K if K consists of all players i with $s_i \geq x_j$. Note $K_{j+1} \subset K_j$ for all $j = 1, ..., k-1$. Also, $K_1 = N$. By the ordering of x, $f_s(K) \geq 0$ for all $s \in \mathbb{R}^n_+$ and all $K \subseteq N$.

According to Lovasz (1983) and Algaba, Bilbao, Fernandez & Jimenez (2004), the Lovasz extension is given by

$$v_{\min}(s) = \sum_{T \in 2^N \setminus \{\emptyset\}} m_v(T) \cdot \min_{i \in T} s_i \tag{XIII.3}$$

$$= \sum_{T \in 2^N \setminus \{\emptyset\}} f_s(T) \, v(T) \tag{XIII.4}$$

and obeys the following properties:

- (p.i) $v_{\min}$ is positively homogeneous, i.e., $v_{\min}(\lambda s) = \lambda v_{\min}(s)$ for all $\lambda \geq 0$.
- (p.ii) $(v + w)_{\min} = v_{\min} + w_{\min}$.
- (p.iii) $(\lambda v)_{\min} = \lambda v_{\min}$ for all $\lambda \in \mathbb{R}$.
- (p.iv) v is supermodular iff $v_{\min}$ is concave.

Concavity of $v_{\min}$ means: For any $\alpha \in [0, 1]$ and any $s, s' \in \mathbb{R}^n_+$, we have

$$\alpha v_{\min}(s) + (1 - \alpha) v_{\min}(s') \leq v_{\min}(\alpha s + (1 - \alpha)(s'))$$

Now, it is not difficult to show the following lemma:

Lemma XIII.1. *Let v be a (monotonic and non-trivial) coalition function. We obtain*

- *(c.i) $s \leq s' \Rightarrow v_{\min}(s) \leq v_{\min}(s')$ for all $s, s' \in \mathbb{R}^n_+$.*
- *(c.ii) If v is convex (supermodular), then*

  $$v_{min}(s + s') \geq v_{min}(s) + v_{min}(s')$$

  for all $s, s' \in \mathbb{R}^n_+$.
- *(c.iii) If v is additive, then*

  $$v_{min}(s + s') = v_{min}(s) + v_{min}(s')$$

  for all $s, s' \in \mathbb{R}^n_+$.

For the proof of this lemma, see the appendix.

We now have two extension candidates, the multilinear extension and the minimum extension. We would like to argue that properties (p.i) through (p.iv) and (c.i) through (c.iii) are plausible enough and do not constitute a case against the minimum extension. In order to argue for the minimum and against the multilinear extension, we reason as follows: The multilinear extension, v_{MLE}, has a probabilistic interpretation (as noted by Owen (1972, p. 64)): Inside a firm, the players work together only if their time schedules happen to coincide. For the above part-time coalition $\left(\frac{1}{2}, \frac{1}{3}\right)$, chocolate bread will be produced for $\frac{1}{2} \cdot \frac{1}{3}$ time units, only. Of course, from an organizational point of view, this interpretation can easily be criticized. After all, the two agents could show up at the same time. Also, it may be possible that the baker bakes his bread which is coated by chocolate later. Because of these interpretational difficulties, we will work with the minimum extension although it is less tractable.

The minimum extension, v^ℓ, allows a very different interpretation. In case of $\left(\frac{1}{2}, \frac{1}{3}\right)$ chocolate bread will be produced for $\min\left(\frac{1}{2}, \frac{1}{3}\right)$ time units. That is, the baker and the chocolate maker's time are perfect complements in the production of chocolate bread. However, the baker has some time left, $\frac{1}{2} - \min\left(\frac{1}{2}, \frac{1}{3}\right)$, and will spend this time producing bread. Since chocolate bread is more valuable than bread (or chocolate) it is efficient to allocate $\min\left(\frac{1}{2}, \frac{1}{3}\right)$ time units to chocolate-bread production and to use the remainder for bread.

In the general case,

$$v_{\min}(s) = \sum_{T \in 2^N \backslash \{\emptyset\}} f_s(T)\, v(T)$$

is constructed similarly. $v(N)$ is to be multiplied by

$$f_s(N) = f(K_1) = x_1 - x_0 = x_1 = \min_{i \in N} s_i.$$

Players $j \in N$ who put all their time s_j into the production of $v(N)$ (i.e. $s_j = \min_{i \in N} s_i$) cannot contribute to coalitions $K \neq N$. We have $J := \{j \in N : s_j = \min_{i \in N} s_i\}$ and $K_2 = N \backslash J$. Given that the players from J cannot contribute anymore, $v(K_2)$ is multiplied with the maximal time budget possible:

$$f(K_2) = x_2 - x_1 = \min_{i \in K_2} s_i - \min_{i \in N} s_i.$$

For us, these interpretations show that the Lovasz extension makes intuitive sense for our purposes.

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3. The model

3.1. The employment relation. Let us consider a quadratic $n \times n$-matrix:

$$\mathcal{A} : N^2 \to [0, 1].$$

$\mathcal{A}(i,j)$ represents the time spent by agent j in agent i's firm. $\mathcal{A}(i,\cdot)$ is the i's row in $\mathcal{A}$ and stands for the players employed by player i. $\mathcal{A}(\cdot,j)$ is the j's column in $\mathcal{A}$ and represents the agents player j works for. We normalize the time budgets of players to one (and obtain what is also called a stochastic matrix, in the theory of random processes).

DEFINITION XIII.2. $\mathcal{A}: N^2 \to [0,1]$ *is called an employment matrix or an employment relation if*

$$\sum_{i=1}^{n} \mathcal{A}(i,j) = 1 \text{ for any } j = 1,...,n$$

holds.

An example is provided by

$$\mathcal{A} = \begin{pmatrix} 1 & \frac{3}{8} & 1 \\ 0 & 0 & 0 \\ 0 & \frac{5}{8} & 0 \end{pmatrix}$$

Here player 1 uses all his time in his own firm. This means that he is not a worker. Since he employs players 2 and 3, with shares of time $\frac{3}{8}$ and 1, respectively, he is an employer. Player 2 spends his time in the firms of player 1 ($\frac{3}{8}$) and player 3 ($\frac{5}{8}$) but he does not employ other players (the second row contains three zeros). Player 3 is both a worker (he spends all his time in player 1's firm) and an employer (he uses $\frac{5}{8}$ of player 2's time).

If nobody employs any other, all players spend all their time in their own one-man firm. Then, we have an employment relation $\mathcal{A}$ obeying $\mathcal{A}(i,i) = 1$, $i \in N$. For three players, this relation is given by the unit matrix

$$\mathcal{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3.2. The employment coalition function. On the basis of the employment relation $\mathcal{A}$, we define a part-time coalition

$$s_K^{\mathcal{A}} := \left(\sum_{i \in K} \mathcal{A}(i,1), ..., \sum_{i \in K} \mathcal{A}(i,n) \right).$$

Given an employment relation $\mathcal{A}$ and a coalition K, we sum the time spent by agent 1 (agent 2, ...) in any of the firms owned by players from K. For the unit matrix $\mathcal{A}$, we have $K = \mathbf{K}\left(s_K^{\mathcal{A}}\right)$.

We can now construct the employment coalition function $v_{\text{ext}}^{\mathcal{A}}$ by

$$v_{\text{ext}}^{\mathcal{A}}(K) := v^{ext}\left(s_K^{\mathcal{A}}\right).$$

An employment coalition function is a coalition function defined for any given extension (Owen, or Lovasz extension) and for any given employment relation. Take a coalition K. The players from K employ themselves and/or other players within and outside K. These players are summarized in the part-time coalition $s_K^{\mathcal{A}}$. The worth of K is then the worth of this part-time coalition under the given extension.

Note that for $\mathcal{A}$ representing the market, we have $v_{\text{ext}}^{\mathcal{A}}(K) = v(K)$. From now on, we will write $v(s)$ instead of $v^{\ell}(s)$ and $v^{\mathcal{A}}(K)$ instead of $v_{\min}^{\mathcal{A}}(K)$.

3.3. Inefficiency.

3.3.1. *Organizational inefficiency.* We now consider organizational inefficiencies. Firms do not obtain efficient solutions for two different kinds of organizational costs:

- quasi-fixed cost for employing players other then oneself,
- variable costs of surmounting team-production problems, principal-agent problems and the like.

We build the fixed costs, f ($f \geq 0$), and the team production costs, t ($0 \leq t \leq 1$), into the employment coalition function as follows. For $K \subseteq N$, we define

$$\begin{aligned} s^K \quad : \quad = & \left((1-t) \sum_{i \in K, i \neq 1} \mathcal{A}(i,1) + \mathcal{A}(1,1) \cdot \begin{cases} 1, & 1 \in K \\ 0, & 1 \notin K \end{cases}, \right. \\ & \left. ..., (1-t) \sum_{i \in K, i \neq n} \mathcal{A}(i,n) + \mathcal{A}(n,n) \cdot \begin{cases} 1, & n \in K \\ 0, & n \notin K \end{cases} \right) \end{aligned}$$

and

$$v^{\mathcal{A},t,f}(K) := v\left(s^K\right) - f \cdot \sum_{i \in K} \begin{cases} 1, & \mathcal{A}(i,j) > 0 \text{ for some } j \in N, j \neq i \\ 0, & \text{otherwise} \end{cases}$$

First, players who employ at least one other player have to pay the fixed costs of organization, f. Second, employing players other than oneself is reflected by the factor $1-t$. A player j working for another player i contributes effective time $(1-t)\mathcal{A}(i,j)$. Our measure of welfare that incorporates organizational inefficiencies is given by

$$v^{\mathcal{A},t,f}(N).$$

In our two-player case, we obtain

$$
\begin{aligned}
v^{\mathcal{A},t,f}(B) = {} & \underbrace{v(B,C)}_{\substack{\text{worth of}\\ \text{chocolate bread}}} \underbrace{\min\left(\underbrace{\mathcal{A}(B,B)}_{\substack{\text{time spent by B}\\ \text{in his own firm}}}, \underbrace{(1-t)\,\mathcal{A}(B,C)}_{\substack{\text{effective time spent by C}\\ \text{in B's firm}}}\right)}_{\substack{\text{effective time use of both players}\\ \text{for chocolate-bread production}}} \\
& + \underbrace{v(B)}_{\substack{\text{worth of}\\ \text{bread}}} \underbrace{\left[\mathcal{A}(B,B) - \min\left(\mathcal{A}(B,B),(1-t)\,\mathcal{A}(B,C)\right)\right]}_{\substack{\text{time spent by B in his own firm}\\ \text{not used for chocolate-bread production}}} \\
& + \underbrace{v(C)}_{\substack{\text{worth of}\\ \text{chocolate}}} \left[(1-t)\underbrace{\left(\mathcal{A}(B,C) - \frac{1}{1-t}\min\left(\mathcal{A}(B,B),(1-t)\,\mathcal{A}(B,C)\right)\right)}_{\substack{\text{time spent by C in B's firm}\\ \text{not used for chocolate-bread production}}}\right] \\
& - f \cdot \begin{cases} 1, & \mathcal{A}(B,C) > 0 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

for the baker. The baker can, if he employs the chocolate maker ($\mathcal{A}(B,C) > 0$), produce chocolate bread. The effective employment time by the chocolate maker is $(1-t)\,\mathcal{A}(B,C)$ and the quantity of chocolate bread produced is $\min(\mathcal{A}(B,B),(1-t)\,\mathcal{A}(B,C))$. This term also describes the time used for chocolate-bread production by the baker. He might have some time left over ($\mathcal{A}(B,B) - \min(\mathcal{A}(B,B),(1-t)\,\mathcal{A}(B,C)) \geq 0$) to produce bread. The chocolate maker's effective time for chocolate bread production is $\min(\mathcal{A}(B,B),(1-t)\,\mathcal{A}(B,C))$ and he uses up $\frac{1}{1-t}\min(\mathcal{A}(B,B),(1-t)\,\mathcal{A}(B,C))$ of his employment time. Hence, $\mathcal{A}(B,C)$ minus this employment time can be used for chocolate production which he can put to effective use only by a factor of $(1-t)$.

Before turning to the worth for both players, $v^{\mathcal{A},t}(B,C)$, we define the effective production time, T^{CB}, spent by the players for production of chocolate bread:

$$\begin{aligned}
T^{CB} &:= \min\left(\underbrace{\mathcal{A}(B,B)}_{\substack{\text{time spent by B} \\ \text{in his own firm}}} + \underbrace{(1-t)\,\mathcal{A}(C,B)}_{\substack{\text{effective time spent by B} \\ \text{in C's firm}}}, \right. \\
&\qquad \left. \underbrace{\mathcal{A}(C,C)}_{\substack{\text{time spent by C} \\ \text{in his own firm}}} + \underbrace{(1-t)\,\mathcal{A}(B,C)}_{\substack{\text{effective time spent by C} \\ \text{in B's firm}}} \right) \\
&= \min\left(1 - \underbrace{t\mathcal{A}(C,B)}_{\substack{\text{time spoiled by B} \\ \text{in C's firm}}}, 1 - \underbrace{t\mathcal{A}(B,C)}_{\substack{\text{time spoiled by C} \\ \text{in B's firm}}} \right)
\end{aligned}$$

Now, the baker is effective T^{CB} time units contributing to chocolate bread. He spoils $t\mathcal{A}(C,B)$ time units. Therefore, he will produce

$$T^B = 1 - t\mathcal{A}(C,B) - T^{CB}$$

units of bread. Similarly, the chocolate maker will produce

$$T^C = 1 - t\mathcal{A}(B,C) - T^{CB}$$

units of chocolate. Summarizing, we obtain

$$
\begin{aligned}
v^{\mathcal{A},t,f}(B,C) = & \underbrace{v(B,C)}_{\substack{\text{worth of}\\ \text{chocolate bread}}} T^{CB} \\
& + \underbrace{v(B)}_{\substack{\text{worth of}\\ \text{simple bread}}} \underbrace{T^{B}}_{\substack{\text{effective time spent by B}\\ \text{producing simple bread}}} \\
& + \underbrace{v(C)}_{\substack{\text{worth of}\\ \text{chocolate}}} \underbrace{T^{C}}_{\substack{\text{effective time spent by C}\\ \text{producing chocolate}}} \\
& - f \cdot \begin{cases} 2, & \mathcal{A}(B,C) > 0 \text{ and } \mathcal{A}(C,B) > 0, \\ 1, & \mathcal{A}(B,C) > 0 \text{ and } \mathcal{A}(C,B) = 0, \\ 1, & \mathcal{A}(B,C) = 0 \text{ and } \mathcal{A}(C,B) > 0, \\ 0, & \mathcal{A}(B,C) = 0 \text{ and } \mathcal{A}(C,B) = 0 \end{cases}
\end{aligned}
$$

3.3.2. *Market inefficiency.* We formalize market inefficiency through partitions on N. The underlying idea is this: If we have partition $\{N\}$, all players know each other and trust each other. Therefore, they can realize any gains from trade without employing each other. We will say that the market, defined by $\mathcal{A}(i,i) = 1$, $i \in N$, is efficient. Taking the other extreme, partition $\{\{1\}, \{2\}, ..., \{n\}\}$ indicates that players do not know or do not trust each other. Then, the market is highly inefficient.

We will assume a probability distribution on the set of partitions on N. Denoting this set by $\mathfrak{P}$ and denoting the probability of a partition $\mathcal{P}$ by $prob(\mathcal{P})$, we define welfare by

$$\sum_{\mathcal{P} \in \mathfrak{P}} prob(\mathcal{P}) \sum_{C \in \mathcal{P}} v^{\mathcal{A}}(C).$$

For any given partition $\mathcal{P}$, we sum the worths $v^{\mathcal{A}}(C)$ where C is a component in $\mathcal{P}$.

It is certainly helpful to have a one-parameter measure for inefficiency. We will now construct such a measure. Inspired by the rank-order formulation of the Shapley value, we consider a rank order

$$\rho = (\rho_1, \rho_2, ..., \rho_n).$$

A coalition K is called cohesive in ρ, if $K = \{\rho_l, ..., \rho_{j-1}, \rho_j\}$ for some l and j, $1 \leq l \leq j \leq n$. In order to split vectors ρ into subvectors, we introduce vectors $(b_0, b_1, ..., b_n) \in \{0,1\}^{n+1}$ obeying $b_0 = b_n = 1$. If b_j $(j = 1, ..., n-1)$ is equal to 1, we say that there is a break after player ρ_j. $b_0 = 1$ $(b_n = 1)$ signifies a "break" before player ρ_1 (after player ρ_n). A cohesive coalition $K = \{\rho_l, ..., \rho_{j-1}, \rho_j\}$ is called effective if $b_{l-1} = 1 = b_j$

and $b_l = ... = b_{j-1} = 0$. For every ρ, a vector $(b_1, ..., b_{n-1})$ specifies a partition of the players,

$$\mathcal{P} := \{C_1, ..., C_m\},$$

into effective coalitions.

We assume a constant probability p for the b_j $(j = 1, ..., n-1)$ being equal to 0. As shown in Wiese (2005a) and in the appendix (section 8), $\mathcal{P}'s$ probability under p, $prob(\mathcal{P}, p)$, is given by

$$prob(\mathcal{P}, p) = \frac{1}{n!} \cdot \prod_{j=1}^{m} z_j! \cdot m! \cdot (1-p)^{m-1} \cdot p^{n-m}$$

where z_j denotes the cardinality of C_j, $j = 1, ..., m$.

For example, for two and three players we obtain

$$\begin{aligned} prob(\{\{1,2\}\}, p) &= p, \\ prob(\{\{1\}, \{2\}\}, p) &= 1-p \end{aligned}$$

and

$$\begin{aligned} prob(\{\{1,2,3\}\}, p) &= \frac{1}{6} \cdot 6 \cdot 1 \cdot 1 \cdot p^2 = p^2, \\ prob(\{\{1\}, \{2,3\}\}, p) &= \frac{1}{6} \cdot 2 \cdot 2 \cdot (1-p) \cdot p = \frac{2}{3}(1-p)p, \\ prob(\{\{1\}, \{2\}, \{3\}\}, p) &= \frac{1}{6} \cdot 1 \cdot 6 \cdot (1-p)^2 \cdot 1 = (1-p)^2. \end{aligned}$$

In general, the higher p, the more likely components with many players. Thus, p is a measure of the market's efficiency.

3.3.3. *Putting market and organizational ineffiencies together.* In the obvious manner, we can now define a welfare measure that takes both market and organizational inefficiencies into account:

$$\pi := \sum_{\mathcal{P} \in \mathfrak{P}} prob(\mathcal{P}) \sum_{C \in \mathcal{P}} v^{\mathcal{A},t,f}(C).$$

If $\mathcal{A}$ represents the market, we obtain

$$\pi = \sum_{\mathcal{P} \in \mathfrak{P}} prob(\mathcal{P}) \sum_{C \in \mathcal{P}} v(C) \begin{cases} \leq v(N), & v \text{ superadditive}, \\ = v(N), & v \text{ additive}. \end{cases}$$

For general employment relations and for additive (inessential) coalition functions, we have

$$\begin{aligned} \pi &= \sum_{\mathcal{P} \in \mathfrak{P}} prob(\mathcal{P}) \sum_{C \in \mathcal{P}} \left(\sum_{i \in C} v(\{i\}) \cdot s_i^C - f \cdot \sum_{i \in C} \begin{cases} 1, & \mathcal{A}(i,j) > 0 \text{ for some } j \in N, j \neq i \\ 0, & \text{otherwise} \end{cases} \right) \\ &= \sum_{\mathcal{P} \in \mathfrak{P}} prob(\mathcal{P}) \sum_{i \in N} v(\{i\}) \cdot s_i^C - f \cdot \sum_{i \in N} \begin{cases} 1, & \mathcal{A}(i,j) > 0 \text{ for some } j \in N, j \neq i \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

If we have only two players (baker and chocolate maker), we obtain

$$\begin{aligned}\pi &= prob\left(\left\{\left\{B\right\},\left\{C\right\}\right\}\right)\left[v^{\mathcal{A},t,f}\left(B\right)+v^{\mathcal{A},t,f}\left(C\right)\right]\\ &\quad +prob\left(\left\{\left\{B,C\right\}\right\}\right)v^{\mathcal{A},t,f}\left(B,C\right).\end{aligned}$$

3.4. Welfare maximization. We assume that employers and workers will agree on wages that exploit all welfare potential while organizational and market inefficiencies persist. That is, we look for

$$\arg\max_{\mathcal{A}}\sum_{\mathcal{P}\in\mathfrak{P}} prob\left(\mathcal{P}\right)\sum_{C\in\mathcal{P}} v^{\mathcal{A},t,f}\left(C\right).$$

4. The two-player case

We now consider a generalization of the baker-chocolate maker game presented in the introduction which is also a special case of the general setup developed in sections **??** and 3. We keep on denoting the players by B and C. The coalition function is assumed to obey superadditivity:

$$u := v\left(B,C\right)-v\left(B\right)-v\left(C\right)\geq 0.$$

Without loss of generality, we assume $v\left(B\right)\geq v\left(C\right)$. By way of tedious calculations (supported by Mathematica), we arrive at the following results:

PROPOSITION XIII.1. *In the two player case without fixed costs of organization ($f=0$), we get the following results:*

- *Total cross employment ($\mathcal{A}\left(B,C\right)=1=\mathcal{A}\left(C,B\right)$) is never the unique best outcome.*
- *If v is inessential (i.e. $u=0$), the market outcome ($\mathcal{A}\left(B,B\right)=1$, $\mathcal{A}\left(B,C\right)=0$) is the unique best solution for $t>0$, and $v\left(C\right)>0$.*
- *If the market is efficient (i.e., $p=1$), the market outcome ($\mathcal{A}\left(B,B\right)=1$, $\mathcal{A}\left(B,C\right)=0$) is the unique best solution for $t>0$, and one of the following conditions:*
  - $v\left(C\right)>0$,
  - $u>0$.
- *If there are no team-production inefficiencies ($t=0$), and if both $p<1$ and $u>0$ hold, the set of best solutions is the continuum defined by*

$$\mathcal{A}\left(B,B\right)=\mathcal{A}\left(B,C\right).$$

- *If we have both market and team-production efficiency (i.e., $p=1$ and $t=0$), every employment matrix is optimal.*
- *If we have both team-production efficiency (i.e. $t=0$) and an inessential game (i.e., $u=0$), every employment matrix is optimal.*
- *If v is essential (i.e. $u>0$) and we have some market and team-production inefficiency (i.e., $p<1$ and $t>0$), we need to distinguish two cases. Either, we have $v\left(B,C\right)\leq 2v\left(B\right)$. Then (see fig.*

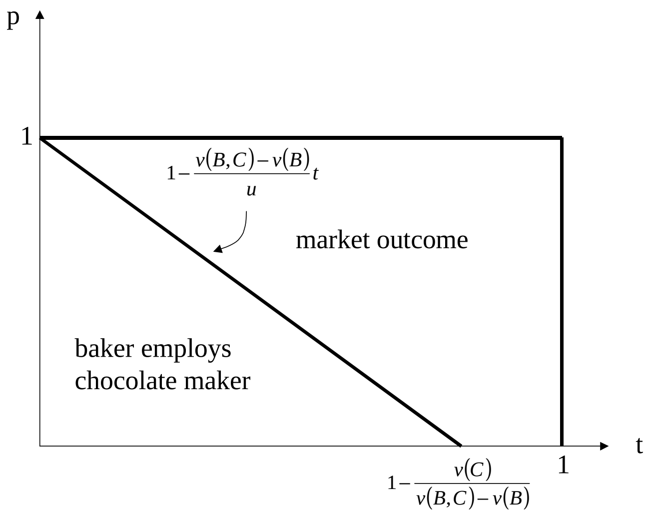

FIGURE 3. Market or firm

3), the market is the unique optimal outcome for sufficiently high values of p and t. Otherwise, the baker employs the chocolate maker (note that $u > 0$ and $v(B,C) \leq 2v(B)$ imply $v(B) > v(C)$). If, however, we have $v(B,C) > 2v(B)$, fig. 4 applies. In that case, partial cross-employment at $\mathcal{A}(B,B) = \frac{1-t}{2-t}$ and $\mathcal{A}(B,C) = \frac{1}{2-t}$ is the unique optimal outcome for $p < 1 - \frac{tv(B,C)}{u(2-t)}$ and $0 < t < 1 - \frac{v(B)}{v(B,C)-v(B)}$. Otherwise, fig. 4 looks like fig. 3. However, in case of $v(B) = v(C)$, both $\mathcal{A}(B,B) = 1$ and $\mathcal{A}(B,C) = 1$, and $\mathcal{A}(B,B) = 0$ and $\mathcal{A}(B,C) = 0$, are optimal.

Some comments are in order. The middle point in fig. 4 is given by $\mathcal{A}(B,B) = \frac{1-t}{2-t}$ and $\mathcal{A}(B,C) = \frac{1}{2-t}$. For this employment matrix, we have

- $$\underbrace{\mathcal{A}(B,B)}_{\substack{\text{time spent by B}\\ \text{in his own firm}}} = \underbrace{(1-t)\,\mathcal{A}(B,C)}_{\substack{\text{effective time spent by C}\\ \text{in B's firm}}},$$
- $$\underbrace{1-\mathcal{A}(B,C)}_{\substack{\text{time spent by C}\\ \text{in his own firm}}} = \underbrace{(1-t)\,(1-\mathcal{A}(B,B))}_{\substack{\text{effective time spent by B}\\ \text{in C's firm}}} \quad \text{and}$$
- $$\underbrace{\mathcal{A}(B,B)}_{\substack{\text{time spent by B}\\ \text{in his own firm}}} + \underbrace{(1-t)\,(1-\mathcal{A}(B,B))}_{\substack{\text{effective time spent by B}\\ \text{in C's firm}}}$$

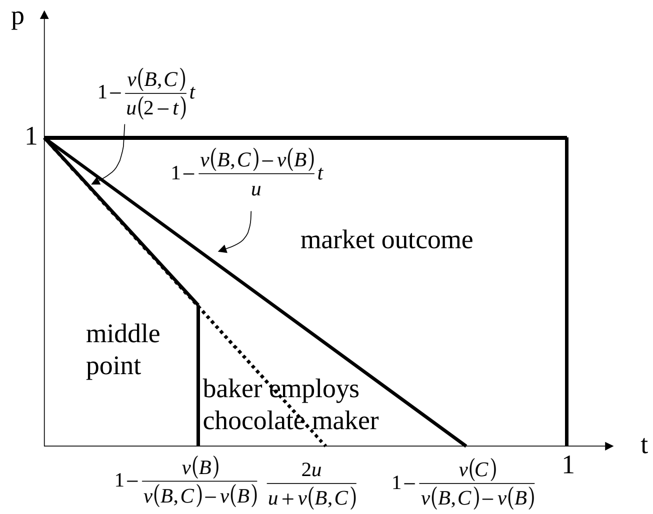

FIGURE 4

$$= \underbrace{1-\mathcal{A}(B,C)}_{\substack{\text{time spent by C} \\ \text{in his own firm}}} + \underbrace{(1-t)\,\mathcal{A}(B,C)}_{\substack{\text{effective time spent by C} \\ \text{in B's firm}}}$$

so that neither bread nor chocolate, but chocolate bread only, is produced by

- the one-player coalition B,
- the one-player coalition C, and
- the grand coalition $\{B,C\}$.

Therefore, this employment matrix is as good as the one given by $\mathcal{A}(B,B) = 1$ and $\mathcal{A}(B,C) = 1$ if there are not team-production inefficiencies and no fixed costs of organization.

If we have both market and organizational inefficiencies the middle point may also be optimal. The reason is this: For the production of chocolate bread one needs both the baker's and the chocolate maker's effective time. By concentrating production in the baker's firm ($\mathcal{A}(B,B) = 1 = \mathcal{A}(B,C)$) there is a relative shortage of chocolate production time. Similarly, if the chocolate maker employs the baker, there is a shortage of baking time. The middle point avoids these shortages thus ensuring that chocolate bread is produced.

5. Conclusions

Cooperative game theory has its obvious weaknesses. It does not model actions, beliefs, or preferences. Instead, it directly heads for payoffs. On the positive side, cooperative game theory can be useful for situations where an analysis by way of noncooperative game theory would need many specific assumptions. This is one of the reasons why bargaining problems are more often dealt with in terms of cooperative than noncooperative game theory. We like to argue that cooperative game theory will see more interesting applications in the near future. In this paper, we presented one area where cooperative game theory can be put to use. It would indeed by a huge challenge to seriously attack the boundaries-of-the-firm problem without elements from cooperative game theory.

While we got concrete results for the two-player case only, it is surely tempting to conjecture about the general case:

- More productive agents tend to employ less productive agents rather than the other way around.
  Because of organizational inefficiencies, the opportunity costs of employing a productive agent are higher than the opportunity costs of employing an unproductive one.
- Cross employment can happen, but tends towards zero under replication.
  In a large economy, the fixed costs of cross employment can be avoided by still guaranteeing a fit of workers.
- Part-time employment can happen, but tends towards zero under replication.
  Part-time employment results in high fixed costs of employment. Again, in a large economy this effect should be eliminated.
- Inessential agents will not be employed.
  There is no reason to incur costs for agents without productive use.
- If v is inessential and there are some organizational inefficiencies, the market outcome obtains.
  For an inessential v, cooperation yields no benefits.

In this paper, we interpret $\mathcal{A}$ as an employment matrix. More generally, one could consider $\mathcal{A}$ an availability relation. For example, a woman can avail of her spouse and young children can avail of their parents. Slaves do not avail of themselves. A love relation may said to exist if both partners avail of each other. Thus, the availability approach may be useful in very different applications.

Finally, while we did not actually use the Shapley value or other common concepts from cooperative game theory, it is obvious how to define their availability variants. For example, given a coalition function v and an availability relation $\mathcal{A}$, the Shapley-availability value should be defined by $\varphi(v, \mathcal{A}) := \varphi\left(v^{\mathcal{A}}\right)$.

6. Appendix

7. Proof of the lemma

For the proof of corollary (c.i) we consider a vector $s \in \mathbb{R}^n_+$ and its strict ordering x,

$$0 \leq x_1 < x_2 < ... < x_k.$$

We now consider one player i and distinguish the four cases:

- $\{x_j\} = \{s_i\}$, $j = 1, ..., k-1$,
- $\{x_k\} = \{s_i\}$,
- $\{x_j\} = \{s_i, ...\}$, $j = 1, ..., k-1$, and
- $\{x_k\} = \{s_i, ...\}$.

The transition from s to s' can be constructed as a finite sequence of steps involving some players $i \in N$. In the first case, we consider s''_i obeying $s_i < s''_i := \min(x_{j+1}, s'_i)$. (If such an s''_i does not exist, we are done with player i.) Also, we let $s''_j = s_j$, $j \neq i$. We then have a vector $s'' > s$. $v_{\min}(s'') \geq v_{\min}(s)$ can now be shown by applying definition XIII.4. In order to compare $v_{\min}(s'')$ and $v_{\min}(s)$, we can restrict attention to $v(K_j)$ and $v(K_{j+1})$. By monotonicity of v we have

$$v(K_{j+1}) \leq v(K_j)$$

and the proof is now affected by showing

$$\begin{aligned} & v(K_j)\left(s''_i - x_{j-1}\right) + v(K_{j+1})\left(x_{j+1} - s''_i\right) \\ = \;& v(K_j)(s_i - x_{j-1}) + v(K_{j+1})(x_{j+1} - s_i) \\ & + [v(K_j) - v(K_{j+1})]\left(s''_i - s_i\right) \\ \geq \;& v(K_j)(s_i - x_{j-1}) + v(K_{j+1})(x_{j+1} - s_i). \end{aligned}$$

The other cases can be treated in a similar fashion. At each step, we get an increase in $v_{\min}(s'')$ until we arrive at s'.

In order to show (c.ii), we consider a convex coalition function v and $s, s' \in \mathbb{R}^n_+$. We obtain

$$\begin{aligned} & v_{\min}\left(s + s'\right) \\ = \;& 2v_{\min}\left(\frac{1}{2}s + \frac{1}{2}s'\right) \text{ (positive homogeneity, p.i)} \\ \geq \;& 2\left(\frac{1}{2}v_{\min}(s) + \frac{1}{2}v_{\min}\left(s'\right)\right) \text{ (convexity of } v, \text{ p.iv)} \\ = \;& v_{\min}(s) + v_{\min}\left(s'\right) \text{ (positive homogeneity, p.i).} \end{aligned}$$

Finally, if v is additive, have

$$\begin{aligned} v_{\min}\left(s+s'\right) &= \sum_{T\in 2^N\backslash\{\emptyset\}} m_v\left(T\right)\cdot\min_{i\in T}\left(s_i+s_i'\right) \\ &= \sum_{i\in N} v\left(\{i\}\right)\cdot\left(s_i+s_i'\right) \\ &= \sum_{i\in N} v\left(\{i\}\right)\cdot s_i + \sum_{i\in N} v\left(\{i\}\right)\cdot s_i' \\ &= \sum_{T\in 2^N\backslash\{\emptyset\}} m_v\left(T\right)\cdot\min_{i\in T} s_i + \sum_{T\in 2^N\backslash\{\emptyset\}} m_v\left(T\right)\cdot\min_{i\in T} s_i' \\ &= v_{\min}\left(s\right)+v_{\min}\left(s'\right). \end{aligned}$$

8. The probability of a given partition

The probability of

$$\mathcal{P} := \{C_1, ..., C_m\}$$

to be made up of effective coalitions is equal to

$$\begin{aligned} & prob\left(\mathcal{P},p\right) \\ = \ & \frac{1}{n!} z_1!\cdot ...\cdot z_m! m! p^{z_1-1}\cdot(1-p)\cdot p^{z_2-1}\cdot(1-p)\cdot ...\cdot p^{z_{m-1}-1}\cdot(1-p)\cdot p^{z_m-1}. \end{aligned}$$

$z_j!$ is the number of permutations of the players in C_j, $m!$ is the number of permutations of the coalitions $C_1, ..., C_m$. Therefore

$$\frac{z_1!\cdot ...\cdot z_m! m!}{n!}$$

is the probability that ρ from R partitions N into cohesive coalitions $C_1, ..., C_m$. For the C_j to be effective, we need $m-1$ breaks between the coalitions (probability $(1-p)^{m-1}$) and no breaks within coalitions (probability p^{z_j-1} for coaliton C_j).

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9. Topics and literature

The main topics in this chapter are

- production set
- production function
- no-free-lunch property

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We recommend the textbook by Wiese (2005c).

10. Solutions

Exercise VIII.3

For the first partition, we obtain $\mathcal{P}(2) = \{2\}, \mathcal{P}(\{2,3\}) = \{\{2\}, \{3,4\}\}$, $\mathcal{P}(\{2\}) = \{\{2\}\}$ and $\mathcal{P}(N \backslash \{2,3\}) = \{\{1\}, \{3,4\}\}$, the second partition yields $\mathcal{P}(2) = \{2,3\}, \mathcal{P}(\{2,3\}) = \{\{2,3\}\}, \mathcal{P}(\{2\}) = \{\{2,3\}\}$ and $\mathcal{P}(N \backslash \{2,3\}) = \{\{1\}, \{4\}\}$. $\mathcal{P}(\{2,3\})$, $\mathcal{P}(\{2\})$ and $\mathcal{P}(N \backslash \{2,3\})$ are subsets of the partitions and partitions in their own right, albeit of different sets.

11. Further exercises without solutions

CHAPTER XIV

The Size of Government

1. Introduction

Most economies are mixed economies, with a private sector and a public sector. There are many reasons, good ones and bad ones, for the use of civil servants working in the public sector. Arguably, judges and policemen should be civil servants with no profit interest attached to their duties. However, in most economies, public servants are also employed in sectors alongside, or instead, of private firms (e.g., in education, transport, energy, and water supply) although exclusively private activity may well be more efficient. In general, the services provided by the public sector benefit some private actors more than others while all of them pay taxes which, by definition, do not need to be in line with the benefit obtained. Thus, private actors can be expected to disagree on the optimal extent of the public sector. Of course, the very same disagreement pertains to cash redistributions (social welfare, tax exemptions etc.).

The very influential paper by Meltzer & Richard (1981, p. 916) discusses a rational theory of the size of government where

- voters know that governmental redistribution or services have to be paid for by taxes (now or in the future),
- the public-good argument for publicly provided services is neglected.

The median voter in Meltzer & Richard's (1981) approach determines the size of government in his own interest. Indeed, the rent-seeking approach to the size-of-government question has also been pursued in the empirical papers by Mueller & Murrell (1986) and Becker & Mulligan (2003). This paper also explores the possibility that the institution of a public sector is a rent-seeking device. We show that civil servants may be employed even if they are less productive than private-sector employees. Our main question concerns the extension of the public sector, or, differently put: Are there limits to government?

To the best of our knowledge, this paper is the first to use cooperative game theory to elucidate the boundaries of the public sector. In particular, we consider a game in coalition-function form (the economy) and a group C of agents (the civil servants). We assume that these agents obtain a prespecified payoff. The other agents – the private sector – have to pay these payoffs but can also benefit from the services rendered by the C-group. We call this the pay-and-use setup.

Our model has two parts. First, we axiomatize a value for a given set of civil servants and given payments π to these agents. The tuple (C, π) is called the public-service vector. In particular, we modify the most famous concept of cooperative game theory, the Shapley value so as to incorporate (C, π). This is a somewhat complicated endeavor because we also want to allow for "lazy" civil servants.

In the second part, we prefix a noncooperative game to the pay-and-use Shapley value. Mixing cooperative and noncooperative games in this manner has been dubbed biform games by Brandenburger & Stuart (2007) (who use the core rather than the Shapley value). In our game, the players determine the public-service vector. An equilibrium public-service vector (C^*, π^*) obeys several conditions. First, in the spirit of Tiebout's (1956) voting by feet, every (private or public) agent is free to go abroad if he prefers. Second, there is a majority of agents that prefer (C^*, π^*) over a purely private economy. Third, every civil servant has a salary not below the payoff he would get in the private sector (or abroad).

We introduce private-sector and mixed-sector coalition functions in section 2 and show how to incorporate lazy civil servants in section 3. The pay-and-use value is axiomaized in section 4. This concludes the first part of the paper. We then endogenize the public-service vectors. Section 5 is devoted to the definition of a suitable equilibrium concept and section 6 works out a simple example. Section 4 concludes the paper.

2. Private-sector and mixed-economy coalition functions

In this section, we assume that the civil servants work as hard as the private agents. A TU game (N, v) is our model of an economy which can be enriched by a public sector:

Definition XIV.1. *An economy with a public sector is a tuple* (N, v, C, π) *where*

- $v \in \mathbb{V}_N$ *is a TU game (the economy),*
- C *is a proper subset of* N*, and*
- $\pi = (\pi_c)_{c \in C}$ *is a vector specifying an exogenous payoff for every member of* C*.*

(C, π) *is called the public-service vector.*

The reason for $C \subsetneqq N$ is that a purely-public economy is hard to imagine. In our setting, the payments for the civil servants are exogenous and budget balancing requires the existence of a private sector.

On the basis of (N, v, C, π), we define a mixed-economy game where both private and public officials are present, and an equivalent private-sector game where the players consist of private agents, only.

DEFINITION XIV.2. *Given an economy with a public sector* (N, v, C, π), *a mixed-economy game is a TU game* $\left(N, m^{v,C,\pi}\right)$ *given by* $m^{v,C,\pi} : 2^N \to \mathbb{R}$ *and*

$$\begin{aligned} m^{v,C,\pi}(K) &= \begin{cases} (v(K \cup C) - \pi_C) + \sum_{c \in K \cap C} \pi_c, & K \backslash C \neq \emptyset \\ \sum_{c \in K \cap C} \pi_c, & K \backslash C = \emptyset \end{cases} \\ &= \begin{cases} v(K \cup C) - \sum_{c \in C \backslash K} \pi_c, & K \backslash C \neq \emptyset \\ \sum_{c \in K} \pi_c, & K \backslash C = \emptyset \end{cases} \end{aligned}$$

A private-sector game is a TU game $\left(N \backslash C, p^{v,C,\pi}\right)$ *given by* $p^{v,C,\pi} : 2^{N \backslash C} \to \mathbb{R}$ *and*

$$p^{v,C,\pi}(S) = \begin{cases} v(S \cup C) - \pi_C, & S \neq \emptyset \\ 0, & S = \emptyset \end{cases}$$

Both the mixed-economy coalition function and the private-sector coalition function incorporate the idea that the private sector (the players from $N \backslash C$) has to pay π_C while at the same time benefitting from the C-players. As a matter of consistency, we have

$$m^{v,C,\pi}(K) = p^{v,C,\pi}(K \backslash C) + \sum_{c \in K \cap C} \pi_c, K \subseteq N \tag{XIV.1}$$

and $C = \emptyset$ implies $m^{v,C,\pi} = p^{v,C,\pi} = v$.

p is close to coalition functions defined in Aumann & Drèze (1974) and in Peleg (1986). The most important difference is that these authors assume that players from S can choose the players from C they want to use and pay for. However, since all people have to pay taxes irrespective of whether they do actually use the services, we opted for the above, simpler, coalition functions.

Our aim is to define a Shapley-like value for economies with a public sector. Before introducing inefficient civil servants and before dealing with the axiomatization of the new value, we offer a lemma whose proof which can be found in the appendix. The lemma confirms that $p^{v,C,\pi}$ and $m^{v,C,\pi}$ are basically equivalent.

LEMMA XIV.1. *Every player* $c \in C$ *is a dummy player in* $\left(N, m^{v,C,\pi}\right)$. *For all players* $i \in N \backslash C$, *we have* $Sh_i\left(m^{v,C,\pi}\right) = Sh_i\left(p^{v,C,\pi}\right)$.

See the appendix for a proof.

3. Introducing inefficiency

Taking up the civil-service example, we are interested in defining the worth of a coalition consisting of private and public agents. We assume that public agents are less hard working than private ones. Let t, $0 \leq t \leq 1$, be the work effort exercised by a typical civil servant who belongs to the coalition under consideration. Formally, a mixed workforce is a function

$$s : N \to [0,1]$$

obeying $s(i) \in \{0,1\}$ for all $i \notin C$ and $s(i) \in \{0,t\}$ for all $i \in C$. Private or public agents that do not belong to the coalition at hand obey $s(i) = 0$. We now apply the extensions known from chapter XIII. We remind the reader of the multi-linear extension which is defined by

$$u_T^{MLE}(s) := \prod_{i \in T} s_i, T \subseteq N, T \neq \emptyset$$

for unanimity games u_T and

$$v^{MLE}(s) := \sum_{T \in 2^N \backslash \{\emptyset\}} h_v(T) \cdot \prod_{i \in T} s_i$$

for any games $v \in \mathbb{V}_N$.

Thus, for usual coalitions, where $s \in \{0,1\}^N$ can be identified with $\mathbf{K}(s)$, the multilinear extension of v^{MLE} coincides with v. When effort levels are between 0 and 1, the multilinear extension, v^{MLE}, has a probabilistic interpretation (as noted by Owen (1972, p. 64)). For example, two productive players in the unanimity game $u_{\{1,2\}}$ with $T = N = \{1,2\}$ and $s = \left(\frac{1}{2}, \frac{1}{3}\right)$ can produce $\frac{1}{2} \cdot \frac{1}{3}$, only.

We now extend the definitions from section 2 to take care of the efficiency parameter t :

Definition XIV.3. *An economy with lazy public servants is a tuple* (N, v, C, π, t) *where* (N, v, C, π) *is an economy with a public sector and* t *is the efficiency parameter for the civil servants obeying* $0 \leq t \leq 1$.

Definition XIV.4. *Given an economy with lazy public servants* (N, v, C, π, t), *a private-sector game is a TU game* $\left(N\backslash C, p^{v,C,\pi,t}\right)$ *given by* $p^{v,C,\pi,t} : 2^{N\backslash C} \to \mathbb{R}$ *and*

$$p^{v,C,\pi,t}(S) = \begin{cases} \sum_{T \in 2^{S \cup C} \backslash \{\emptyset\}} h_v(T)\, t^{|C \cap T|} - \pi_C, & S \neq \emptyset \\ 0, & S = \emptyset \end{cases}$$

The mixed mixed-economy coalition function is obtained by the obvious application of eq. XIV.1.

Again, the proof of the next lemma can be found in the appendix.

Lemma XIV.2. *For* $\gamma \in \mathbb{R}$, *we have*

$$p^{\gamma u_T, C, \pi, t}(N \backslash C) = \gamma t^{|C \cap T|} - \pi_C. \tag{XIV.2}$$

For any game v, *the marginal contribution of a null player* $i \notin C$ *is*

$$MC_i^S\left(p^{v,C,\pi,t}\right) = \begin{cases} \sum_{T \in 2^C \backslash \{\emptyset\}} h_v(T) \cdot t^{|C \cap T|} - \pi_C, & S = \emptyset \\ 0, & S \neq \emptyset \end{cases}$$

4. The pay-and-use value: definition and axiomatization

A ps-value (public-service value) φ assigns a payoff vector to every economy with lazy public servants (N, v, C, π, t), $\varphi(N, v, C, \pi, t) \in \mathbb{R}^n$. Values φ may obey the following axioms (or families of axioms):

Axiom X (exogenous payments): For all $i \in C$, we have

$$\varphi_i(N, v, C, \pi, t) = \pi_i.$$

Axiom N (Null player): For any player $i \in N \backslash C$ that is a null player in (N, v),

$$\varphi_i(N, v, C, \pi, t) = \frac{v^{MLE}\left(\underbrace{t, ..., t}_{\text{civil servants}}, \underbrace{0, ..., 0}_{\text{private agents}}\right) - \pi_C}{|N \backslash C|}.$$

Axiom E (efficiency): We have

$$\varphi_N(N, v, C, \pi, t) = v(N)$$

Axiom S (symmetry): For all symmetric players $i, j \in N \backslash C$,

$$\varphi_i(N, v, C, \pi, t) = \varphi_j(N, v, C, \pi, t).$$

Axiom A (additivity): For any coalition functions $v', v'' \in \mathbb{V}_N$, any payments π', $\pi'' \in \mathbb{R}^{|C|}$ and any player i from N,

$$\varphi_i\left(N, v' + v'', C, \pi' + \pi'', t\right) = \varphi_i\left(N, v', C, \pi', t\right) + \varphi_i\left(N, v'', C, \pi'', t\right).$$

Axioms E, S, and N for $C = \emptyset$ are the efficiency, symmetry, and null-player axioms due to Shapley. A null player is better off under N than under the usual null-player axiom if the C-players' worth is higher than their aggregate fixed payment. Axiom X imposes the exogenous payments π for the players in C. Note that the additivity axiom adds both the coalition functions and the civil-service payments on the left-hand side of the equation.

We now provide the axiomatization of our PU-value; the proof can be found in the appendix.

THEOREM XIV.1. *There exists one and only one ps-value that satisfies the axioms X, N, E, S, and A. It is called PU-value and is given by*

$$PU_i(N, v, C, \pi, t) = \begin{cases} \pi_i, & i \in C \\ Sh_i\left(N \backslash C, p^{v,C,\pi,t}\right), & i \notin C \end{cases}$$

An interpretation in terms of rank orders can be given. Consider the private-sector game $\left(N \backslash C, p^{v,C,\pi}\right)$. The players from C are gathered in a room. They are able to produce $v(C)$ but demand payment $\pi_C = \sum_{c \in C} \pi_c$.

Now, the players from $N\backslash C$ enter the room, one after the other. The first player j from $N\backslash C$ to join the C-players obtains the marginal contribution

$$p^{v,C,\pi,t}(j) - 0 = v^{MLE}\left(\underbrace{t,...,t}_{\text{civil servants}}, \underbrace{0,...,0}_{\text{private agents}}\right).$$

He pays the civil servants. The following players, all from $N\backslash C$, obtain the marginal contributions with respect to v^{MLE} (the π_C-term always cancels). Since every private agent has the same chance of being the first to enter, the "tax" payable by each private-sector player is equitable and equal to $\frac{\pi_C}{|N\backslash C|}$.

LEMMA XIV.3. *Let (N, v, C, π, t) be an economy. For every $i \in N\backslash C$ and every $c \in C$, we have*

$$\partial \frac{PU_i(N, v, C, \pi, t)}{\partial \pi_c} = \frac{1}{|N\backslash C|}.$$

5. Suggesting public-service vectors: a noncooperative game

The PU-value rests on a given public-service vector which we are now to endogenize. Basically, we employ three conditions. First, every civil servant is free to join the private sector; this is the no-slavery condition. Second, an equilibrium public-service vector has to survive a majority test. Third, all agents are free to emigrate and earn a foreign reservation payoff. Of course, we cannot "allow" all private agents to emigrate:

DEFINITION XIV.5. *The tuple $\left(N, v, (r_i)_{i\in N}\right)$ is called an economy (with emigration) where $r_i \in \mathbb{R}$ is the foreign reservation payoff for agent $i \in N$. An outcome for an economy is a tuple (C^*, π^*, E^*) where (C^*, π^*) is a public-service vector and $E^* \subsetneqq N\backslash C^*$ the set of emigrants.*

In rough terms, we define a five-stage game:

- Nature picks an agenda setter $\hat{\imath}$ from the set N with equal chance $\frac{1}{n}$ for each player.
- The chosen player $\hat{\imath}$ suggests a public-service vector (C, π).
- All agents $i \in N$ consider to emigrate or to stay, $e(i) \in \{s, g\}$ with s for "stay", i.e., not emigrate and g for "go", i.e., emigrate. Let $E := \{i \in N : e(i) = g\}$. In case of $C = \emptyset$, $E \cap C \neq \emptyset$, or $E = N\backslash C$ the game is over and $C^* := \emptyset$ (together with any π^*) and $E^* := E$ the outcome.
- If $C \neq \emptyset$ and $E \subsetneqq N\backslash C$ (this last condition is equivalent to $E \cap C = \emptyset$ and $E \neq N\backslash C$), all agents $i \in N\backslash E$ cast a vote, yes or no, with respect to (C, π), $a(i) \in \{yes, no\}$. If more than 50% vote "yes", the proposal (C, π) is adopted, otherwise, the game is over and $C^* := \emptyset$ (together with any π^*) and $E^* := E$ is the outcome.

- If $C \neq \emptyset$, $E \subsetneqq N \backslash C$, and more than 50% of the players from $N \backslash E$ have voted "yes", the agents from C decide whether or not to accept, $p(c) \in \{acc, dec\}$. The outcome is $C^* := \begin{cases} C, & p(c) = acc \text{ for all } i \in C \\ \emptyset, & \text{otherwise} \end{cases}$ together with $\pi^* := \pi$ and $E^* := E$.

If several players are to move within a stage, they decide simultaneously. We assume that an agenda setter will propose a public-service vector (C, π) such that

- no prospective civil servant emigrates,
- more than 50% of the non-emigrants vote "yes", and
- no prospective civil servant declines.

We also assume that indifferent agents say "yes" and "accept" at stages 4, and 5, respectively. Also, the emigration decisions follow the suggestion of the proposer unless the agents strictly prefer otherwise. An agenda setter proposes $C = \emptyset$ unless a set of civil servants $C \neq \emptyset$ exists which is strictly better for him. If an agenda setter is indifferent between several public-service vectors with $C \neq \emptyset$ surviving stages 3, 4 and 5, he chooses any of these with equal probability.

Our definition of an equilibrium is somewhat similar to the ones found in principal-agent theory. The proposer chooses an outcome subject to certain constraints. Along with the public-service vector (C^*, π^*) the principal chooses the emigrating agents E^* (in case of (C^*, π^*), with $C^* \neq \emptyset$) and $E^\emptyset$ (in case of $C^* = \emptyset$).

DEFINITION XIV.6. *Let $(N, v, (r_i)_{i \in N})$ be an economy and let $\hat{\imath} \in N$ be the proposer chosen at stage* 1. *$(E^*, C^*, \pi^*, E^{\emptyset *})$ constitutes an equilibrium if this vector is from*

$$\arg \max_{\substack{(E, C, \pi, E^\emptyset), \\ C \subsetneqq N \backslash E}} PU_{\hat{\imath}} \left(N \backslash E, v|_{N \backslash E}, C, \pi \right)$$

subject to the consistency requirement $C^* = \emptyset \Rightarrow E^* = E^{\emptyset *}$ and subject to

(1) *the strict-preference constraint S:*

$$\begin{aligned} C^* &\neq \emptyset \text{ and } \hat{\imath} \notin E^* \\ &\Rightarrow PU_{\hat{\imath}} \left(N \backslash E^*, v|_{N \backslash E^*}, C^*, \pi^* \right) > \begin{cases} PU_{\hat{\imath}} \left(N \backslash E^{\emptyset *}, v|_{N \backslash E^{\emptyset *}}, \emptyset, \cdot \right), & \hat{\imath} \notin E^{\emptyset *} \\ r_i, & \hat{\imath} \in E^{\emptyset *} \end{cases} \end{aligned}$$

(2) *the emigration constraints E:*

$$\begin{aligned} i &\notin E^* \Rightarrow PU_i \left(N \backslash E^*, v|_{N \backslash E^*}, C^*, \pi^* \right) \geq r_i \text{ and} \\ i &\in E^* \Rightarrow PU_i \left((N \backslash E^*) \cup \{i\}, v|_{(N \backslash E^*) \cup \{i\}}, C^*, \pi^* \right) \leq r_i \end{aligned}$$

and

$$i \notin E^{\emptyset *} \Rightarrow PU_i\left(N\backslash E^{\emptyset *}, v|_{N\backslash E^{\emptyset *}}, \emptyset, \cdot\right) \geq r_i \text{ and}$$

$$i \in E^{\emptyset *} \Rightarrow PU_i\left(\left(N\backslash E^{\emptyset *}\right) \cup \{i\}, v|_{\left(N\backslash E^{\emptyset *}\right)\cup\{i\}}, \emptyset, \cdot\right) \leq r_i$$

(3) *the majority constraint M: For all $i \in N\backslash E^*$, we have*

$$a(i) = \begin{cases} yes, & PU_i\left(N\backslash E^*, v|_{N\backslash E^*}, C^*, \pi^*\right) \geq PU_i\left(N\backslash E^*, v|_{N\backslash E^*}, \emptyset, \cdot\right) \\ no, & otherwise \end{cases}$$

together with

$$|\{i \in N\backslash E^* : a(i) = yes\}| > |\{i \in N\backslash E^* : a(i) = no\}|,$$

(4) *the civil-service participation constraint P: For every $c \in C^*$, we obtain $c \neq E^*$ and*

$$\pi_c \geq PU_c\left(N\backslash E^*, v|_{N\backslash E^*}, \emptyset, \cdot\right).$$

Note that constaints M and P rest on a comparison with $PU_c\left(N\backslash E^*, v|_{N\backslash E^*}, \emptyset, \cdot\right)$. This is a reflection of the sequence given above where the emigration decision precedes the voting and participation decions.

6. A simple three-player example

6.1. Payoffs. We consider $N = \{1,2,3\}$, the unanimity game $u_{\{1,2\}}$ with the two productive players 1 and 2. If no player emigrates, we need to distinguish seven cases:

- A: no civil servants:

$$PU\left(\{1,2,3\}, u_{\{1,2\}}, \emptyset, \pi, t\right) = \left(\frac{1}{2}, \frac{1}{2}, 0\right).$$

- B: productive player 1 is civil servant:

$$PU\left(\{1,2,3\}, u_{\{1,2\}}, \{1\}, \pi, t\right) = \left(\pi_1, t - \frac{\pi_1}{2}, -\frac{\pi_1}{2}\right).$$

- C: productive player 2 is civil servant:

$$PU\left(\{1,2,3\}, u_{\{1,2\}}, \{2\}, \pi, t\right) = \left(t - \frac{\pi_2}{2}, \pi_2, -\frac{\pi_2}{2}\right).$$

- D: unproductive player 3 is civil servant:

$$PU\left(\{1,2,3\}, u_{\{1,2\}}, \{3\}, \pi, t\right) = \left(\frac{1}{2} - \frac{\pi_3}{2}, \frac{1}{2} - \frac{\pi_3}{2}, \pi_3\right).$$

- E: two productive players 1 and 2 are civil servants:

$$PU\left(\{1,2,3\}, u_{\{1,2\}}, \{1,2\}, \pi, t\right) = \left(\pi_1, \pi_2, t^2 - \pi_1 - \pi_2\right).$$

- F: productive player 1 and unproductive player 3 are civil servants:

$$PU\left(\{1,2,3\}, u_{\{1,2\}}, \{1,3\}, \pi, t\right) = \left(\pi_1, t - \pi_1 - \pi_3, \pi_3\right).$$

- G: productive player 2 and unproductive player 3 are civil servants:

$$PU\left(\{1,2,3\}, u_{\{1,2\}}, \{2,3\}, \pi, t\right) = (t - \pi_2 - \pi_3, \pi_2, , \pi_3).$$

Note

$$\begin{aligned} PU\left(\{1,2\}, u_{\{1,2\}}\big|_{\{1,2\}}, \emptyset, \pi, t\right) &= \left(\frac{1}{2}, \frac{1}{2}\right), \\ PU\left(\{1,3\}, u_{\{1,2\}}\big|_{\{1,3\}}, \emptyset, \pi, t\right) &= (0,0), \text{ and} \\ PU\left(\{i\}, u_{\{1,2\}}\big|_{\{i\}}, \emptyset, \pi, t\right) &= 0. \end{aligned}$$

6.2. A special case. To show the workings of our model, we report some results in a special case:

LEMMA XIV.4. *Assume that player* 1 *is the agenda setter (proposer) and let* $r_1 = \frac{1}{6}, r_2 = \frac{7}{12}$, *and* $r_3 = -\frac{1}{3}$ *be the foreign reservation payoffs. Then, we obtain* $E^{\emptyset *} = \{1,2\}$ *as the only emigration set fulfilling constraint* $E^{\emptyset}$. *Depending on the efficiency parameter* t, *we obtain the best proposals given by table 1 or figure 1 (where the numbers on the axes are correct in an ordinal sense, only).*

parameter t	*proposal* $\left(E^*, C^*, \pi^*, E^{\emptyset *}\right)$	*payoff*
$t < \frac{19}{24}$	$(\{1,2\}, \emptyset, \cdot, \{1,2\})$	$r_1 = \frac{1}{6}$
$\frac{19}{24} \leq t < \frac{7}{8}$	$\left(\emptyset, \{2\}, \frac{7}{12}, \{1,2\}\right)$	$t - \frac{7}{24}$
$\frac{7}{8} \leq t < \frac{11}{12}$	$\left(\emptyset, \{1\}, 2t - \frac{7}{6}, \{1,2\}\right)$	$2t - \frac{7}{6}$
$\frac{11}{12} \leq t < \sqrt{\frac{11}{12}}$	$\left(\emptyset, \{1\}, \frac{2}{3}, \{1,2\}\right)$	$\frac{2}{3}$
$\sqrt{\frac{11}{12}} \leq t \leq 1$	$\left(\emptyset, \{1,2\}, \left(\pi_1^* = t^2 - \frac{1}{4}, \pi_2^* = \frac{7}{12}\right), \{1,2\}\right)$	$t^2 - \frac{1}{4}$

Table 1: Best proposals and maximal proposer payoffs

Figure 1: Best proposals and maximal proposer payoffs

The proof of this lemma is given in the appendix. For $t \leq \frac{19}{24}$ (public servants are very or somewhat inefficient), all the productive members of the economy emigrate, leaving the economy in a desperate state.

Note, also, the discontinuity in the proposer's payoff at $t = \frac{19}{24}$. For t-values below this treshold, the proposer's payoff drops to the foreign reservation payoff $\frac{1}{6}$. This is due to the time structure of the model. For small t-values the proposer cannot guarantee himself a payoff of at least $\frac{1}{2}$. Of course, given his low foreign reservation payoff $\frac{1}{6} < \frac{1}{2}$, he would be happy to accept less than $\frac{1}{2}$ at the proposing stage. However, at the voting stage, the emigration decision has already been made and voter 1 (the former proposer) rejects any offer of less than 1/2. In a subgame-perfect equilibrium, the proposer cannot bind himself as a voter or as a prospective civil servant.

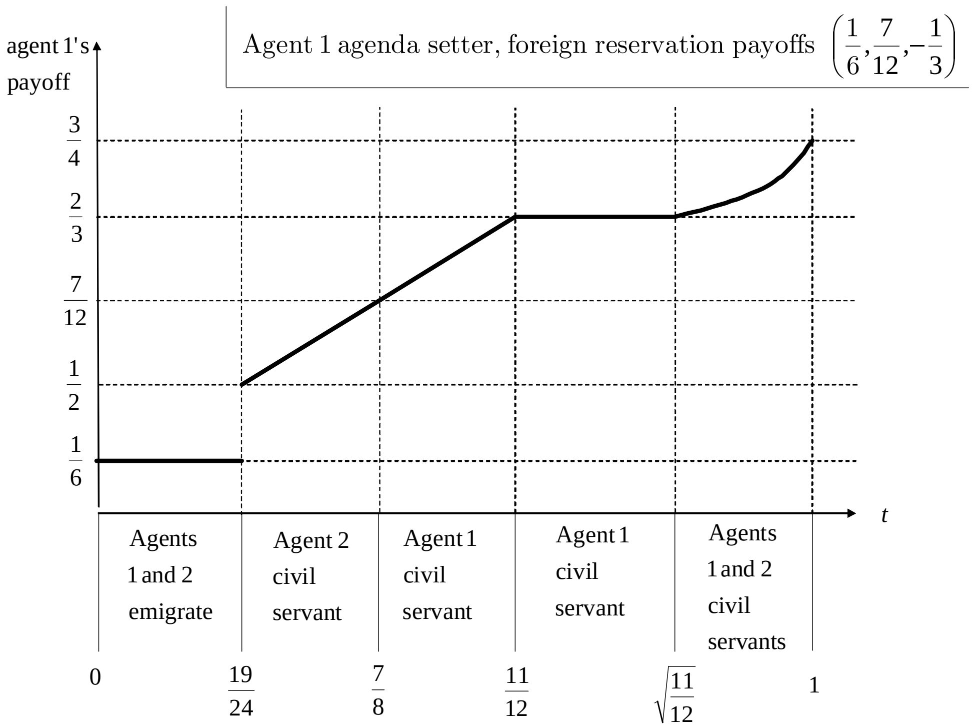

FIGURE 1

The proof of the lemma also shows that the proposer overcomes the majority constraint with the help of player 2. Therefore, if no player emigrates (at stage 3), player 1 and 2 expect a payoff of at least $\frac{1}{2}$.

In the t-interval defined by $\frac{19}{24} \leq t < \frac{7}{8}$, player 1 proposes player 2 as the civil servant who is to obtain a payment (at least) equal to his foreign reservation payoff of $\frac{7}{12}$. The taxes are borne equally by players 1 and 3 so that null player 3 obtains $-\frac{\pi_2}{2} = -\frac{\frac{7}{12}}{2} = -\frac{7}{24} > -\frac{1}{3} = r_3$. Player 1 obtains $t - \frac{\pi_2}{2} = t - \frac{7}{24}$.

For $\frac{7}{8} \leq t < \sqrt{\frac{11}{12}}$, player 1 proposes himself as a civil servant who obtains payoff π_1. By player 3's emigration constraint, we have $-\frac{\pi_1}{2} \geq r_3$ and hence $\pi_1 \leq \frac{2}{3}$. Indeed, $\pi_1 = \frac{2}{3}$ is the payoff achievable for $\frac{11}{12} \leq t < \sqrt{\frac{11}{12}}$. In the other subinterval ($\frac{7}{8} \leq t < \frac{11}{12}$), player 2's emigration constraint $t - \frac{\pi_1}{2} \geq r_2$ (which is equivalent to $\pi_1 \leq 2t - \frac{7}{6}$) is binding.

In the last interval, given by $\sqrt{\frac{11}{12}} \leq t \leq 1$, player 1 proposes both himself and player 2 as civil servants. The inefficiency resulting from two civil servants with overall product t^2 rather than just one civil servant with product t (if there is no civil servant, we see both players 1 and 2 emigrate) is less severe for high t-values. Player 1 can fix both π_1 and π_2. Respecting the two emigration constraints, the payments have to obey $\pi_2 \geq \frac{7}{12}$ and $t^2 - \pi_1 - \pi_2 \geq -\frac{1}{3}$. Therefore, player 1 can propose $\pi_1 = t^2 - \frac{1}{4}$ for himself.

7. Conclusions

Microeconomic analysis of the limits of government, the limits of the firm or other economic institutions is well established. This paper is part of an ongoing research program where similar analyses are done by way of cooperative game theory. The size-of-government question is a companion problem to the famous boundaries-of-the-firm question posed by Ronald Coase (1937): What kind of economic activity is conducted through markets and what kind is conducted through firms? Wiese (2005b) is an attempt to approach that question with cooperative means.

The discussion of our special case makes clear that the model bears out meaningful and interpretable results. A major drawback of the analysis (as done so far) is, of course, that there are no general results. Also, the sequence of events in our non-cooperative model is up to criticism. After all, it is responsible for the inability of the proposer to commit himself to voting for his own proposal. Alternatively, the emigration decisions could be the last to be made, after voting and after the civil servants participation decision. However, in that case, the voters may vote for a constellation that is partly or totally made obsolete by the emigration decisions to follow. For this reason, we opted for the sequence as presented in the paper.

Finally, our model suffers from a basic asymmetry. While the agents are free to emigrate, immigration has no role to play. Indeed, in order to close the model with respect to immigration, we would need to build a total model comprising a home and a foreign country. The methodology presented in this paper gives a clear indication of how this can be achieved.

8. Appendix

Proof of lemma XIV.1

Note that $c \in C$ implies $S \backslash C = (S \backslash c) \backslash C$ and $S \cup C = (S \backslash c) \cup C$. For any $S \subseteq N$ with $c \in C \cap S$ we have

$$\begin{aligned}
& MC_c^S \left(m^{v,C,\pi}\right) \\
= \; & m^{v,C,\pi}(S) - m^{v,C,\pi}(S \backslash c) \\
= \; & \begin{cases} v(S \cup C) - \sum_{d \in C \backslash S} \pi_d, & S \backslash C \neq \emptyset \\ \sum_{d \in S} \pi_d, & S \backslash C = \emptyset \end{cases} \\
& - \begin{cases} v((S \backslash c) \cup C) - \sum_{d \in C \backslash (S \backslash c)} \pi_d, & (S \backslash c) \backslash C \neq \emptyset \\ \sum_{d \in (S \backslash c)} \pi_d, & (S \backslash c) \backslash C = \emptyset \end{cases} \\
= \; & \begin{cases} v(S \cup C) - \sum_{d \in C \backslash S} \pi_d, & S \backslash C \neq \emptyset \\ \sum_{d \in S} \pi_d, & S \backslash C = \emptyset \end{cases} \\
& - \begin{cases} v(S \cup C) - \sum_{d \in C \backslash (S \backslash c)} \pi_d, & S \backslash C \neq \emptyset \\ \sum_{d \in S \backslash c} \pi_d, & S \backslash C = \emptyset \end{cases} \\
= \; & \pi_c
\end{aligned}$$

so that c is a dummy player in $\left(N, m^{v,C,\pi}\right)$.

Let i be from $N \backslash C$. Then

$$\begin{aligned}
& Sh_i\left(m^{v,C,\pi}\right) \\
= \; & \frac{1}{n!} \sum_{\rho \in RO(N)} MC_i^{K_i(\rho)}\left(m^{v,C,\pi}\right) \text{ (definition Shapley value)} \\
= \; & \frac{1}{n!} \sum_{\rho \in RO(N)} \left[m^{v,C,\pi}(K_i(\rho)) - m^{v,C,\pi}(K_i(\rho) \backslash i)\right] \text{ (definition marginal contribution)} \\
= \; & \frac{1}{n!} \sum_{\rho \in RO(N)} \left[p^{v,C,\pi}(K_i(\rho) \backslash C) + \sum_{c \in K_i(\rho) \cap C} \pi_c - \left(p^{v,C,\pi}((K_i(\rho) \backslash i) \backslash C) + \sum_{c \in (K_i(\rho) \backslash i) \cap C} \pi_c\right)\right] \\
& \text{(equation XIV.1)} \\
= \; & \frac{1}{n!} \sum_{\rho \in RO(N)} \left[p^{v,C,\pi}(K_i(\rho) \backslash C) - p^{v,C,\pi}((K_i(\rho) \backslash i) \backslash C)\right] \; (i \notin C) \\
= \; & \frac{1}{n!} \frac{n!}{(n - |C|)!} \sum_{\rho \in RO(N \backslash C)} \left[p^{v,C,\pi}(K_i(\rho) \backslash C) - p^{v,C,\pi}((K_i(\rho) \backslash i) \backslash C)\right] \; (*) \\
= \; & \frac{1}{(n - |C|)!} \sum_{\rho \in RO(N \backslash C)} \left[p^{v,C,\pi}(K_i(\rho)) - p^{v,C,\pi}(K_i(\rho) \backslash i)\right] \; (K_i(\rho) \backslash C = K_i(\rho)) \\
= \; & \frac{1}{(n - |C|)!} \sum_{\rho \in RO(N \backslash C)} MC_i^{K_i(\rho)}\left(p^{v,C,\pi}\right) \text{ (definition marginal contribution)} \\
= \; & Sh_i\left(p^{v,C,\pi}\right) \text{ (definition Shapley value)}
\end{aligned}$$

where the main step, at (*), consists of putting together all those rank orders from $\rho \in RO(N)$ that leave the order of the players from $N \backslash C$ intact. We have $\frac{n!}{(n-|C|)!}$ such orders.

Proof of lemma XIV.2

For $i \notin S$, we find

$$\begin{aligned}
& \sum_{\substack{T\in 2^{S\cup i\cup C}\backslash\{\emptyset\},\\ i\in T}} h_v(T)\cdot t^{|C\cap T|} \\
= & \sum_{\substack{T\in 2^{S\cup i\cup C}\backslash\{\emptyset\},\\ i\in T}} \sum_{K\in 2^T\backslash\{\emptyset\}} (-1)^{|T|-|K|}\, v(K)\cdot t^{|C\cap T|} \\
= & \sum_{\substack{T\in 2^{S\cup i\cup C}\backslash\{\emptyset\},\\ i\in T}} \sum_{\substack{K\in 2^T\backslash\{\emptyset\},\\ i\notin K}} \underbrace{\left[(-1)^{|T|-|K|}\, v(K) + (-1)^{|T|-|K\cup\{i\}|}\, v(K\cup\{i\})\right]}_{0}\cdot t^{|C\cap T|} \\
= & \ 0.
\end{aligned}$$

Therefore, $S \neq \emptyset$ with $i \notin S$ implies

$$\begin{aligned}
& p^{v,C,\pi,t}(S\cup i) - p^{v,C,\pi,t}(S) \\
= & \sum_{T\in 2^{S\cup i\cup C}\backslash\{\emptyset\}} h_v(T)\cdot t^{|C\cap T|} - \sum_{T\in 2^{S\cup C}\backslash\{\emptyset\}} h_v(T)\cdot t^{|C\cap T|} \\
= & \sum_{\substack{T\in 2^{S\cup i\cup C}\backslash\{\emptyset\},\\ i\in T}} h_v(T)\cdot t^{|C\cap T|} + \sum_{\substack{T\in 2^{S\cup i\cup C}\backslash\{\emptyset\},\\ i\notin T}} h_v(T)\cdot t^{|C\cap T|} - \sum_{T\in 2^{S\cup C}\backslash\{\emptyset\}} h_v(T)\cdot t^{|C\cap T|} \\
= & \sum_{\substack{T\in 2^{S\cup i\cup C}\backslash\{\emptyset\},\\ i\in T}} h_v(T)\cdot t^{|C\cap T|} \\
= & \ 0
\end{aligned}$$

while $S = \emptyset$ leads to

$$\begin{aligned}
& p^{v,C,\pi,t}(i) - p^{v,C,\pi,t}(\emptyset) \\
= & \sum_{T\in 2^{i\cup C}\backslash\{\emptyset\}} h_v(T)\cdot t^{|C\cap T|} - \pi_C \\
= & \sum_{\substack{T\in 2^{i\cup C}\backslash\{\emptyset\},\\ i\in T}} h_v(T)\cdot t^{|C\cap T|} + \sum_{\substack{T\in 2^{i\cup C}\backslash\{\emptyset\},\\ i\notin T}} h_v(T)\cdot t^{|C\cap T|} - \pi_C \\
= & \sum_{\substack{T\in 2^{i\cup C}\backslash\{\emptyset\},\\ i\notin T}} h_v(T)\cdot t^{|C\cap T|} - \pi_C \\
= & \sum_{T\in 2^{C}\backslash\{\emptyset\}} h_v(T)\cdot t^{|C\cap T|} - \pi_C.
\end{aligned}$$

Proof of theorem XIV.1

We follow the proof outlined by Aumann (1989, pp. 30). In order to show uniqueness, let φ be any ps-value satisfying the axioms mentioned in the theorem. The reader is reminded of two facts concerning the unanimity games u_T, $T \neq \emptyset$, $T \subseteq N$. First, they form a basis of the vector space $\mathbb{V}_N$. Eq. **??** shows that the Harsanyi dividends are the coefficients. Second, players from $N \backslash T$ are null players in γu_T, $\gamma \in \mathbb{R}$. Hence, for any $\gamma \in \mathbb{R}$, any $T \subseteq N$, $T \neq \emptyset$, any $C \neq N$ and any $\pi^T \in \mathbb{R}^{|C|}$, we find

$$\begin{aligned}
\gamma t^{|C\cap T|} &= p^{\gamma u_T, C, \pi^T, t}(N\backslash C) + \pi_C^T \text{ (eq. XIV.2)} \\
&= \varphi_N\left(N, \gamma u_T, C, \pi^T, t\right) \text{ (axiom E)} \\
&= \sum_{i\in C} \varphi_i\left(N, \gamma u_T, C, \pi^T, t\right) + \sum_{i\in T\backslash C} \varphi_i\left(N, \gamma u_T, C, \pi^T, t\right) \\
&\quad + \sum_{i\in N\backslash(T\cup C)} \varphi_i\left(N, \gamma u_T, C, \pi^T, t\right) \\
&= \pi_C^T + \sum_{i\in T\backslash C} \varphi_i\left(N, \gamma u_T, C, \pi^T, t\right) \\
&\quad + \sum_{i\in N\backslash(T\cup C)} \frac{\sum_{T\in 2^C\backslash\{\emptyset\}} h_v(T)\cdot t^{|C\cap T|} - \pi_C}{|N\backslash C|} \text{ (axioms X, N)}
\end{aligned}$$

Axiom S now implies

$$\varphi_i\left(N, \gamma u_T, C, \pi^T, t\right) = \begin{cases} \pi_i^T, & i \in C \\ \frac{\gamma t^{|C\cap T|} - \pi_C^T - \sum_{i\in N\backslash(T\cup C)} \frac{\sum_{T\in 2^C\backslash\{\emptyset\}} h_v(T)\cdot t^{|C\cap T|} - \pi_C}{|N\backslash C|}}{|T\backslash C|}, & i \in T\backslash C \\ \frac{\sum_{T\in 2^C\backslash\{\emptyset\}} h_v(T)\cdot t^{|C\cap T|} - \pi_C}{|N\backslash C|}, & i \in N\backslash(T\cup C) \end{cases}$$

Letting

$$\pi^T := \begin{cases} \pi, & T = N \\ 0, & \text{otherwise} \end{cases}$$

and using **??**, axiom A yields

$$\begin{aligned}
\varphi_i(N, v, C, \pi, t) &= \varphi_i\left(N, \sum_{T\in 2^N\backslash\{\emptyset\}} h_v(T)\, u_T, C, \pi^T, t\right) \\
&= \sum_{T\in 2^N\backslash\{\emptyset\}\backslash N} \varphi_i(N, h_v(T)\, u_T, C, 0, t) + \varphi_i(N, \gamma u_N, C, \pi, t)
\end{aligned}$$

Thus, the axioms determine the payoffs.

It is not difficult to show that the PU-value satisfies all the axioms. Axiom X is obviously fulfilled. Axiom N follows from lemma XIV.2 and the fact that every player $i \in N\backslash C$ has a chance of 1 over $|N\backslash C|$ of being "the first player to enter". Efficiency follows from

$$\begin{aligned}
\sum_{i\in N} PU_i\left(N,v,C,\pi,t\right) &= \sum_{i\in C} PU_i\left(N,v,C,\pi,t\right) + \sum_{i\in N\backslash C} PU_i\left(N,v,C,\pi,t\right) \\
&= \pi_C + \sum_{i\in N\backslash C} Sh_i\left(p^{v,C,\pi,t}\right) \text{ (definition of } PU\text{)} \\
&= \pi_C + p^{v,C,\pi,t}\left(N\backslash C\right) \text{ (efficiency of Shapley value)} \\
&= \sum_{T\in 2^N\backslash\{\emptyset\}} h_v\left(T\right) t^{|C\cap T|} \text{ (definition of } p^{v,C,\pi,t}\text{)} \\
&= v^{MLE}\left(\underbrace{t,...,t}_{\text{civil servants}}, \underbrace{1,...,1}_{\text{private agents}}\right).
\end{aligned}$$

Axiom S is true for PU because the payments for players outside C are not affected by PU.

We now turn to axiom A. Additivity obviously holds for $i \in C$. Assume $i \notin C$. Consider any coalition functions $v', v'' \in \mathbb{V}_N$ and any payments π', $\pi'' \in \mathbb{R}^{|C|}$. Then, additivity follows from the additivity of the Harsanyi dividend (eq. **??**):

$$\begin{aligned}
& PU_i\left(N, v'+v'', C, \pi'+\pi'', t\right) \\
= \; & Sh_i\left(N\backslash C, p^{v'+v'',C,\pi'+\pi'',t}\right) \\
= \; & \frac{1}{(n-|C|)!}\left(\sum_{\substack{\rho\in RO(N\backslash C),\\ K_i(\rho)\backslash i\neq\emptyset}} \left[p^{v'+v'',C,\pi'+\pi'',t}\left(K_i\left(\rho\right)\right) - p^{v'+v'',C,\pi'+\pi'',t}\left(K_i\left(\rho\right)\backslash i\right)\right]\right. \\
& \left. + \sum_{\substack{\rho\in RO(N\backslash C),\\ K_i(\rho)\backslash i=\emptyset}} \left[p^{v'+v'',C,\pi'+\pi'',t}\left(K_i\left(\rho\right)\right) - p^{v'+v'',C,\pi'+\pi'',t}\left(K_i\left(\rho\right)\backslash i\right)\right]\right) \\
= \; & \frac{1}{(n-|C|)!}\left(\sum_{\substack{\rho\in RO(N\backslash C),\\ K_i(\rho)\backslash i\neq\emptyset}} \left[\sum_{T\in 2^{K_i(\rho)\cup C}\backslash\{\emptyset\}} h_{v'+v''}\left(T\right) t^{|C\cap T|} - \pi'_C - \pi''_C\right.\right. \\
& \left. - \left(\sum_{T\in 2^{K_i(\rho)\backslash i\cup C}\backslash\{\emptyset\}} h_{v'+v''}\left(T\right) t^{|C\cap T|} - \pi'_C - \pi''_C\right)\right] \\
& \left. + \sum_{\substack{\rho\in RO(N\backslash C),\\ K_i(\rho)\backslash i=\emptyset}} \left[\sum_{T\in 2^{\{i\}\cup C}\backslash\{\emptyset\}} h_{v'+v''}\left(T\right) t^{|C\cap T|} - \pi'_C - \pi''_C - 0\right]\right)
\end{aligned}$$

$$
\begin{aligned}
= \ & \frac{1}{(n-|C|)!}\left(\sum_{\substack{\rho\in RO(N\backslash C),\\ K_i(\rho)\backslash i\neq\emptyset}}\left[\sum_{T\in 2^{K_i(\rho)\cup C}\backslash\{\emptyset\}} h_{v'}(T)\, t^{|C\cap T|}-\pi'_C\right.\right.\\
& \left.-\left(\sum_{T\in 2^{K_i(\rho)\backslash i\cup C}\backslash\{\emptyset\}} h_{v'}(T)\, t^{|C\cap T|}-\pi'_C\right)\right]\\
& +\sum_{\substack{\rho\in RO(N\backslash C),\\ K_i(\rho)\backslash i=\emptyset}}\left[\sum_{T\in 2^{\{i\}\cup C}\backslash\{\emptyset\}} h_{v'}(T)\, t^{|C\cap T|}-\pi'_C-0\right]\\
& +\sum_{\substack{\rho\in RO(N\backslash C),\\ K_i(\rho)\backslash i\neq\emptyset}}\left[\sum_{T\in 2^{K_i(\rho)\cup C}\backslash\{\emptyset\}} h_{v''}(T)\, t^{|C\cap T|}-\pi''_C-\left(\sum_{T\in 2^{K_i(\rho)\backslash i\cup C}\backslash\{\emptyset\}} h_{v''}(T)\, t^{|C\cap T|}-\pi''_C\right)\right]\\
& \left.+\sum_{\substack{\rho\in RO(N\backslash C),\\ K_i(\rho)\backslash i=\emptyset}}\left[\sum_{T\in 2^{\{i\}\cup C}\backslash\{\emptyset\}} h_{v''}(T)\, t^{|C\cap T|}-\pi''_C-0\right]\right)\\
= \ & Sh_i\left(N\backslash C, p^{v',C,\pi',t}\right)+Sh_i\left(N\backslash C, p^{v'',C,\pi'',t}\right)\\
= \ & PU_i\left(N,v',C,\pi',t\right)+PU_i\left(N,v'',C,\pi'',t\right) \ \text{(definition of } PU\text{)}
\end{aligned}
$$

Thus, the PU-value fulfills all the axioms mentioned in the theorem and is the only value to do so.

Proof of lemma XIV.4

For $E^* = \emptyset$, tables A1 and A2 help to find the equilibria. In the leftmost column of both tables, we see the proposal made by 1 together with the payoffs in terms of π. The status quo A is not listed. If another proposal is successful, it needs to fulfill the constraints S (strict preference), E (emigration), M (majority), and P (civil-service participation). Player 1 now needs to check which of these proposals (if any) is best. Tables A1 and A2 report the constraints depending on whether player 1 enlists player 2 to ensure a majority (table A1) or player 3 (table A2).

	$\{1,2\}:\left(\frac{1}{2},\frac{1}{2}\right)$
B $PU\left(\{1,2,3\},u_{\{1,2\}},\{1\},\pi,t\right)$ $=\left(\pi_1,t-\frac{\pi_1}{2},-\frac{\pi_1}{2}\right)$	S: $\pi_1>\begin{cases}PU_1\left(N\backslash E^{\emptyset},v\vert_{N\backslash E^{\emptyset}},\emptyset,\cdot\right), & 1\notin E^{\emptyset}\\ r_1, & 1\in E^{\emptyset}\end{cases}$ E: $\pi_1\geq r_1, t-\frac{\pi_1}{2}\geq r_2$ $-\frac{\pi_1}{2}\geq r_3$ M: $\pi_1\geq\frac{1}{2}, t-\frac{\pi_1}{2}\geq\frac{1}{2}$ P: $\pi_1\geq\frac{1}{2}$
C $PU\left(\{1,2,3\},u_{\{1,2\}},\{2\},\pi,t\right)$ $=\left(t-\frac{\pi_2}{2},\pi_2,-\frac{\pi_2}{2}\right)$	S: $t-\frac{\pi_2}{2}>\begin{cases}PU_1\left(N\backslash E^{\emptyset},v\vert_{N\backslash E^{\emptyset}},\emptyset,\cdot\right), & 1\notin E^{\emptyset}\\ r_1, & 1\in E^{\emptyset}\end{cases}$ E: $t-\frac{\pi_2}{2}\geq r_1,\pi_2\geq r_2$ $-\frac{\pi_2}{2}\geq r_3$ M: $t-\frac{\pi_2}{2}\geq\frac{1}{2},\pi_2\geq\frac{1}{2}$ P: $\pi_2\geq\frac{1}{2}$
D $PU\left(\{1,2,3\},u_{\{1,2\}},\{3\},\pi,t\right)$ $=\left(\frac{1}{2}-\frac{\pi_3}{2},\frac{1}{2}-\frac{\pi_3}{2},\pi_3\right)$	S: $\frac{1}{2}-\frac{\pi_3}{2}>\begin{cases}PU_1\left(N\backslash E^{\emptyset},v\vert_{N\backslash E^{\emptyset}},\emptyset,\cdot\right), & 1\notin E^{\emptyset}\\ r_1, & 1\in E^{\emptyset}\end{cases}$ E: $\frac{1}{2}-\frac{\pi_3}{2}\geq r_1,\frac{1}{2}-\frac{\pi_3}{2}\geq r_2$ $\pi_3\geq r_3$ M: $\frac{1}{2}-\frac{\pi_3}{2}\geq\frac{1}{2},\frac{1}{2}-\frac{\pi_3}{2}\geq\frac{1}{2}$ P: $\pi_3\geq 0$
E $PU\left(\{1,2,3\},u_{\{1,2\}},\{1,2\},\pi,t\right)$ $=\left(\pi_1,\pi_2,t^2-\pi_1-\pi_2\right)$	S: $\pi_1>\begin{cases}PU_1\left(N\backslash E^{\emptyset},v\vert_{N\backslash E^{\emptyset}},\emptyset,\cdot\right), & 1\notin E^{\emptyset}\\ r_1, & 1\in E^{\emptyset}\end{cases}$ E: $\pi_1\geq r_1,\pi_2\geq r_2$ $t^2-\pi_1-\pi_2\geq r_3$ M: $\pi_1\geq\frac{1}{2},\pi_2\geq\frac{1}{2}$ P: $\pi_1\geq\frac{1}{2},\pi_2\geq\frac{1}{2}$
F $PU\left(\{1,2,3\},u_{\{1,2\}},\{1,3\},\pi,t\right)$ $=(\pi_1,t-\pi_1-\pi_3,\pi_3)$	S: $\pi_1>\begin{cases}PU_1\left(N\backslash E^{\emptyset},v\vert_{N\backslash E^{\emptyset}},\emptyset,\cdot\right), & 1\notin E^{\emptyset}\\ r_1, & 1\in E^{\emptyset}\end{cases}$ E: $\pi_1\geq r_1,t-\pi_1-\pi_3\geq r_2$ $\pi_3\geq r_3$ M: $\pi_1\geq\frac{1}{2},t-\pi_1-\pi_3\geq\frac{1}{2}$ P: $\pi_1\geq\frac{1}{2},\pi_3\geq 0$
G $PU\left(\{1,2,3\},u_{\{1,2\}},\{2,3\},\pi,t\right)$ $=(t-\pi_2-\pi_3,\pi_2,\pi_3)$	S: $t-\pi_2-\pi_3>\begin{cases}PU_1\left(N\backslash E^{\emptyset},v\vert_{N\backslash E^{\emptyset}},\emptyset,\cdot\right), & 1\notin E^{\emptyset}\\ r_1, & 1\in E^{\emptyset}\end{cases}$ E: $t-\pi_2-\pi_3\geq r_1,\pi_2\geq r_2$ $\pi_3\geq r_3$ M: $t-\pi_2-\pi_3\geq\frac{1}{2},\pi_2\geq\frac{1}{2}$ P: $\pi_2\geq\frac{1}{2},\pi_3\geq 0$

Table A1: Constraints for proposer 1 enlisting player 2 and any foreign reservation payoffs

	$\{1,3\}:\left(\frac{1}{2},0\right)$
B $PU\left(\{1,2,3\},u_{\{1,2\}},\{1\},\pi,t\right)$ $=\left(\pi_1,t-\frac{\pi_1}{2},-\frac{\pi_1}{2}\right)$	S: $\pi_1 > \begin{cases} PU_1\left(N\backslash E^{\emptyset}, v\vert_{N\backslash E^{\emptyset}},\emptyset,\cdot\right), & 1\notin E^{\emptyset} \\ r_1, & 1\in E^{\emptyset}\end{cases}$ E: $\pi_1 \geq r_1, t-\frac{\pi_1}{2}\geq r_2$ $-\frac{\pi_1}{2}\geq r_3$ M: $\pi_1\geq\frac{1}{2},-\frac{\pi_1}{2}\geq 0$ P: $\pi_1\geq\frac{1}{2}$
C $PU\left(\{1,2,3\},u_{\{1,2\}},\{2\},\pi,t\right)$ $=\left(t-\frac{\pi_2}{2},\pi_2,-\frac{\pi_2}{2}\right)$	S: $t-\frac{\pi_2}{2} > \begin{cases} PU_1\left(N\backslash E^{\emptyset}, v\vert_{N\backslash E^{\emptyset}},\emptyset,\cdot\right), & 1\notin E^{\emptyset} \\ r_1, & 1\in E^{\emptyset}\end{cases}$ E: $t-\frac{\pi_2}{2}\geq r_1,\pi_2\geq r_2$ $-\frac{\pi_2}{2}\geq r_3$ M: $t-\frac{\pi_2}{2}\geq\frac{1}{2},-\frac{\pi_2}{2}\geq 0$ P: $\pi_2\geq\frac{1}{2}$
D $PU\left(\{1,2,3\},u_{\{1,2\}},\{3\},\pi,t\right)$ $=\left(\frac{1}{2}-\frac{\pi_3}{2},\frac{1}{2}-\frac{\pi_3}{2},\pi_3\right)$	S: $\frac{1}{2}-\frac{\pi_3}{2} > \begin{cases} PU_1\left(N\backslash E^{\emptyset}, v\vert_{N\backslash E^{\emptyset}},\emptyset,\cdot\right), & 1\notin E^{\emptyset} \\ r_1, & 1\in E^{\emptyset}\end{cases}$ E: $\frac{1}{2}-\frac{\pi_3}{2}\geq r_1,\frac{1}{2}-\frac{\pi_3}{2}\geq r_2$ $\pi_3\geq r_3$ M: $\frac{1}{2}-\frac{\pi_3}{2}\geq\frac{1}{2},\pi_3\geq 0$ P: $\pi_3\geq 0$
E $PU\left(\{1,2,3\},u_{\{1,2\}},\{1,2\},\pi,t\right)$ $=\left(\pi_1,\pi_2,t^2-\pi_1-\pi_2\right)$	S: $\pi_1 > \begin{cases} PU_1\left(N\backslash E^{\emptyset}, v\vert_{N\backslash E^{\emptyset}},\emptyset,\cdot\right), & 1\notin E^{\emptyset} \\ r_1, & 1\in E^{\emptyset}\end{cases}$ E: $\pi_1\geq r_1,\pi_2\geq r_2$ $t^2-\pi_1-\pi_2\geq r_3$ M: $\pi_1\geq\frac{1}{2},t^2-\pi_1-\pi_2\geq 0$ P: $\pi_1\geq\frac{1}{2},\pi_2\geq\frac{1}{2}$
F $PU\left(\{1,2,3\},u_{\{1,2\}},\{1,3\},\pi,t\right)$ $=(\pi_1,t-\pi_1-\pi_3,\pi_3)$	S: $\pi_1 > \begin{cases} PU_1\left(N\backslash E^{\emptyset}, v\vert_{N\backslash E^{\emptyset}},\emptyset,\cdot\right), & 1\notin E^{\emptyset} \\ r_1, & 1\in E^{\emptyset}\end{cases}$ E: $\pi_1\geq r_1,t-\pi_1-\pi_3\geq r_2$ $\pi_3\geq r_3$ M: $\pi_1\geq\frac{1}{2},\pi_3\geq 0$ P: $\pi_1\geq\frac{1}{2},\pi_3\geq 0$
G $PU\left(\{1,2,3\},u_{\{1,2\}},\{2,3\},\pi,t\right)$ $=(t-\pi_2-\pi_3,\pi_2,\pi_3)$	S: $t-\pi_2-\pi_3 > \begin{cases} PU_1\left(N\backslash E^{\emptyset}, v\vert_{N\backslash E^{\emptyset}},\emptyset,\cdot\right), & 1\notin E^{\emptyset} \\ r_1, & 1\in E^{\emptyset}\end{cases}$ E: $t-\pi_2-\pi_3\geq r_1,\pi_2\geq r_2$ $\pi_3\geq r_3$ M: $t-\pi_2-\pi_3\geq\frac{1}{2},\pi_3\geq 0$ P: $\pi_2\geq\frac{1}{2},\pi_3\geq 0$

Table A2: Constraints for proposer 1 enlisting player 3 and any foreign reservation payoffs

As a next step, we consider the special case $r_1 = \frac{1}{6}, r_2 = \frac{7}{12}$, and $r_3 = -\frac{1}{3}$. If the proposer 1 suggests the status quo, $E^{\emptyset} = \{1,2\}$ is the only emigration set fulfilling constraint $\mathrm{E}^{\emptyset}$. If the proposer does not want to emigrate, he aims for $E^* \neq \emptyset$. This follows from the fact that an emigration by player 2 induces player 1 to emigrate himself while player 3 (with a reservation payoff below his Shapley value) offers himself for exploitation. Thus, we are justified in working with tables A1 and A2. We use the specific foreign reservation payoffs and find contradictions or best civil-service vectors:

	$\{1,2\} : \left(\frac{1}{2}, \frac{1}{2}\right)$
B $PU\left(\{1,2,3\}, u_{\{1,2\}}, \{1\}, \pi, t\right)$ $= \left(\pi_1, t - \frac{\pi_1}{2}, -\frac{\pi_1}{2}\right)$	$\frac{1}{2} \leq \pi_1 \leq 2t - \frac{7}{6}$ $\pi_1 \leq \frac{2}{3}$ which implies $t \geq \frac{5}{6}$ and Sol: $\pi_1^* = \min\left(2t - \frac{7}{6}, \frac{2}{3}\right)$ $= \begin{cases} \frac{2}{3}, & \frac{11}{12} \leq t \leq 1 \\ 2t - \frac{7}{6}, & \frac{5}{6} \leq t < \frac{11}{12} \approx 0.916\,67 \end{cases}$
C $PU\left(\{1,2,3\}, u_{\{1,2\}}, \{2\}, \pi, t\right)$ $= \left(t - \frac{\pi_2}{2}, \pi_2, -\frac{\pi_2}{2}\right)$	$\frac{1}{2} \leq t - \frac{\pi_2}{2}$ $\frac{7}{12} \leq \pi_2 \leq \frac{2}{3}$ which implies $t \geq \frac{19}{24} \approx 0.79$ Sol: $\pi_2^* = \frac{7}{12},\ t - \frac{\pi_2^*}{2} = t - \frac{7}{24}$
D $PU\left(\{1,2,3\}, u_{\{1,2\}}, \{3\}, \pi, t\right)$ $= \left(\frac{1}{2} - \frac{\pi_3}{2}, \frac{1}{2} - \frac{\pi_3}{2}, \pi_3\right)$	contradiction between E and P
E $PU\left(\{1,2,3\}, u_{\{1,2\}}, \{1,2\}, \pi, t\right)$ $= \left(\pi_1, \pi_2, t^2 - \pi_1 - \pi_2\right)$	$\pi_1 \geq \frac{1}{2}$ $\pi_2 \geq \frac{7}{12}$ $\pi_1 \leq t^2 + \frac{1}{3} - \pi_2$ $\leq t^2 - \frac{1}{4}$ which implies $t^2 \geq \frac{1}{2} - \frac{1}{3} + \frac{7}{12} = \frac{3}{4}$ and hence $t \geq \sqrt{\frac{3}{4}} \approx 0.87$ Sol: $\pi_2^* = \frac{7}{12}, \pi_1^* = t^2 - \frac{1}{4}$
F $PU\left(\{1,2,3\}, u_{\{1,2\}}, \{1,3\}, \pi, t\right)$ $= (\pi_1, t - \pi_1 - \pi_3, \pi_3)$	contradiction between E and P: $\frac{7}{12} \leq t - \pi_1 - \pi_3$ $\leq t - \frac{1}{2} - 0$ $\leq 1 - \frac{1}{2} = \frac{1}{2}$
G $PU\left(\{1,2,3\}, u_{\{1,2\}}, \{2,3\}, \pi, t\right)$ $= (t - \pi_2 - \pi_3, \pi_2, \pi_3)$	contradiction between E, M, and P: $\frac{1}{2} \leq t - \pi_2 - \pi_3$ $\leq t - \frac{7}{12} - 0$ $\leq 1 - \frac{7}{12} = \frac{5}{12}$

Table A3: Contradictions and optimality for each proposal involving $C^* \neq \emptyset$ (enlisting player 2)

	$\{1,3\} : \left(\frac{1}{2}, 0\right)$
B $PU\left(\{1,2,3\}, u_{\{1,2\}}, \{1\}, \pi, t\right)$ $= \left(\pi_1, t - \frac{\pi_1}{2}, -\frac{\pi_1}{2}\right)$	contradiction in M
C $PU\left(\{1,2,3\}, u_{\{1,2\}}, \{2\}, \pi, t\right)$ $= \left(t - \frac{\pi_2}{2}, \pi_2, -\frac{\pi_2}{2}\right)$	contradiction in M and P
D $PU\left(\{1,2,3\}, u_{\{1,2\}}, \{3\}, \pi, t\right)$ $= \left(\frac{1}{2} - \frac{\pi_3}{2}, \frac{1}{2} - \frac{\pi_3}{2}, \pi_3\right)$	contradiction between E and P
E $PU\left(\{1,2,3\}, u_{\{1,2\}}, \{1,2\}, \pi, t\right)$ $= \left(\pi_1, \pi_2, t^2 - \pi_1 - \pi_2\right)$	contradiction between E and M: $0 \leq t^2 - \pi_1 - \pi_2$ $\leq t^2 - \frac{1}{2} - \frac{7}{12}$ $\leq 1 - \frac{13}{12} < 0$
F $PU\left(\{1,2,3\}, u_{\{1,2\}}, \{1,3\}, \pi, t\right)$ $= (\pi_1, t - \pi_1 - \pi_3, \pi_3)$	contradiction between E and M: $\frac{7}{12} \leq t - \pi_1 - \pi_3$ $\leq t - \frac{1}{2} - 0$ $\leq 1 - \frac{1}{2} = \frac{1}{2}$
G $PU\left(\{1,2,3\}, u_{\{1,2\}}, \{2,3\}, \pi, t\right)$ $= (t - \pi_2 - \pi_3, \pi_2, \pi_3)$	contradiction between E and M: $\frac{1}{2} \leq t - \pi_2 - \pi_3$ $\leq t - \frac{7}{12} - 0$ $\leq 1 - \frac{7}{12} = \frac{5}{12}$

Table A4: Contradictions and optimality for each proposal involving $C^* \neq \emptyset$ (enlisting player 3)

Thus, we have three candidates. The lemma now follows from

$$0 < 1 - \frac{1}{6}\sqrt{3} < \frac{19}{24} < \frac{5}{6} < \sqrt{\frac{3}{4}} < \frac{7}{8} < \frac{11}{12} < \frac{1}{2} + \frac{1}{12}\sqrt{5}\sqrt{6} < \sqrt{\frac{11}{12}} < \frac{23}{24} < 1$$

and the straightforward comparisons

$$\begin{aligned}
&\text{B versus C:} && \frac{2}{3} < t - \frac{7}{24} \Leftrightarrow t > \frac{23}{24}, \\
&\text{B versus C:} && 2t - \frac{7}{6} < t - \frac{7}{24} \Leftrightarrow t < \frac{7}{8} \\
&\text{B versus E:} && \frac{2}{3} < t^2 - \frac{1}{4} \Leftrightarrow t > \sqrt{\frac{11}{12}} \\
&\text{B versus E:} && 2t - \frac{7}{6} < t^2 - \frac{1}{4} \Leftrightarrow t < 1 - \frac{1}{6}\sqrt{3} \\
&\text{C versus E:} && t - \frac{7}{24} < t^2 - \frac{1}{4} \Leftrightarrow t > \frac{1}{2} + \frac{1}{12}\sqrt{5}\sqrt{6}
\end{aligned}$$

CHAPTER XV

A real-estate model

1. Introduction

The aim of cooperative game theory is to suggest and defend payoffs for the players that depend on a coalition function (characteristic function) describing the economic, social, or political situation. In this sense, the players' payoffs are determined endogenously. However, there are situations in real life where some players' payoffs are exogenous. For example, in many countries lawyers or real-estate agents obtain a regulated fee or a regulated percentage of the business involved. Similarly, civil servants who participate in the production of economic goods in different ways are also paid according to official schedules. As a final example, consider the cost allocation problem for a firm's important input such as computing or other facilities. If the firm sells some user rights to outsiders, the cost allocation problem for the firm's units involves exogenous (negative) payments to the outsiders.

As the title suggests, we aim for a value that incorporates the idea of exogenous payments while staying close to the Shapley value. Thus, we are asking the question of how much the business partners or the private-sector agents obtain after paying off the lawyers or civil servants, respectively.

Our characterizations use three important axioms. First of all, we demand that the payoffs under the new value actually give the predetermined payoff to the exogenous players, i.e., the realtor's fee to the realtor and the civil-service payments to the civil servants (axiom X). Second, our value has the following property: If the exogenous payments change and if the coalition function changes by the same amount (to be made precise later), the endogenous players' payoffs do not change. Third, we impose a consistency axiom (axiom C): If the exogenous payments happen to be equal to the payoff determined endogenously (i.e., according to the Shapley value), then the endogenous agents also obtain their Shapley values.

Section 2 introduces and axiomatizes our exogenous-payments value. In that section, we also relate the value to the core and present an application. We show how to incorporate weights for the endogenous players in section 3, again with an application. The final section concludes the paper.

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2. XP values

2.1. XP games. We now introduce the set of exogenous players $X \subseteq N$ (the civil servants, if you like) and the payments $\pi \in \mathbb{R}^{|X|}$ they receive. In order to make the problem interesting, we demand that X be a strict subset of N. The other players are called endogenous players (the private sector) and denoted by $D := N \backslash X$.

Thus, we have the following definition of XP games (where XP stands for eXogenous Payments):

Definition XV.1. *XP games are tuples*

$$(N, v, X, \pi)$$

where

- (N, v) *is a TU game,*
- X *is a strict subset of* N, *and*
- $\pi \in \mathbb{R}^{|X|}$ *is a vector specifying a payoff for every member of* X.

2.2. Axioms. An XP value φ assigns a payoff vector to every XP game, $\varphi(N, v, X, \pi) \in \mathbb{R}^n$. Of course, our value has to fulfill the following axiom:

X (exogenous payments): For all $i \in X$, we have $\varphi_i(N, v, X, \pi) = \pi_i$.

Axiom X expresses the idea that exogenous players $i \in X$ indeed obtain π_i. Given that axiom, most other axioms are restricted to players from D for obvious reasons. Consider now the following five axioms:

E (efficiency): We have $\varphi_N(N, v, X, \pi) = v(N)$.

S (symmetry): For all symmetric players $i, j \in D$, $\varphi_i(N, v, X, \pi) = \varphi_j(N, v, X, \pi)$.

N-$\emptyset$ (null player for $X = \emptyset$): If $i \in N$ is a null player, then $\varphi_i(N, v, \emptyset, \pi) = 0$.

Axioms E, S and N-$\emptyset$ are obvious requirements. Axiom N-$\emptyset$ demands that a null player obtains the payoff zero if there are no exogenous players in the game. If, however, exogenous players exist, null players cannot, in general, have zero payoffs. For example, in the 0-game v (defined by $v(K) = 0$ for all $K \subseteq N$), all players are null players and the endogenous players have to pay π_X for reasons of efficiency. Thus, a null-player axiom is not a reasonable requirement in case of $X \neq \emptyset$. Also, a null-player-out axiom (see Derks & Haller 1999) cannot hold for the value we are to define. If a null player from D is excluded from the game, the other endogenous players have to divide π_X between themselves.

M (marginalism): Assume two coalition functions v and z from $\mathbb{V}_N$. Let i be a player from D obeying

$$v(S \cup \{i\}) - v(S) = z(S \cup \{i\}) - z(S)$$

for all $S \subseteq N \backslash \{i\}$. Then

$$\varphi_i(N, v, X, \pi) = \varphi_i(N, z, X, \pi).$$

BF (Brink fairness): Let i and j be players from D that are symmetric in (N, z). Then

$$\varphi_i(N, v + z, X, \pi) - \varphi_i(N, v, X, \pi) = \varphi_j(N, v + z, X, \pi) - \varphi_j(N, v, X, \pi).$$

Axiom M states that player i from D is affected by a coalition function only insofar as his marginal contributions are concerned. This holds for the Shapley value but not for our value. The reason is that the players from D pay π to the players from X but enjoy the contributions made by these exogenous players by efficiency. In contrast to axiom M, our value fulfills axiom BF. This axiom says that two players are equally affected by adding a coalition function z (to some given coalition function v) if they are symmetric in (N, z).

A (additivity): For any coalition functions $v', v'' \in \mathbb{V}_N$, any payments π', $\pi'' \in \mathbb{R}^{|X|}$ and any player i from N, we obtain

$$\varphi_i\left(N, v' + v'', X, \pi' + \pi''\right) = \varphi_i\left(N, v', X, \pi'\right) + \varphi_i\left(N, v'', X, \pi''\right).$$

Note that axiom A concerns all the players from $X \cup D$ and refers to payments as well as coalition functions. Thus, if a player $i \in X$ is involved in two games, he is to obtain the sum of what he would get in each of them.

Next, we present the shifting axiom. It says that a player from D does not gain or suffer if a change in π_X is balanced by a corresponding change of v by π_X. In a sense, both π_X and v (see eq. (**??**)), are shifted in the same direction. For example, if a lawyer or a civil servant is responsible for an increase (or a decrease) of the social product and if his renumeration is changed by the very same amount, the endogenous players are not affected.

SH (shifting): For all $i \in D$, we have

$$\varphi_i(N, v + \pi_X, X, \pi) = \varphi_i\left(N, v + \pi'_X, X, \pi'\right)$$

for all $\pi, \pi' \in \mathbb{R}^{|X|}$.

It is not difficult to show that axioms X, S, E, and A imply axiom SH. The final axiom is a very important one:

C (consistency): For any player $i \in D$,

$$\varphi_i\left(N, v, X, (\varphi_x(N, v, \emptyset, \pi))_{x \in X}\right) = \varphi_i(N, v, \emptyset, \pi).$$

If the players in X obtain what they would obtain without any exogenous players, the players in D also obtain what they should get without exogenous players. Differently put, if the players in X (happen to) obtain the value dictated by the axioms for games without exogenous players, so do the other players. Consistency axioms have been surveyed by Thomson (1990) and Driessen (1991).

2.3. Axiomatization. In order to compare our value with the Shapley value, we note the following theorem:

THEOREM XV.1. *Assuming $X = \emptyset$ (in which case N-$\emptyset$ and N are equivalent) and ignoring π in that case, the Shapley value is characterized by the following sets of axioms for solution φ:*

- *E, S, N, and A (Shapley (1953a))*
- *E, S, and M (Young (1985))*
- *E, N, and BF (van den Brink (2001))*

The Shapley value with exogenous payments is denoted by $Sh^{X,\pi}$ and given by

$$Sh_i^{X,\pi}(N,v) = \begin{cases} \pi_i, & i \in X \\ Sh_i(N,v) + \frac{1}{|D|}(Sh_X(N,v) - \pi_X), & i \in D \end{cases}$$

We can consider the Shapley value with exogenous payments as an XP value for XP games (N, v, X, π) or, alternatively, as a solution for TU games where X and π enter as parameters.

As the above discussion makes clear, we can look for sets of axioms including the axioms E, S, N-$\emptyset$, and A or including E, N-$\emptyset$, and BF. We prepare our two characterizations with a lemma:

LEMMA XV.1. *Assuming axiom C and any of the two following axiom sets*

- *E, S, N-$\emptyset$, and A or*
- *E, N-$\emptyset$, and BF*

we obtain

$$Sh_i(N,v) = \varphi_i\left(N, v, X, (Sh_x(N,v))_{x\in X}\right)$$

for all players $i \in D$.

Proof. Either one of the set of axioms (E, S, N-$\emptyset$, and A on the one hand or E, N-$\emptyset$, and BF on the other hand) obviously imply

$$\varphi_i(N, v, \emptyset, \pi) = Sh_i(N,v). \tag{XV.1}$$

We then find

$$\begin{aligned} Sh_i(N,v) &= \varphi_i(N,v,\emptyset,\pi) \text{ (eq. (XV.1))} \\ &= \varphi_i\left(N,v,X,(\varphi_x(N,v,\emptyset,\pi))_{x\in X}\right) \text{ (axiom C)} \\ &= \varphi_i\left(N,v,X,(Sh_x(N,v))_{x\in X}\right) \text{ (eq. (XV.1))} \end{aligned}$$

□

THEOREM XV.2. *The Shapley value with exogenous payments is characterized by the axioms X, E, S, N-$\emptyset$, A, and C.*

Proof. It is not difficult to show that $Sh^{X,\pi}$ fulfills all the axioms mentioned in the theorem. Let φ be an XP value. For $i \in X$, axiom X guarantees $\varphi_i(N, v, X, \pi) = \pi_i$. For $i \in D$, we obtain the desired result by

$$\begin{aligned} & \varphi_i(N, v, X, \pi) \\ = \; & \varphi_i\left(N, v, X, (Sh_x(N, v))_{x\in X}\right) \\ & +\varphi_i\left(N, 0, X, (\pi_x)_{x\in X} - (Sh_x(N, v))_{x\in X}\right) \text{ (axiom A)} \\ = \; & Sh_i(N, v) + \varphi_i\left(N, 0, X, (\pi_x)_{x\in X} - (Sh_x(N, v))_{x\in X}\right) \text{ (lemma XV.1)} \\ = \; & Sh_i(N, v) + \frac{1}{|D|}(Sh_X(N, v) - \pi_X) \text{ (axioms E, S)} \end{aligned}$$

□

The axioms are independent. For the necessity of axiom C, see the conclusions.

THEOREM XV.3. *The Shapley value with exogenous payments is characterized by the axioms X, E, BF, N-$\emptyset$, SH, and C.*

Proof. $Sh^{X,\pi}$ also fulfills the axioms BF and SH. Consider the coalition function $z := \pi_X - Sh_X(N, v)$. Then any two players i and j from D are symmetric in (N, z) and Brink fairness implies

$$\varphi_i(N, v+z, X, \pi) - \varphi_i(N, v, X, \pi) = \varphi_j(N, v+z, X, \pi) - \varphi_j(N, v, X, \pi).$$

Fix $i \in D$ and sum this equation for all $j \in D$. Using axioms X and E and hence $\varphi_D(N, v, X, \pi) = v(N) - \pi_X$, we find

$$\varphi_i(N, v, X, \pi) = \varphi_i(N, v+z, X, \pi) + \frac{1}{|D|}(Sh_X(N, v) - \pi_X).$$

The equations

$$\begin{aligned} Sh_i(N, v) &= \varphi_i\left(N, v, X, (Sh_x(N, v))_{x\in X}\right) \text{ (lemma XV.1)} \\ &= \varphi_i\left(N, v - Sh_X(N, v) + \pi_X, X, (\pi_x)_{x\in X}\right) \text{ (axiom SH)} \\ &= \varphi_i(N, v+z, X, \pi) \end{aligned}$$

provide the final bit of our proof.□

The verification of independence is easy with the exception of the shifting axiom.

LEMMA XV.2. *There is an XP value different from the Shapley value with exogenous payments that satisfies the axioms X, E, BF, N-$\emptyset$, and C.*

Proof. The XP value ρ defined by

$$\begin{aligned}&\rho_i(N,v)\\&=\begin{cases}Sh_i^{X,\pi}(N,v)+1, & N\supseteq\{1,2,3\},1,2\in D,3\in X,Sh_3(N,v)\neq\pi_3=0,i=1\\ Sh_i^{X,\pi}(N,v)-1, & N\supseteq\{1,2,3\},1,2\in D,3\in X,Sh_3(N,v)\neq\pi_3=0,i=2\\ Sh_i^{X,\pi}(N,v), & \text{otherwise}\end{cases}\end{aligned}$$

obeys all the axioms mentioned in the lemma.□

2.4. The Shapley value with exogenous payments and the core. The core of a game (N,v) is given by

$$\{x\in\mathbb{R}^n : x_N=v(N) \text{ and } x_K\geq v(K) \text{ for all } K\subseteq N\}.$$

The question arises: under what circumstances does $Sh^{X,\pi}(N,v)$ lie in the core? An immediate requirement is $\pi_K\geq v(K)$ for all $K\subseteq X$. According to a familiar theorem, the Shapley value of convex games is in the core. The following implication is rather immediate:

Corollary XV.1. *Let v be a convex game. If $\pi_X\leq Sh_X(N,v)$ and $\pi_K\geq v(K)$ for all $K\subseteq X$, $Sh^{X,\pi}(N,v)$ lies in the core.*

The inequality $\pi_X\leq Sh_X(N,v)$ cannot, in general, be weakened; just consider inessential games.

2.5. Application: Basic income. In many countries, the introduction of some basic income is vividly discussed. Under such a system of social security and taxation, every agent (often restricted to citizens) – rich or poor – obtains a basic payoff (basic income). In general, basic payoffs may well differ from person to person according to handicaps, nationality or other differences. Of course, the basic income for everybody has to be paid for by taxes of various sorts. The Shapley value for exogenous payments can be used for a simple model.

Following a suggestion made by André Casajus in private communication, we duplicate a TU game (N,v) (which stands for the economy) in the following manner.

- On the basis of player set $N=\{1,...,n\}$, we define a set $N':=\{1',...,n'\}$ with $|N|=|N'|$ and a player set $\hat{N}:=N\cup N'$.
- We define a TU game $\left(\hat{N},\hat{v}\right)$ by $\hat{v}(K)=v(K\cap N)$. Thus, every player from N' is a null player in $\left(\hat{N},\hat{v}\right)$ and we have $Sh_i(N,v)=Sh_i\left(\hat{N},\hat{v}\right)$ for all players $i\in N$.
- Every player $i'\in N'$ is an exogenous player and obtains the payoff (the basic income) $\pi_{i'}$.

Obviously, the dash-player is just a copy of a player from N invented for the purpose of collecting the basic income. We find the payoffs

$$Sh_i^{N',\pi}\left(\hat{N},\hat{v}\right)=\begin{cases}\pi_i, & i\in N'\\ Sh_i(N,v)-\frac{\pi_{N'}}{|N|}, & i\in N\end{cases}$$

Thus, the overall payoff for a player $i\in N$ and his clone $i'\in N$ is

$$\underbrace{Sh_i(N,v)}_{\text{market income}}+\underbrace{\pi_{i'}}_{\text{basic income}}-\underbrace{\frac{\pi_{N'}}{|N|}}_{\text{tax}}.$$

Therefore, the introduction of a basic-income system makes an agent better off iff his basic payoff is greater than the average basic payoff.

3. Weighted XP values

3.1. Axiomatization. Our value can be extended to incorporate weights for the players from D. The weights determine the burden sharing of the D-players with respect to the payments obtained by the X-players. In contrast, the weights in the Kalai-Samet weighted value affect all players' payoffs, depending on the hierarchy level (see Kalai & Samet 1987b).

A weighted XP game is a tuple (N,v,X,π,w) where (N,v,X,π) is an XP game and $w=(w_i)_{i\in D}$ a tuple of strictly positive numbers. A weighted XP value φ assigns a payoff vector to every weighted XP game, $\varphi(N,v,X,\pi,w)\in\mathbb{R}^n$. The weighted Shapley value with exogenous payments (no relation to the weighted Shapley values!) is given by

$$Sh_i^{X,\pi,w}(N,v)=\begin{cases}\pi_i, & i\in X\\ Sh_i(N,v)+\frac{w_i}{\sum_{d\in D}w_d}\left(Sh_X(N,v)-\pi_X\right), & i\in D\end{cases}$$

It can be axiomatized on the basis of (obvious variations of) the axioms X, E, N-$\emptyset$, A, and C from the first axiom set.

X (exogenous payments): For all $i\in X$, we have $\varphi_i(N,v,X,\pi,w)=\pi_i$.

E (efficiency): We have $\varphi_N(N,v,X,\pi,w)=v(N)$.

N-$\emptyset$ (null player for $X=\emptyset$): If $i\in N$ is a null player, then $\varphi_i(N,v,\emptyset,\pi,w)=0$.

A (additivity): For any coalition functions $v',v''\in\mathbb{V}_N$, any payments π', $\pi''\in\mathbb{R}^{|X|}$ and any player i from N, we obtain

$$\varphi_i\left(N,v'+v'',X,\pi'+\pi'',w\right)=\varphi_i\left(N,v',X,\pi',w\right)+\varphi_i\left(N,v'',X,\pi'',w\right).$$

C (consistency): For any player $i\in D$,

$$\varphi_i\left(N,v,X,\left(\varphi_x(N,v,\emptyset,\pi,w)\right)_{x\in X},w\right)=\varphi_i(N,v,\emptyset,\pi,w).$$

The symmetry axiom has to take the weights into account:

S (symmetry): For all symmetric players $i, j \in D$ obeying $w_i = w_j$,

$$\varphi_i(N, v, X, \pi, w) = \varphi_j(N, v, X, \pi, w).$$

Additionally, we need two more axioms:

IR (irrelevance): For all $i \in D$ and all $\pi, \pi' \in \mathbb{R}^{|X|}, w, w' \in \mathbb{R}^{|D|}$, we have

$$\varphi_i(N, v, \emptyset, \pi, w) = \varphi_i\left(N, v, \emptyset, \pi', w'\right).$$

W (weighing): For all players $i, j \in D$,

$$w_i \varphi_j(N, 0, X, \pi, w) = w_j \varphi_i(N, 0, X, \pi, w).$$

Axiom IR states that the exogenous payments and the weights are not relevant for a player i if there are no exogenous players. Axiom W ensures that the ratio of weights is equal to the ratio of payoffs in a zero game. It is similar to the "weighting of treatments" axiom by Haeringer (1999).

THEOREM XV.4. *The weighted Shapley value with exogenous payments is characterized by the axioms (given in **this** section) X, E, S, N-$\emptyset$, A, C, IR, and W.*

Proof. $Sh^{X,\pi,w}$ fulfills all the above axioms. Turning to uniqueness, axiom X ensures $\varphi_i(N, v, X, \pi) = \pi_i$ for all $i \in X$. Note that IR and S imply weight-independent symmetry in case of $X = \emptyset$. Assume two symmetric players $i, j \in D$ that do not (necessarily) obey $w_i = w_j$. We then have

$$\begin{aligned} \varphi_i(N, v, \emptyset, \pi, w) &= \varphi_i(N, v, \emptyset, \pi, (1, ..., 1)) \text{ (axiom IR)} \\ &= \varphi_j(N, v, \emptyset, \pi, (1, ..., 1)) \text{ (axiom S)} \\ &= \varphi_j(N, v, \emptyset, \pi, w) \text{ (axiom IR)} \end{aligned}$$

We now closely follow the proof of lemma XV.1 to show that axioms E, S, IR, N-$\emptyset$, A and C imply

$$Sh_i(N, v) = \varphi_i\left(N, v, X, (Sh_x(N, v))_{x \in X}, w\right).$$

Proceeding as in the proof of theorem XV.2, we easily find

$$\varphi_i(N, v, X, \pi, w) = Sh_i(N, v) + \varphi_i\left(N, 0, X, (\pi_x)_{x \in X} - (Sh_x(N, v))_{x \in X}, w\right).$$

We now apply axiom W:

$$\begin{aligned} &\varphi_i(N, v, X, \pi, w) \\ = \; & Sh_i(N, v) + \frac{1}{\sum_{d \in D} w_d} \sum_{d \in D} w_d \varphi_i\left(N, 0, X, (\pi_x)_{x \in X} - (Sh_x(N, v))_{x \in X}, w\right) \\ = \; & Sh_i(N, v) + \frac{1}{\sum_{d \in D} w_d} \sum_{d \in D} w_i \varphi_d\left(N, 0, X, (\pi_x)_{x \in X} - (Sh_x(N, v))_{x \in X}, w\right) \text{ (axiom W)} \\ = \; & Sh_i(N, v) + \frac{w_i}{\sum_{d \in D} w_d}\left(-\pi_X + Sh_X(N, v)\right) \text{ (axiom E)} \end{aligned}$$

□

One might wonder whether axiom IR is not implied by the other axioms. This is not true as the following lemma shows.

LEMMA XV.3. *There is a weighted XP value different from the weighted Shapley value with exogenous payments that satisfies the axioms X, E, S, N-∅, A, C, and W.*

Proof. We define the ρ-value in three steps:
I. Let T be any nonempty subset of N and u_T a unanimity game. Then, for any $\alpha \in \mathbb{R}$, the weighted XP value ρ for $X = \emptyset$ is defined by

$$\begin{aligned} \rho_i\left(N, \alpha u_T, \emptyset, \pi, w\right) &= \begin{cases} \frac{\alpha}{|M_T|}, & i \in T, w_i = W_T \\ 0, & i \notin T \text{ or } w_i < W_T \end{cases}, \\ \text{where } W_T &: = \max_{i \in T} w_i \text{ and } M_T := \{i \in T : w_i = W\} \end{aligned}$$

II. As any game v can be written as

$$v = \sum_{\substack{T \neq \emptyset, \\ T \subseteq N}} \lambda_T u_T$$

for suitably chosen λ_T, we obtain

$$\rho_i\left(N, v, \emptyset, \pi, w\right) = \sum_{\substack{T \neq \emptyset, \\ T \subseteq N}} \rho_i\left(N, \lambda_T u_T, \emptyset, \pi, w\right)$$

III. Finally, we define

$$\begin{aligned} &\rho_i\left(N, v, X, \pi, w\right) \\ &\begin{cases} \pi_i, & i \in X \\ \rho_i\left(N, v, \emptyset, \pi, w\right) + \frac{w_i}{\sum_{d \in D} w_d}\left(\rho_X\left(N, v, \emptyset, \pi, w\right) - \pi_X\right), & i \in D \end{cases} \end{aligned}$$

It is not difficult to show that the axioms mentioned in the lemma are indeed fulfilled. Also, in general, we have $\rho_i\left(N, \alpha u_T, \emptyset, \pi, w\right) \neq Sh_i\left(N, \alpha u_T\right)$.□

3.2. Application: buying a house in the presence of a realtor.

3.2.1. *The model.* We now turn to the application of our value to a very simple housing market. The three agents are a seller of a house S, a buyer B and a real-estate agent A. (Thus, these pages contribute to intermediation theory, Spulber (1999) being the standard reference.) We assume that the seller's reservation price r is below the buyer's willingness to pay w. Thus, the gains from trade are positive, $w - r > 0$.

In many real-world markets, the realtor charges a fee π which is a fraction f of the house price p for his service, $\pi = fp$. This payoff to the realtor π is payable by the buyer and the seller in proportions $g_S = 0$ and $g_B = 1$, respectively. These are the weights introduced in the previous section and we assume that they are given exogenously. The guiding question for our application concerns the fee fraction f chosen by the realtor in order to maximize π.

The seller and the buyer need the realtor to come into contact. Therefore, the coalition function v is given by $N = \{S, B, A\}$ and

$$v(K) = \begin{cases} w - r, & K = N, \\ 0, & \text{otherwise} \end{cases}$$

- At the first stage, the realtor decides on f.
- At the second stage, the seller and the buyer decide whether they will indeed do business with each other. If not, the game ends with payoffs 0 for every player.
- At the third stage, the seller and the buyer engage in a bargaining process, the outcome of which is determined by the weighted XP value.

3.2.2. *The third stage: bargaining.* We abbreviate $Sh^{\{A\},\pi,(0,1)}(N, v)$ by ξ. The three agents S, B, and A obtain weighted Shapley value with exogenous payments

$$\begin{aligned} \xi &= (\xi_S, \xi_B, \xi_A) \\ &= \left(\frac{w-r}{3}, \frac{w-r}{3} + 1 \cdot \left(\frac{w-r}{3} - \pi\right), \pi\right) \\ &= \left(\frac{w-r}{3}, \frac{2}{3}(w-r) - \pi, \pi\right) \end{aligned}$$

So far, the realtor's fee π is exogenous so that we could apply our formula. However, the model allows to calculate the "equilibrium" house price p^* so that payments to the realtor are now endogenous at fp^*. Indeed, the seller's rent is $p - r = \xi_S$ so that we obtain

$$\begin{aligned} p^* &= \xi_S^*(f) + r = \frac{w-r}{3} + r \\ &= \frac{2}{3}r + \frac{1}{3}w \end{aligned} \tag{XV.2}$$

and

$$\begin{aligned} \xi_B^*(f) &= w - p^* - fp^*, \\ \pi^*(f) &= fp^* \end{aligned}$$

3.2.3. *The second stage: do they have a deal.* The seller is willing to sell his house if $\xi_S \geq 0$ holds which is true by $w - r > 0$. The buyer will buy this house if $w - p^* - fp^* \geq 0$ or

$$f \leq \frac{w - p^*}{p^*}$$

or (use eq. (XV.2))

$$f \leq \frac{w - \left(\frac{w-r}{3} + r\right)}{\frac{w-r}{3} + r} = \frac{2\,(w-r)}{2r+w}$$

hold. For any $f \geq 0$, the realtor is happy to help in the deal. Thus, the deal can be struck for any fee percentage f obeying

$$0 \leq f \leq \frac{2\,(w-r)}{2r+w}.$$

3.2.4. *The first stage: setting f.* Obviously, the real-estate agent maximizes her profit by letting

$$f^* = \frac{2\,(w-r)}{2r+w}$$

As expected, we find $\frac{df^*}{dw} > 0$ and $\frac{df^*}{dr} < 0$.

4. Conclusions

This paper has to aims. First, we introduce and axiomaize a new value where payments are fixed for some players. Second, we show how to apply this value to two quite different fields - basic income (social policy) and renumeration for real-estate agents. The second example belongs to the growing number of hybrid noncooperative-cooperative models which, following Brandenburger & Stuart (2007) (who use the core rather than the Shapley value or the XP Shapley value), can also be called biform games. In our example, the first two stages (setting f and deciding on whether to trade) form an extensive game where the payoffs are calculated by way of cooperative means at the third stage.

The main idea of our paper is to give exogenous payments to some players. Consistency plays a central role in our proofs. It seems a natural requirement. However, for future research, we point to an attractive alternative. Define a TU game $\left(N\backslash X, p^{v,X,\pi}\right)$ by

$$p^{v,X,\pi}(S) = \begin{cases} v\,(S \cup X) - \pi_X, & S \neq \emptyset \\ 0, & S = \emptyset \end{cases}$$

For example, X is the set of civil servants in an economy (N, v) and π_X the taxes to be paid for the civil servants. $p^{v,X,\pi}$ is close to coalition functions defined in Aumann & Drèze (1974) and in Peleg (1986). The most important difference is that these authors assume that players from S can choose the players from X they want to use and pay for. Our more simple definition makes sense for the above interpretation. Assuming that the players from X obtain π and the endogenous players get $Sh\left(p^{v,X,\pi}\right)$, we find efficiency in the sense of $Sh_{N\backslash X}\left(p^{v,X,\pi}\right) + \pi_X = v\,(N)$.

Interestingly, consistency is not fulfilled by our XP value. As an example, consider $N = \{1,2,3\}$, $v = u_{\{1,2\}}$, $X = \{1\}$ and the payoffs

$$\begin{aligned} Sh_2\left(N, u_{\{1,2\}}\right) &= \frac{1}{2} \\ Sh_2^{X,\pi}\left(N, u_{\{1,2\}}\right) &= \frac{1}{2} + \frac{1}{|\{2,3\}|}\left(\frac{1}{2} - \pi_1\right) \\ Sh_2\left(p^{u_{\{1,2\}},\{1\},\pi}\right) &= 1 - \frac{\pi_1}{2} \neq \frac{1}{2} \text{ for } \pi_1 = \frac{1}{2}. \end{aligned}$$

In this example, player 2 takes all the benefit from the services provided by the civil servant 1, but pays half the taxes. In such-like situations, a violation of consistency makes perfect sense.

Vector-measure games

Part F

Vector-measure games

This part deals with nonatomic agents. We need continua of agents for an analysis of growth theory and for a model on evolutionary theory. Growth theory is attacked in two steps. In chapter XVI, we present the standard Solow growth model which takes a central role in any course on growth theory. This model builds on production functions featuring constant returns. For other production functions, the continuous Shapley value can be very helpful. This is the subject matter of chapter XVII.

The second application concerns an evolutionary cooperative game theory which we develop in chapter XVIII.

CHAPTER XVI

The Solow growth model

1. Introduction

This chapter prepares the upcoming one where we make use of the continuous Shapley value. We present the standard Solow (1956) model that uses a constant-returns production function in order to trace the capital-per-head trajectory in terms of the rate of saving, the depreciation rate, the growth rate of the (working) population and the initial capital per head. The first part of this chapter presents the Solow model on the basis of a Cobb-Douglas production function. The second part generalizes to any neoclassical production function.

We will guide the reader

- to an understanding of discrete and continuous growth rates,
- through the dynamics of the Solow model for both Cobb-Douglas and neoclassical production functions, and
- to the equilibrium concept employed by growth theorists.

2. Growth rates

2.1. Discrete-time growth rates. We take some economic (or other) variable y whose evolution we want to consider. By y_t we denote the value of y at time t, $t = 0, 1, \ldots$. Our definition of a growth rate in discrete time presupposes some given time interval, for example a year or a month.

Definition XVI.1. *The discrete-time growth rate of y is defined by*

$$\gamma_y^{\langle 1 \rangle} := \frac{y_{t+1} - y_t}{y_t}.$$

Growth rates are often denoted by γ ("gamma"), the Greek letter for g. Superscript $\langle 1 \rangle$ refers to the full time interval, a year, say. Note that $\gamma_y^{\langle 1 \rangle}$ does not carry the time index. Sometimes, this will mean that $\gamma_y^{\langle 1 \rangle}$ is constant over time, but at other times, the author is just to lazy to write down the time index or does not want to bother the reader with too much notational garbage.

EXERCISE XVI.1. *What are the growth rates of* x_t, y_t, *and* z_t, *given by*

$$\begin{aligned} x_t &: = t, \\ y_t &: = t + 4 \text{ and} \\ z_t &: = 100t? \end{aligned}$$

Multiplying

$$y_t$$

by the growth factor

$$1 + \gamma_y^{\langle 1 \rangle} = 1 + \frac{y_{t+1} - y_t}{y_t}$$

yields, at the end of a year,

$$\begin{aligned} & y_t \left(1 + \frac{y_{t+1} - y_t}{y_t}\right) \\ = \ & y_{t+1}. \end{aligned}$$

t years later, a given y_0 (y at time 0) has become

$$y_t = y_0 \left(1 + \gamma_y^{\langle 1 \rangle}\right)^t. \qquad \text{(XVI.1)}$$

For example, if you take Euro 100,- to the bank to earn an interest of $r = \frac{5}{100} = 5\%$, at the end of five years, you collect

$$100 \left(1 + \frac{5}{100}\right)^5 \approx 100 \cdot 1.276 = 127.6.$$

In growth theory, y_t often denotes income per head at time t, i.e.,

$$y_t = \frac{Y_t}{L_t},$$

where Y_t is the income and L_t the labor force, both at time t. One would, of course, think that the growth rates of y, Y and L are closely connected.

Indeed, we obtain

$$\begin{aligned}
\frac{y_{t+1}-y_t}{y_t} &= \frac{\frac{Y_{t+1}}{L_{t+1}}-\frac{Y_t}{L_t}}{\frac{Y_t}{L_t}} \\
&= \frac{\frac{\frac{Y_{t+1}}{L_{t+1}}-\frac{Y_t}{L_t}}{\frac{Y_t}{L_t}}}{\frac{Y_{t+1}-Y_t}{Y_t}-\frac{L_{t+1}-L_t}{L_t}}\left(\frac{Y_{t+1}-Y_t}{Y_t}-\frac{L_{t+1}-L_t}{L_t}\right) \\
&= \frac{\frac{\frac{Y_{t+1}L_t-Y_tL_{t+1}}{L_{t+1}L_t}}{\frac{Y_t}{L_t}}}{\frac{Y_{t+1}L_t-Y_tL_t-(L_{t+1}Y_t-L_tY_t)}{Y_tL_t}}\left(\frac{Y_{t+1}-Y_t}{Y_t}-\frac{L_{t+1}-L_t}{L_t}\right) \\
&= \frac{\frac{\frac{1}{L_{t+1}L_t}}{\frac{Y_t}{L_t}}}{\frac{1}{Y_tL_t}}\left(\frac{Y_{t+1}-Y_t}{Y_t}-\frac{L_{t+1}-L_t}{L_t}\right) \\
&= \frac{\frac{1}{L_{t+1}Y_t}}{\frac{1}{Y_tL_t}}\left(\frac{Y_{t+1}-Y_t}{Y_t}-\frac{L_{t+1}-L_t}{L_t}\right) \\
&= \frac{L_t}{L_{t+1}}\left(\frac{Y_{t+1}-Y_t}{Y_t}-\frac{L_{t+1}-L_t}{L_t}\right).
\end{aligned}$$

Thus, the growth rate of

$$y=\frac{Y}{L}$$

is close to the growth rate of Y minus the growth rate of L if L_t is close to L_{t+1},

$$\frac{y_{t+1}-y_t}{y_t}\approx\frac{Y_{t+1}-Y_t}{Y_t}-\frac{L_{t+1}-L_t}{L_t}.$$

Very much the same holds for the product of two variables. Let us consider the production function

$$Y_t=L_tK_t,$$

which supposes that income Y_t is the product (in mathematical terms) of labor L_t and capital K_t. We have

$$\begin{aligned}
&\frac{L_{t+1}-L_t}{L_t}+\frac{K_{t+1}-K_t}{K_t} \\
=\ & \frac{(L_{t+1}-L_t)\,K_t+(K_{t+1}-K_t)\,L_t}{L_tK_t} \\
=\ & \frac{L_{t+1}K_{t+1}-L_{t+1}K_{t+1}+(L_{t+1}-L_t)\,K_t+(K_{t+1}-K_t)\,L_t}{L_tK_t} \\
=\ & \frac{L_{t+1}K_{t+1}-L_tK_t-L_{t+1}K_{t+1}+L_{t+1}K_t-L_tK_t+K_{t+1}L_t}{L_tK_t} \\
=\ & \frac{Y_{t+1}-Y_t}{Y_t}+\frac{L_t\,(K_{t+1}-K_t)-L_{t+1}\,(K_{t+1}-K_t)}{L_tK_t} \\
\approx\ & \frac{Y_{t+1}-Y_t}{Y_t}.
\end{aligned}$$

The growth rate of the product of two variables is approximately equal to the sum of the growth rates of the two factors. Now, if the time intervals are "very small", both approximations are very good. Indeed, if we define growth in continuous time, they will turn out to be exact.

2.2. From discrete to continuous time. Let us consider half-yearly instead of yearly growth rates. For example, the bank may pay out interest every six months. To make up for the half-yearly interest payment, the interest rate is halfed. Instead of the growth factor

$$\left(1+\gamma^{\langle 1\rangle}\right)^t$$

for the yearly growth rate $\gamma_y^{\langle 1\rangle}$, we have the growth factor

$$\left(\left(1+\frac{\gamma^{\langle 1\rangle}}{2}\right)^2\right)^t = \left(1+\frac{\gamma^{\langle 1\rangle}}{2}\right)^{2t}.$$

for the half-yearly growth rate $\frac{\gamma^{\langle 1\rangle}}{2}$.

Food for thought: Would you prefer an interest payment of $\frac{\gamma^{\langle 1\rangle}}{2}$, two times a year, to an interest rate of $\gamma^{\langle 1\rangle}$, paid out only once a year?

Since we earn interest on the interest, these two factors are not equal:

$$\left(1+\frac{\gamma^{\langle 1\rangle}}{2}\right)^{2t} > \left(1+\gamma^{\langle 1\rangle}\right)^t.$$

We now look for a growth rate that makes the investor indifferent between half-yearly payments and yearly payments. That is, we define $\gamma^{\left\langle \frac{1}{2}\right\rangle}$ implicitly by

$$\left(1+\frac{\gamma^{\left\langle \frac{1}{2}\right\rangle}}{2}\right)^{2t} = \left(1+\gamma^{\langle 1\rangle}\right)^t.$$

Food for thought: Would you expect $\gamma^{\left\langle \frac{1}{2}\right\rangle} > \gamma^{\langle 1\rangle}$ or $\gamma^{\left\langle \frac{1}{2}\right\rangle} < \gamma^{\langle 1\rangle}$?

From

$$\left(1+\frac{\gamma^{\left\langle \frac{1}{2}\right\rangle}}{2}\right)^{2t} = \left(1+\gamma^{\langle 1\rangle}\right)^t$$

we obtain

$$1+\frac{\gamma^{\left\langle \frac{1}{2}\right\rangle}}{2} = \left(\left(1+\frac{\gamma^{\left\langle \frac{1}{2}\right\rangle}}{2}\right)^{2t}\right)^{\frac{1}{2t}} = \left(\left(1+\gamma^{\langle 1\rangle}\right)^t\right)^{\frac{1}{2t}} = \left(1+\gamma^{\langle 1\rangle}\right)^{\frac{1}{2}}$$

and then

$$\gamma^{\left\langle \frac{1}{2}\right\rangle} = -2+2\sqrt{1+\gamma^{\langle 1\rangle}}.$$

We can now conclude

$$\begin{aligned}
\gamma^{\langle 1\rangle} &> 0 \\
&\Rightarrow \left(\gamma^{\langle 1\rangle}\right)^2 > 0 \\
&\Rightarrow \left(\gamma^{\langle 1\rangle}\right)^2 + 4(1+\gamma) > 4(1+\gamma) \\
&\Rightarrow \left(\gamma^{\langle 1\rangle} + 2\right)^2 > 4(1+\gamma) \\
&\Rightarrow \gamma^{\langle 1\rangle} + 2 > 2\sqrt{1+\gamma} \\
&\Rightarrow \gamma^{\langle 1\rangle} > -2 + 2\sqrt{1+\gamma} = \gamma^{\left\langle \frac{1}{2}\right\rangle}
\end{aligned}$$

We now decrease the time interval even further. Generally, we consider an interest payment m times a year with interest rate $\gamma^{\langle 1\rangle}/m$. Then, at the end of t years, we obtain

$$\begin{aligned}
&\left(\left(1+\frac{\gamma^{\langle 1\rangle}}{m}\right)^m\right)^t \\
= &\left(1+\frac{\gamma^{\langle 1\rangle}}{m}\right)^{mt}.
\end{aligned}$$

It can be shown (but we will not do that here) that this growth factor is an increasing function of m. The sequence $\left(\left(1+\frac{\gamma^{\langle 1\rangle}}{m}\right)^{mt}\right)_{m\in N}$ converges (gets closer and closer to some value) and we have

$$\lim_{m\to\infty}\left(1+\frac{\gamma^{\langle 1\rangle}}{m}\right)^{mt} = e^{\gamma^{\langle 1\rangle}t}.$$

Again, because of the interest on the interest, one prefers to obtain continuous interest payments. Analogous to $\gamma^{\left\langle \frac{1}{2}\right\rangle}$, we are now looking for $\gamma^{\langle 0\rangle}$, which is the rate at which indifference to a yearly interest rate obtains:

$$e^{\gamma^{\langle 0\rangle}t} = \left(1+\gamma^{\langle 1\rangle}\right)^t.$$

Applying the natural logarithm on both sides and deviding by t yields

$$\gamma^{\langle 0\rangle} = \ln\left(1+\gamma^{\langle 1\rangle}\right) \qquad \text{(XVI.2)}$$

We would like to confirm $\gamma^{\langle 0\rangle} < \gamma^{\langle 1\rangle}$. Indeed, it is well-known that

$$\ln x < x - 1 \text{ for } x > 0, x \neq 1$$

holds. In fig. 1, the reader can see the logarithm which cuts the abscisse at $x = 1$. So does $x - 1$. Replacing x by $1 + y$, we obtain

$$\ln(1+y) < y \text{ for } y > -1, y \neq 0$$

from where we find the desired inequality:

$$\gamma^{\langle 0\rangle} = \ln\left(1+\gamma^{\langle 1\rangle}\right) < \gamma^{\langle 1\rangle} \text{ for } \gamma^{\langle 1\rangle} > -1, \gamma^{\langle 1\rangle} \neq 0$$

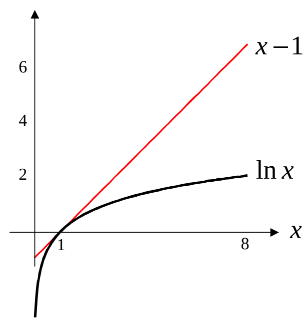

FIGURE 1. The natural logarithm

The growth rates $\gamma^{\langle 1\rangle}$ and $\gamma^{\langle 0\rangle}$ are close for small rates, as can be seen from the following table:

$\gamma^{\langle 1\rangle}$	$\gamma^{\langle 0\rangle}$ (approximation)
$0,001$ (one-tenth of a percent)	$0,0009995$
$0,01$ (one percent)	$0,0099503$
$0,1$ (10 percent)	$0,09531$
$0,2$ (20 percent)	$0,18232$
$0,3$ (30 percent)	$0,26236$

2.3. Continuous-time growth rates. In discrete time, the growth rate of y is defined by

$$\gamma_y^{\langle\Delta t\rangle} := \frac{\frac{y_{t+\Delta t}-y_t}{(t+\Delta t)-t}}{y_t}.$$

Taking the limit with respect to Δt yields

$$\begin{aligned}\lim_{\Delta t\to 0}\gamma_y^{\langle\Delta t\rangle} &= \lim_{\Delta t\to 0}\frac{\frac{y_{t+\Delta t}-y_t}{(t+\Delta t)-t}}{y_t}\\ &= \lim_{\Delta t\to 0}\frac{\frac{\Delta y_t}{\Delta t}}{y_t}\\ &= \frac{\frac{dy_t}{dt}}{y_t}.\end{aligned}$$

DEFINITION XVI.2. *The continuous-time growth rate of y is defined by*

$$\gamma_y := \gamma_{y,t} := \frac{\frac{dy_t}{dt}}{y_t}$$

where the time index is often suppressed.

It is important for the reader to understand that y_t is just another way to write $y(t)$, i.e., we have a function y which takes one argument, t. Therefore, we could as well have written $\frac{dy(t)}{dt}$ or $\frac{dy}{dt}$ instead of $\frac{dy_t}{dt}$. Writing time as an index rather than a functional argument is the usual procedure in growth theory.

Assuming a constant growth rate g, we obtain

$$g = \frac{\frac{dy_t}{dt}}{y_t},$$

which does not define $\gamma_{y,t}$ (as in the definition above) but claims that the growth rate of y, i.e., $\frac{dy_t}{dt}/y_t$, is equal to the constant g. This is a differential equation, i.e., an equation that contains a function y (with argument t) together with the first (or higher order) derivative of y. "Solving a differential equation" means to state the function y explicitly. In this case, exponential growth given by

$$y_t = y_0 e^{gt} \tag{XVI.3}$$

does the trick.

Exercise XVI.2. *Calculate* $\frac{dy_t}{dt}/y_t$ *for* $y_t = y_0 e^{gt}$. *Hint: the derivative of* e^x *is* e^x, *but do not forget the chain rule.*

The upshot of this exercise is $\gamma_y = g$ so that we can (and will) write

$$y_t = y_0 e^{\gamma_y t}.$$

2.4. Using the natural logarithm to express growth. In analyzing the growth of some x, it is sometimes expedient not to consider

$$y_t$$

directly, but rather take recourse to

$$\widehat{y}_t = \ln y_t.$$

The reason is this: The derivative of $\widehat{y}$ with respect to t is equal to the growth rate of y. To see this, note $\frac{d \ln x}{dx} = \frac{1}{x}$. We obtain

$$\begin{aligned} \frac{d\widehat{y}}{dt} &= \frac{d \ln y_t}{dt} \\ &= \frac{1}{y_t} \frac{dy}{dt} \text{ (chain rule!)} \\ &= \frac{\dot{y}_t}{y_t} \end{aligned}$$

Therefore, if $\widehat{y}$ is plotted against t, the growth rate of y can be seen directly from the slope of the $\widehat{y}$-graph.

Lemma XVI.1. *The growth rate of* y *is given by*

$$\frac{\dot{y}_t}{y_t} = \frac{d \ln y_t}{dt}.$$

Exercise XVI.3. *Try to find the relationship between the (continuous-time) growth rates of* Y, K *and* L *for* $Y_t = L_t K_t$. *Hint: apply the product rule of differentiation and use* $\ln(LK) = \ln L + \ln K$.

Of course, the solution to our exercise cannot surprise the reader who has gone through subsection 2.1 on pp. 275. Also, for

$$y = \frac{Y}{L}$$

we find

$$\gamma_y = \gamma_Y - \gamma_L.$$

To sum up, in continuous time we obtain:

- The growth rate of a product is equal to the sum of the growth rates of its factors.
- The growth rate of a ratio is equal to the difference of the growth rates of nominator and denominator.

An application concerns the relationship between the monetary interest rate and the real interest rate. By r we denote the monetary interest rate which is the growth rate for an asset K_m, a deposit in a bank or a government bond. If π denotes the rate of inflation, the real interest rate is given by $r - \pi$. Let us explain, why.

Now, if we denote the price level by P and the real capital by K, we have

$$K := \frac{K_m}{P}$$

and, applying the above rule, the real interest rate is given by

$$\gamma_K = \gamma_{K_m} - \gamma_P = r - \pi.$$

Exercise XVI.4. *Assuming a constant growth rate, apply the natural logarithm to the exponential-growth formula*

$$y_t = y_0 e^{\gamma_y t}$$

in order to confirm

$$\gamma_y = \frac{\ln y_t - \ln y_0}{t - 0} = \frac{1}{t} \ln \frac{y_t}{y_0}.$$

Hint: $\ln x$ *is the inverse of* e^x*, i.e.,* $\ln e^x = x$.

Finally, we can use the natural logarithm to justify a handy rule of thumb. According to this rule the number of years needed to double some variable y is approximately equal to

$$\frac{70}{\gamma_y \cdot 100}.$$

For example, if you take some money to the bank and you get an interest rate of 2%, you need 35 years to double your capital.

Inversely, in order to achieve a doubling in t years, a growth rate (in percentage points) of

$$\frac{70}{t}$$

is needed. If you hope to double your capital within 10 years, you have to ask for an interest rate of 7%.

The confirmation of this rule is not difficult. We are looking for the growth rate γ_y and/or the time span needed to double y, i.e., we need to solve

$$y_0 e^{\gamma_y t} = 2y_0.$$

Deviding by y_0 and taking the logarithm leads to

$$\gamma_y t = \ln\left(e^{\gamma_y t}\right) = \ln 2 \approx 0,69315.$$

Solving for t or γ_y, we obtain

$$t \approx \frac{70}{\gamma_y \cdot 100}$$

and

$$\gamma_y \cdot 100 \approx \frac{70}{t},$$

respectively. This approximation formula yields the following table:

Growth rate in percentage points	Years needed for doubling (approximation)	Years needed for doubling (correct, continuous time)	Years needed for doubling (correct, yearly interest)
0.1 (one-tenth of a percent)	700	≈ 693.15	≈ 693.49
1 (one percent)	70	≈ 69.31	≈ 69.66
10 (ten percent)	7	≈ 6.93	≈ 7.27
20 (twenty percent)	$3\frac{1}{2}$	≈ 3.46	≈ 3.80
30 (thirty percent)	$2\frac{1}{3}$	≈ 2.31	≈ 2.64

The reader will note that we used two approximations for this formula. First, we have $\ln 2$ instead of 0.7 (no problem), second, we use the continuous-time growth rate instead of the more usual yearly one.

3. Convergence

One of the central questions of growth theory is whether or not different economies converge over time. In this section, we will stress the need to distinguish between weak and strong convergence. Consider two variables x_t and y_t with $0 < x_0 < y_0$. Weak convergence means that x grows faster than y. Alternatively, the ratio of y over x decreases in time. Put formally:

DEFINITION XVI.3. *Weak convergence between x_t and y_t is said to hold if, whenever $0 < x_0 < y_0$, the growth rates obey $\gamma_x > \gamma_y$ for all $t \geq 0$.*

LEMMA XVI.2. *Weak convergence between x_t and y_t hold iff, whenever $0 < x_0 < y_0$, $\frac{d\frac{y_t}{x_t}}{dt} < 0$.*

"Iff" is short for "if and only if". Differently put, the lemma provides a criterium for weak convergence. The proof is easy:

$$
\begin{aligned}
& \frac{d\frac{y}{x}}{dt} < 0 \\
\Leftrightarrow \quad & \frac{\frac{dy}{dt}x - \frac{dx}{dt}y}{x^2} < 0 \\
\Leftrightarrow \quad & \frac{\frac{dy}{dt}}{x} - \frac{\frac{dx}{dt}}{x}\frac{y}{x} < 0 \\
\Leftrightarrow \quad & \frac{\frac{dy}{dt}}{y} - \frac{\frac{dx}{dt}}{x} < 0 \text{ (multiply by } \frac{x}{y}\text{)} \\
\Leftrightarrow \quad & \gamma_y < \gamma_x
\end{aligned}
$$

Weak convergence may hold even if x and y never get close. For example, weak convergence exists between

$$
\begin{aligned}
x_t &= t \text{ and} \\
y_t &= 2t + 2.
\end{aligned}
$$

EXERCISE XVI.5. *Show that weak convergence holds between x_t and y_t.*

Strong convergence requests that the two variables do indeed get closer and closer.

DEFINITION XVI.4. *Strong convergence between x_t and y_t is said to hold if weak convergence between x_t and y_t holds and if*

$$\lim_{t\to\infty} \frac{y_t}{x_t} = 1.$$

EXERCISE XVI.6. *Show that strong convergence does not hold between $x_t = t$ and $y_t = 2t + 2$.*

4. Cobb-Douglas production functions

The Cobb-Douglas production function F is given by

$$Y = F(K, L) = AK^\alpha L^{1-\alpha},\ A > 0, 0 < \alpha < 1.$$

Y is total output, K and L denote the amount of capital and labor that enter into production, and A is a technological coefficient that can be used to model and discuss technological progress. Since we do not deal with technological progress in this chapter and the next, we disregard the parameter A:

$$Y = F(K, L) = K^\alpha L^{1-\alpha},\ 0 < \alpha < 1.$$

The Cobb-Douglas (for short, CD) production function exhibits a number of interesting properties. First of all, if all inputs are increase by some factor, output grows by that same factor.

DEFINITION XVI.5. *A production function F exhibits constant returns to scale, if we have*

$$F(\tau K, \tau L) = \tau F(K, L), K \geq 0, L \geq 0$$

for any $\tau \geq 0$.

EXERCISE XVI.7. *Can you prove that the CD production function is of constant returns? Hint: you will use $(a_1 a_2)^b = a_1^b a_2^b$ and $a^b a^c = a^{b+c}$.*

Second, the marginal productivity of each factor is positive and decreasing. The more capital (or labor) we employ, the higher the output, but the additional output of additional input of capital gets smaller and smaller. Indeed, for $L > 0$, we obtain

$$\begin{aligned} \frac{\partial F}{\partial K} &= \alpha K^{\alpha-1} L^{1-\alpha} \\ &= \alpha \frac{L^{1-\alpha}}{K^{1-\alpha}} \\ &= \alpha \left(\frac{L}{K}\right)^{1-\alpha} > 0 \end{aligned}$$

and it is easy to see that the marginal product of capital decreases with increasing K. In other words: F is a concave function of K (and of L, too). In expressing the marginal product of capital, we have written $\frac{\partial F}{\partial K}$ instead of $\frac{dF}{dK}$, because F carries two arguments, K and L, and we need to apply the partial derivative with respect to K, while holding L constant.

CD production functions feature positive and decreasing marginal productivities in an extreme fashion. On one hand, if we keep on increasing K, the marginal product finally becomes zero,

$$\lim_{K \to \infty} \frac{\partial F}{\partial K} = 0.$$

It cannot decrease any further. On the other hand, if we let K vanish, the marginal product becomes very high,

$$\lim_{K \to 0} \frac{\partial F}{\partial K} = \infty.$$

This two properties are called Inada conditions.

Third, the production elasticity of capital is constant and equal to α while the production elasticity of labor is equal to $1-\alpha$. Thus, a one percent change in the quantity of capital (labor) results in an α percent ($1 - \alpha$ percent) change in output. Formally, the production elasticity of capital is given by

$$\varepsilon_{Y,K} = \frac{\frac{\partial Y}{Y}}{\frac{\partial K}{K}} = \frac{\partial Y}{\partial K} \frac{K}{Y}.$$

EXERCISE XVI.8. *Can you confirm that the production elasticity of capital is equal to α?*

Assume that factors of production are paid their marginal product. Then, factor payments equal output. This is the exhaustion theorem, due to Euler.

EXERCISE XVI.9. *Prove Euler's theorem for CD production functions:*

$$\frac{\partial F}{\partial K} \cdot K + \frac{\partial F}{\partial L} \cdot L = F(K, L).$$

EXERCISE XVI.10. *Assuming a CD production function, show how the growth rate of output depends on the growth rates of capital and labor. Hint: you will use the product and chain rule of differentiation (first growth-rate definition) or the rules for manipulating the natural logarithm (second growth-rate definition).*

In growth theory, we are often concerned with per-head variables. In particular, per-head consumption might be taken as a measure of welfare. CD production functions are very suitable in this context because they allow to express per-head output

$$y := \frac{Y}{L}$$

as a function of capital per head,

$$k := \frac{K}{L}.$$

Indeed, we find

$$y = \frac{K^\alpha L^{1-\alpha}}{L} = \frac{K^\alpha}{L^\alpha} = k^\alpha =: f(k).$$

f is called the production function in intensive form. Its (one) argument is capital per head.

5. Dynamics (CD production function)

We are now ready to introduce the Solow growth model for CD production functions. At every point in time, output is devided between consumption and investment. We assume that output can be used for both purposes. For example, animals such as cows (output) can be slaughtered and eaten (consumption) or used to produce additional animals.

In the standard Solow model, one works with the plausible consumption function

$$C := (1-s)Y$$

where C is overall consumption and $s \geq 0$ the constant saving rate. This is the behaviorist tradition. Alternatively, and closer to microeconomics, one assumes a representative agent who chooses his consumption path over his whole life time. We take up this optimizing tradition in chapter **??**.

Pursuing the behaviorist tradition, per-head consumption is given by

$$c := \frac{C}{L} = (1-s)\frac{Y}{L} = (1-s)y.$$

Now, we can put down the changes in the stock of capital:

$$\dot{K} = sY - \delta K \tag{XVI.4}$$

where sY is the income's share not consumed and hence invested, and δ is the constant depreciation rate. Eq. XVI.4 is based on the assumption that savings and investments are equal so that s can be addressed as the rate of savings or the rate of gross investments.

Since per-head output y depends on per-head capital endowment k, we are interested in knowing the dynamics of capital per head. In case of zero investments, capital endowments per head decrease for two reasons. First, depreciation reduces the amount of capital available in our economy. Second, if the population grows, per-head endowment of capital is reduced even if overall capital stays constant. Our calculations will show the increase of capital needed in order to make up for these two effects.

We proceed in two steps. First, we apply the quotient rule:

$$\begin{aligned}
\dot{k} &= \dot{\left(\frac{K}{L}\right)} = \frac{\dot{K}L - \dot{L}K}{L^2} \\
&= \frac{\dot{K}}{L} - \frac{\dot{L}}{L}\frac{K}{L} \\
&= \frac{\dot{K}}{L} - nk
\end{aligned}$$

and find how the change in per-head capital depends on the change of capital $\dot{K}$, labor supply L, the growth rate of labor $n := \gamma_L$ and the capital per head k.

Second, we insert eq. XVI.4 and find

$$\begin{aligned}
\dot{k} &= \frac{sY - \delta K}{L} - nk \\
&= s\frac{Y}{L} - \delta\frac{K}{L} - nk \\
&= sk^{\alpha} - (\delta + n)\,k.
\end{aligned}$$

By deviding with $k > 0$, we obtain the growth rate of per-head capital

$$\gamma_k = \frac{\dot{k}}{k} = \frac{s}{k^{1-\alpha}} - (\delta + n)\,. \tag{XVI.5}$$

Capital per head increases if the actual investment per head

$$s\frac{Y}{L}$$

lies above the break-even investment per head

$$(\delta + n)\,k.$$

Note that the break-even investment per head reflects the two effects discussed above.

Let us consider two economies that differ only in capital endowment per head but have equal parameters s, δ, and n. Then, the capital-poor economy (measured in capital per head) will witness a greater growth of capital per head. This can easily be seen from eq. XVI.5. This is the property of β-convergence that we will look at in greater detail in chapter **??**.

6. Steady state (CD production function)

Apart from the dynamics, we are interested in wether our variables will settle down in the long run. If they do, a so-called steady state has been achieved.

DEFINITION XVI.6. *A steady state is a tuple of relevant economic variables that grow at constant rates.*

The reader will note that this equilibrium concept is very different from microeconomic equilibrium concepts. Indeed, the definition does not refer to any economic actors that have preferences, endowments and actions or strategies. So far, growth theory is devoid of preferences, optimization, and other ingredients typical of economic theory. We will turn to a more actor-based growth theory in later chapters and, in a very restrictive manner, in the next section.

For the Solow model, one might consider the tuples (Y, K, L) or (Y, y, k, L). Since the per-head capital endowment is a central variable, a steady state implies that $\frac{\dot{k}}{k}$ and hence (see eq. XVI.5)

$$\frac{s}{k^{1-\alpha}} - (\delta + n)$$

is constant. However, since s, δ and n are constant, this term, can be constant only if k does not change. Formally, this can be seen from $\frac{d\left(\frac{s}{k_t^{1-\alpha}} - (\delta+n)\right)}{dt} = 0$.

Indeed, we obtain

$$\begin{aligned}
\frac{d\left(\frac{s}{k^{1-\alpha}} - (\delta + n)\right)}{dt} &= \frac{d\left(sk^{-1+\alpha}\right)}{dt} \\
&= s\left(-1+\alpha\right) k^{-2+\alpha} \frac{dk}{dt} \\
&= s\left(-1+\alpha\right) \frac{1}{k^{2-\alpha}} \frac{dk}{dt}
\end{aligned}$$

which is equal to zero

- for $s = 0$, in which case the (constant!) growth rate of per-head capital is equal to $-(\delta + n)$ and per-head capital approaches zero, or

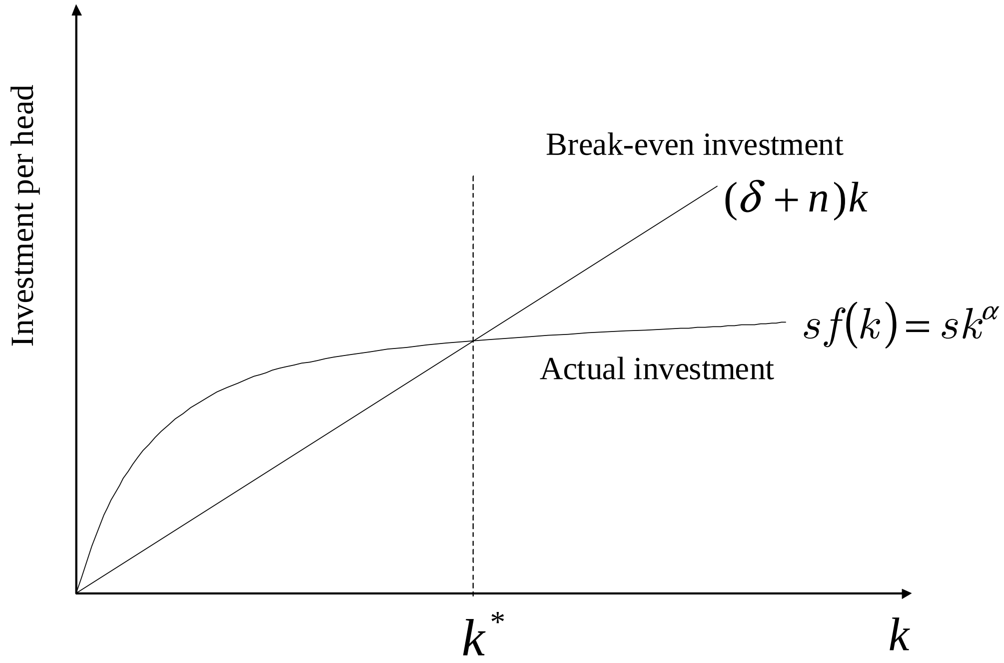

FIGURE 2. Break-even versus actual investment

- for $\frac{dk}{dt}=0$, so that the growth rate of per-head capital is zero and the steady state is characterized by

$$\frac{s}{(k^*)^{1-\alpha}} = \delta+n, \quad \text{(XVI.6)}$$

$$s\,(k^*)^{\alpha} = (\delta+n)\,k^*, \text{ or} \quad \text{(XVI.7)}$$

$$k^* = \left(\frac{s}{\delta+n}\right)^{\frac{1}{1-\alpha}}. \quad \text{(XVI.8)}$$

In the steady state, k^* is constant and so are

$$y^* = f(k^*) = (k^*)^{\alpha} = \left(\frac{s}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}} \text{ and}$$

$$c^* = (1-s)\left(\frac{s}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}.$$

On the other hand, K, Y, and C grow at rate n.

EXERCISE XVI.11. *Show this. Hint: remember $K=kL, Y=yL$, and $C=cL$.*

The dynamics and the steady state can be visualized as in fig. 2 and 3. In both figures, k is the abscisse variable. The first depicts the change in per-head capital, the second the growth rate.

Both figures suggest that the per-head endowment of capital increases as long as it is smaller than the steady-state value. This can also be shown

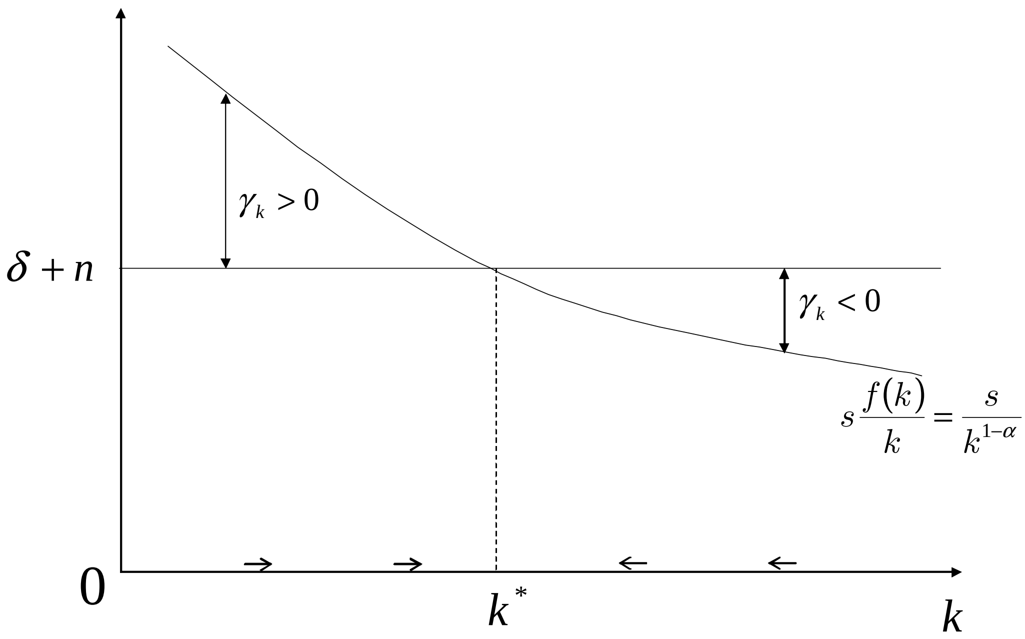

FIGURE 3. Positive and negative growth rates

algebraically: $0 < k < k^* = \left(\frac{s}{\delta+n}\right)^{\frac{1}{1-\alpha}}$ implies

$$\begin{aligned}
\gamma_k &= \frac{s}{k^{1-\alpha}} - (\delta + n) \\
&> \frac{s}{(k^*)^{1-\alpha}} - (\delta + n) \\
&= \frac{s}{\left(\left(\frac{s}{\delta+n}\right)^{\frac{1}{1-\alpha}}\right)^{1-\alpha}} - (\delta + n) \\
&= 0.
\end{aligned}$$

Inversely, capital per head goes down if it is larger than the steady-state value. In this sense, the steady state at k^* is stable.

Another steady state exists at $k = 0$. Here, output is zero and investment, too. However, for a small $k > 0$ capital per head increases (if $s > 0$ holds) and converges towards k^*.

Both figures show how changes in s, δ, n, and α influence the steady-state capital per head. This is the topic of the next section.

An alternative way to obtain the steady state is to solve the differential equation

$$\dot{k} = sk^{\alpha} - (n + \delta)k.$$

The solution is

$$k_t = \left(\frac{s}{n+\delta} + \left(k_0^{1-\alpha} - \frac{s}{n+\delta}\right) e^{-(1-\alpha)(n+\delta)t}\right)^{\frac{1}{1-\alpha}}.$$

EXERCISE XVI.12. *Can you confirm that the solution is correct? You need to form the time derivative of k_t and find the above differential equation. Hint: this is not easy, so this time you will be pardoned for not trying for yourself.*

Now, by letting the time index go towards infinity, we find

$$\lim_{t\to\infty} e^{-(1-\alpha)(n+\delta)t} = \lim_{t\to\infty} \frac{1}{e^{(1-\alpha)(n+\delta)t}} = 0$$

and see that k_t converges towards its steady state:

$$\begin{aligned}
& \lim_{t\to\infty} k_t \\
= & \left(\frac{s}{n+\delta} + \left(k_0^{1-\alpha} - \frac{s}{n+\delta}\right) \lim_{t\to\infty} e^{-(1-\alpha)(n+\delta)t}\right)^{\frac{1}{1-\alpha}} \\
= & \left(\frac{s}{n+\delta} + \left(k_0^{1-\alpha} - \frac{s}{n+\delta}\right) \cdot 0\right)^{\frac{1}{1-\alpha}} \\
= & \left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\alpha}} \\
= & k^*.
\end{aligned}$$

7. Comparative statics and the golden rule (CD production function)

Comparative statics means: How do the (exogenous) parameters of our model influence the (endogenous) variables? In the Solow model, the parameters are s, δ, n, and α. The central variable is capital per head,

$$k^* = \left(\frac{s}{\delta+n}\right)^{\frac{1}{1-\alpha}}.$$

It is a positive function of

- the production elasticity of capital α and
- the saving rate s

but depends negatively on

- the rate of depreciation δ and
- the growth rate of the population n.

It is important to note that a change in these parameters does indeed change capital per head, but does not change the growth rates of the most important variables (which are 0 and n, respectively).

With respect to developing countries, Robert Solow posed the famous question: "Why are we so rich and they so poor?" His model does provide a partial answer. Consumption per head may be taken as an indicator of richness. Indeed, looking at

$$c^* = (1-s)\left(\frac{s}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}},$$

we find that a high rate of depreciation (no repair of public and private capital) and a high rate of population growth depress consumption per head. Also, a high production elasticity of capital increases consumption because it increases capital per head and output per head. However, while saving increases capital per head, it does not unambiguously increase consumption per head. On one hand, increasing s decreases consumption immediately, on the other hand, increasing s leads to a higher per-head capital and income.

This brings us to the question of which saving rate is optimal, where optimality is defined in terms of steady-state consumption per head c^*. Setting the derivative of

$$\begin{aligned} c^* &= (1-s)\, y^* \\ &= (k^*(s))^\alpha - s\,(k^*(s))^\alpha \\ &= (k^*(s))^\alpha - (\delta+n)\, k^* \text{ (eq. XVI.7)} \end{aligned}$$

with respect to s equal to zero yields

$$\alpha\,(k^*(s))^{\alpha-1}\frac{dk^*}{ds} - (\delta+n)\frac{dk^*}{ds} = 0,$$

hence

$$k_{gold} \overset{!}{=} \left(\frac{\alpha}{\delta+n}\right)^{\frac{1}{1-\alpha}}.$$

k_{gold} is the capital per head in the steady state that maximizes steady-state consumption per head. The optimal capital per head is positive function of α (the production elasticity of capital) and a negative function of both δ (depreciation rate) and n (rate of growth of the population). Comparing,

$$k_{gold} \overset{!}{=} \left(\frac{\alpha}{\delta+n}\right)^{\frac{1}{1-\alpha}} \text{ and } k^*(s) = \left(\frac{s}{\delta+n}\right)^{\frac{1}{1-\alpha}}$$

yields

$$s_{gold} \overset{!}{=} \alpha.$$

Here, the exclamation marks ! expresses the fact that k_{gold} and s_{gold} are the result of an optimization. A capital endowment per head above k_{gold} is "dynamically inefficient", i.e., it is possible to consume more in every period by saving less.

Fig. 4 helps to visualize the golden rule. s_{gold} leads to a steady-state capital per head k_{gold} and a steady-state consumption per head c_{gold}. Note that steady-state consumption c^* is the difference of

$$(k^*(s))^\alpha \text{ and } (\delta+n)\, k^*.$$

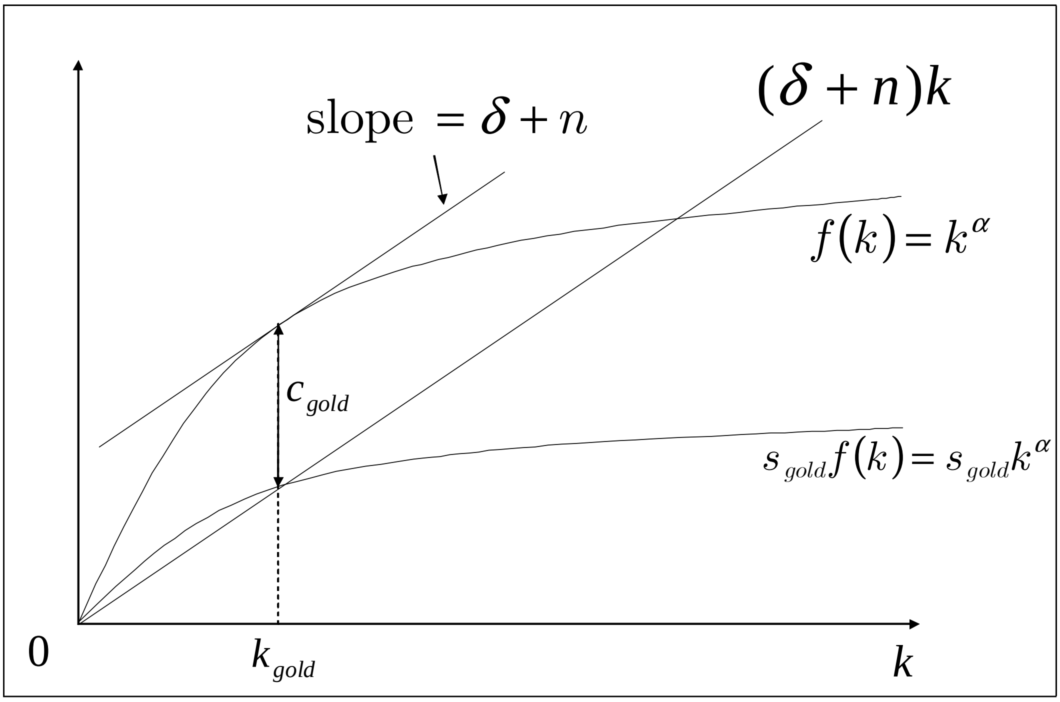

FIGURE 4. The golden rule of capital accumulation

Hence, in order to maximize this difference, the slope of $(k^*(s))^{\alpha}$ has to equal the slope of $(\delta + n) k^*$ which is $\delta + n$.

8. Neoclassical production function

8.1. Constant returns to scale. So far, we looked at the Solow model where the production function is of the Cobb-Douglas variety. This is unduly restrictive, motivated by didactic considerations, only. In general terms, the Solow model presupposes a neoclassical production function (the most prominent example being Cobb-Douglas). A production function $Y = F(K, L)$ is called neoclassical if F has two properties:

(1) constant returns to scale and
(2) decreasing marginal productivities obeying the Inada conditions.

Neoclassical production functions are of constant returns which will turn out to be a very powerful property. Constant returns to scale is a special sort of homogeneity:

DEFINITION XVI.7. *A production function F is homogeneous of degree d, if we have*

$$F(\tau K, \tau L) = \tau^d F(K, L), K \geq 0, L \geq 0$$

or any $\tau \geq 0$. A production function F exhibits constant returns to scale if it is homogeneous of degree 1.

EXERCISE XVI.13. *Prove that the production function given by*

$$F(K,L) = \left[\alpha K^{-\rho} + (1-\alpha) L^{-\rho}\right]^{-1/\rho}, 0 < \alpha < 1, \rho > -1, \rho \neq 0$$

exhibits constant returns to scale.

EXERCISE XVI.14. *Can you show that the Leontief production function, given by*

$$Y = F(K,L) = \min(AK, BL),$$

also obeys constant returns to scale?

The production function of the above exercise (which is not neoclassical!) is called CES production function because it shows a "constant elasticity of substitution" which we will demonstrate below on in section **??**.

Constant returns to scale are not a very plausible assumption. Indeed, for very low endowments of capital and labor, constant returns will not hold because gains from specialization need a certain size of the economy. Also, the reader might think of public goods such as software (production) which can be used by everyone in a small economy and a huge economy alike. On the other hand, if all gains from specialization have been exhausted and no public goods exist, constant returns to scale might hold for $\tau \geq 1$.

EXERCISE XVI.15. *Can you prove* $F(0,0) = 0$ *for any constant-returns production function* F*?*

The expedience of constant returns lies in the possibility of expressing the output per head as a function of capital per head. Indeed, for $\tau := \frac{1}{L}$, we obtain

$$F\left(\frac{K}{L}, 1\right) = F\left(\frac{1}{L}K, \frac{1}{L}L\right) = \frac{1}{L}F(K,L).$$

By defining

$$\begin{aligned} k &:= \frac{K}{L}, \\ y &:= \frac{Y}{L}, \text{ and} \\ f(k) &:= F(k,1) \end{aligned}$$

we find

$$y = \frac{F(K,L)}{L} = F\left(\frac{K}{L}, 1\right) = f(k). \tag{XVI.9}$$

Hence, $f(k)$ is the output per head for a per-head endowment of capital k. Romer (1996, S. 9) suggests the following interpretation of this equation. Imagine that the economy is devided in L small economies, each of which endowed with 1 unit of labor and $k = \frac{K}{L}$ units of capital. Because of constant returns, each of these small economies produces an Lth part of the total economy. f is called the production function in intensive form.

EXERCISE XVI.16. *Determine the intensive form of the CES production function.*

Constant returns is a powerful restriction that turns out to have many interesting consequences concerning the marginal productivities

$$\frac{\partial F}{\partial K} \text{ and } \frac{\partial F}{\partial L}.$$

We assume that our production functions are "twice continuously differentiable". Note that every differentiable function is continuous. Therefore, a twice differentiable function is continuous and its first derivative is continuous, too. Inserting "continuously" in "twice continuously differentiable" ensures that the second-order derivatives are also continuous. Our production function has four second-order derivates, two of them mixed:

$$\begin{aligned}
\frac{\partial^2 F}{(\partial K)^2} &:= \frac{\partial \frac{\partial F}{\partial K}}{\partial K}, \\
\frac{\partial^2 F}{(\partial L)^2} &:= \frac{\partial \frac{\partial F}{\partial L}}{\partial L}, \\
\frac{\partial^2 F}{\partial K \partial L} &:= \frac{\partial \frac{\partial F}{\partial L}}{\partial K}, \text{ and} \\
\frac{\partial^2 F}{\partial L \partial K} &:= \frac{\partial \frac{\partial F}{\partial K}}{\partial L}
\end{aligned}$$

The two mixed derivates are equal. The effect of a marginal increase of capital on the marginal productivity of labor is equal to the effect of a marginal increase of labor on the marginal productivity of capital.

Sometimes, it is necessary to explicitly state where (for which capital and labor values) a derivative is calculated. If no such values are given, (K, L) is assumed, for example,

$$\frac{\partial F}{\partial K} = \left.\frac{\partial F}{\partial K}\right|_{(K,L)}.$$

We can now prove some identities that follow from the fact that F is constant-returns.

LEMMA XVI.3. *Let F be homogeneous of degree* 1. *Then,*

(1) *the marginal productivities are homogeneous of degree* $0:$

$$\left.\frac{\partial F}{\partial K}\right|_{(\tau K, \tau L)} = \left.\frac{\partial F}{\partial K}\right|_{(K,L)} \quad \textit{and} \tag{XVI.10}$$

$$\left.\frac{\partial F}{\partial L}\right|_{(\tau K, \tau L)} = \left.\frac{\partial F}{\partial L}\right|_{(K,L)} \tag{XVI.11}$$

(2) *the second-order derivatives are homogenous of degree* -1 :

$$\tau \left. \frac{\partial^2 F}{(\partial K)^2} \right|_{(\tau K, \tau L)} = \left. \frac{\partial^2 F}{(\partial K)^2} \right|_{(K,L)} \text{ and} \tag{XVI.12}$$

$$\tau \left. \frac{\partial^2 F}{(\partial L)^2} \right|_{(\tau K, \tau L)} = \left. \frac{\partial^2 F}{(\partial L)^2} \right|_{(K,L)} \tag{XVI.13}$$

(3) *the marginal productivities can be expressed as functions of capital per head,* k :

$$\frac{\partial F}{\partial K} = \frac{df}{dk} \text{ and} \tag{XVI.14}$$

$$\frac{\partial F}{\partial L} = f(k) - k\frac{df}{dk} =: \omega(k) \tag{XVI.15}$$

(4) *Euler's theorem holds:*

$$F(K, L) = \frac{\partial F}{\partial K} K + \frac{\partial F}{\partial L} L \tag{XVI.16}$$

and, finally,

(5) *the second-order derivatives relate to each other in a simple manner:*

$$\frac{\partial^2 F}{\partial K \partial L} = -k \frac{\partial^2 F}{(\partial K)^2}, \tag{XVI.17}$$

$$\frac{\partial^2 F}{\partial K \partial L} = -\frac{1}{k} \frac{\partial^2 F}{(\partial L)^2}, \text{ and} \tag{XVI.18}$$

$$\frac{\partial^2 F}{(\partial K)^2} \frac{\partial^2 F}{(\partial L)^2} = \left(\frac{\partial^2 F}{\partial K \partial L} \right)^2. \tag{XVI.19}$$

The first item of the above lemma can be solved by the reader.

Exercise XVI.17. *Prove*

$$\left. \frac{\partial F}{\partial K} \right|_{(\tau K, \tau L)} = \left. \frac{\partial F}{\partial K} \right|_{(K,L)} \text{ and}$$

$$\left. \frac{\partial F}{\partial L} \right|_{(\tau K, \tau L)} = \left. \frac{\partial F}{\partial L} \right|_{(K,L)}$$

for a constant-returns production function F, *i.e., show that the marginal productivities are homogeneous of degree* 0 *(note* $\tau^0 = 1$*). Hint: form the derivative of* $F(\tau K, \tau L) = \tau F(K, L)$ *with respect to* K *and* L *and use the chain rule of differentiation.*

The first item can be generalized as we note, without proof, in the following lemma:

Lemma XVI.4. *Let* F *be a production function that is homogeneous of degree* d. *Then,*

$$\left. \frac{\partial F}{\partial K} \right|_{(\tau K, \tau L)} = \tau^{d-1} \left. \frac{\partial F}{\partial K} \right|_{(K,L)}$$

The proof of the second item is quite analogous to the proof of the first. The trick is to form the derivative of $\frac{\partial F}{\partial K}\big|_{(\tau K,\tau L)} = \frac{\partial F}{\partial K}\big|_{(K,L)}$ with respect to K and to proceed analogously with labor.

Turning to the third item, we find

$$\begin{aligned} \frac{\partial F(K,L)}{\partial K} &= \frac{\partial\left[Lf\left(\frac{K}{L}\right)\right]}{\partial K} = L\frac{df}{d\left(\frac{K}{L}\right)}\frac{\partial\left(\frac{K}{L}\right)}{\partial K} \\ &= L\frac{df}{dk}\frac{1}{L} = \frac{df}{dk}. \end{aligned}$$

In case of constant returns, the marginal product of labor is designated by $\omega(k)$. Here, ω is reminiscent of w as in *wage* rate. Indeed, if factors are paid their marginal products, the wage rate of a worker who uses k units of capital is equal to

$$\omega(k) = \underbrace{f(k)}_{\substack{\text{output by one worker}\\ \text{with capital } k}} - \underbrace{\underbrace{k}_{\substack{\text{capital}\\ \text{used by worker}}} \underbrace{\frac{df}{dk}}_{\substack{\text{marginal-product}\\ \text{price for capital}}}}_{\substack{\text{payments for capital}\\ \text{used by worker}}}$$

EXERCISE XVI.18. *Show* $\frac{\partial F}{\partial L} = f(k) - k\frac{df}{dk}$. *Hint: Beginn with* $\frac{\partial F(K,L)}{\partial L} = \frac{\partial\left(Lf\left(KL^{-1}\right)\right)}{\partial L}$ *and apply the product rule of differentiation.*

EXERCISE XVI.19. *Prove Euler's theorem (item* 4*). Hint: You need the results from item* 3.

Euler's theorem claims that factor payments according to marginal products exhaust the total product. Also useful: Euler's theorem implies item 5. Forming the derivative with respect to K and L yield

$$\begin{aligned} \frac{\partial F}{\partial K} &= \left(\frac{\partial^2 F}{(\partial K)^2}K + \frac{\partial F}{\partial K}\right) + \frac{\partial^2 F}{\partial K \partial L}L \text{ and} \\ \frac{\partial F}{\partial L} &= \frac{\partial^2 F}{\partial K \partial L}K + \left(\frac{\partial^2 F}{(\partial L)^2}L + \frac{\partial F}{\partial L}\right) \end{aligned}$$

and hence the desired equalities

$$\begin{aligned} \frac{\partial^2 F}{\partial K \partial L} &= -k\frac{\partial^2 F}{(\partial K)^2} \text{ and} \\ \frac{\partial^2 F}{\partial K \partial L} &= -\frac{1}{k}\frac{\partial^2 F}{(\partial L)^2} \end{aligned}$$

which imply the third one,

$$\frac{\partial^2 F}{(\partial K)^2}\frac{\partial^2 F}{(\partial L)^2} = \left(\frac{\partial^2 F}{\partial K \partial L}\right)^2.$$

This concludes the proof of lemma XVI.3.

8.2. Decreasing marginal productivities and Inada conditions. The second property of neoclassical production functions concern the marginal productivities of the factors. For capital (and analogously, for labor), neoclassical production functions require positive and decreasing marginal productivity:

$$\frac{\partial F}{\partial K} > 0 \text{ for } L > 0, \tag{XVI.20}$$

$$\frac{\partial^2 F}{(\partial K)^2} < 0. \tag{XVI.21}$$

That is, F is a concave function of K (and of L, too). Neoclassical production functions are also defined by the Inada conditions. These require that the marginal product (being positive and decreasing) finally vanishes:

$$\lim_{K \to \infty} \frac{\partial F}{\partial K} = 0 \tag{XVI.22}$$

If, on the other hand, K tends to 0, the marginal product (having increased all the way with decreasing K) gets infinite:

$$\lim_{K \to 0} \frac{\partial F}{\partial K} = \infty \tag{XVI.23}$$

The Inada conditions also require the corresponding properties for labor.

We now need to know whether f inherits these properties from F. Indeed, we find that

- the marginal product per head of capital per head is positive by XVI.20:
$$\frac{df}{dk} = \frac{\partial F(k,1)}{\partial k} > 0, \tag{XVI.24}$$
- the marginal product per head of capital per head decreases by XVI.21:
$$\frac{d^2 f}{(dk)^2} = \frac{\partial^2 F(k,1)}{(\partial k)^2} < 0, \tag{XVI.25}$$
and also
- the Inada conditions hold by XVI.22 and XVI.23:
$$\lim_{k \to \infty} \frac{df}{dk} = \lim_{k \to \infty} \frac{\partial F(k,1)}{\partial k} = 0, \tag{XVI.26}$$
$$\lim_{k \to 0} \frac{df}{dk} = \lim_{k \to 0} \frac{\partial F(k,1)}{\partial k} = \infty. \tag{XVI.27}$$

We have shown above that constant returns to scale imply $F(0,0) = 0$. Furthermore, using the Inada conditions, Barro & Sala-i-Martin (1999, p. 52) show that $F(0,L) = F(K,0) = 0$ hold for any L and K. Hence, we also have

$$f(0) = F(0,1) = 0. \tag{XVI.28}$$

9. Dynamics and steady state (neoclassical production function)

Very similar to the procedure in the first part of this chapter, we derive the dynamics for capital per head. First of all, we have

$$\dot{K}_t = sY_t - \delta K_t \tag{XVI.29}$$

where

- K_t is the economy's stock of capital,
- $\dot{K}_t$ is the change of this stock
- due to sY_t, the share s (=saving rate) of income Y_t invested and
- due to δK_t, the depreciated capital at a depreciation rate δ.

Since output per head y can be expressed as a function of capital per head k, we consider the dynamics of k :

$$\begin{aligned}
\dot{k} &= \dot{\left(\frac{K}{L}\right)} \\
&= \frac{\dot{K}L - \dot{L}K}{L^2} \\
&= \frac{\dot{K}}{L} - nk \\
&= \frac{sY - \delta K}{L} - nk \\
&= sf(k) - (n+\delta)k
\end{aligned} \tag{XVI.30}$$

where n is the (working) population's growth rate. In order to grow (in terms of capital per head), the actual investment per head

$$sf(k)$$

has to make up for both per-head depreciation of capital δk and the dilution effect through population growth nk.

The growth rate of per-capita endowment of capital is given by

$$\gamma_k = \frac{\dot{k}}{k} = s\frac{f(k)}{k} - (\delta + n), k > 0. \tag{XVI.31}$$

In a steady state, growth rates of all relevant economic variables have to be constant so that

$$\frac{sf(k)}{k} - (\delta + n)$$

needs to be constant, too. Therefore, if k is the steady-state capital per head, $\frac{sf(k)}{k} - (\delta + n)$ may not change. Therefore according to the quotient rule and the chain rule,

$$\begin{aligned}
0 &= \frac{d\left[\frac{sf(k)}{k} - (\delta+n)\right]}{dt} \\
&= s\frac{d\frac{f(k)}{k}}{dt} \\
&= s\frac{\frac{df}{dk}\frac{dk}{dt}k - \frac{dk}{dt}f(k)}{k^2} \\
&= s\frac{\frac{df}{dk}k - f(k)}{k}\frac{\frac{dk}{dt}}{k} \\
&= -s\frac{f(k) - \frac{df}{dk}k}{k}\gamma_k \\
&= -s\frac{\frac{\partial F}{\partial L}}{k}\gamma_k \text{ (eq. ??).}
\end{aligned}$$

Since the marginal product of labor is positive, a steady state occurs at $s = 0$ or $\gamma_k = 0$. Leaving aside zero savings for the moment, eq. XVI.31 implies

$$sf(k^*) = (\delta + n)k^* \tag{XVI.32}$$

for the steady-state variable k^*.

Output and consumption per head are also constant,

$$\begin{aligned}
y^* &= f(k^*) \text{ and} \\
c^* &= (1-s)y^*,
\end{aligned}$$

while $K = kL$, $Y = yL$, and $C = cL$ grow at rate n.

We depict the dynamics and the steady state in figures 5 and 6. The first depicts the change in per-head capital. When the actual investment outpasses the break-even investment, capital per head increases by eq. XVI.30. However, a second steady state exists at $k = 0$, with output and investment equal to zero. This steady state is not stable. Indeed, for a small $k > 0$ capital per head increases (if $s > 0$ holds) and converges towards k^*. Fig. 6 shows the development towards k^* by depicting eq. XVI.31.

Both figures assume $s > 0$. If we have $s = 0$, according to eq. XVI.31, the growth rate of capital per head is negative and constant at $-(\delta+n)$.

EXERCISE XVI.20. *Draw the equivalents of figures 5 and 6 for $s = 0$.*

Returning to $s > 0$, we now want to show more formally that $k = 0$ and $k^* > 0$ are the two values of capital per head where the actual investment equals the break-even investment.

- For sufficiently small endowments of capital per head $k > 0$, actual investment per head $sf(k)$ is greater than the break-even investment $(\delta+n)k$ by the Inada condition XVI.27. Hence, $\dot{k} = sf(k) - (n+\delta)k$ is positive and capital per head increases.

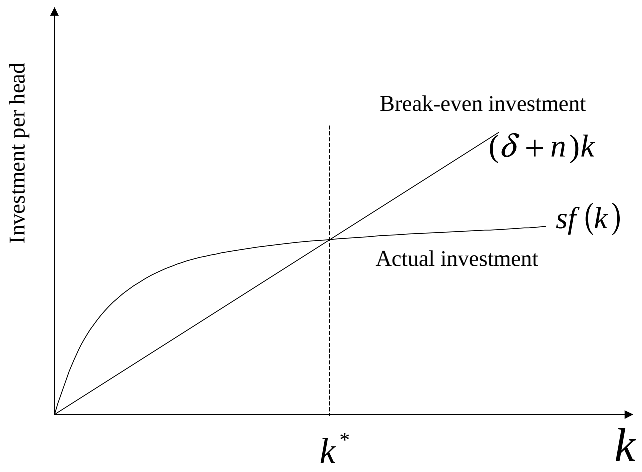

FIGURE 5. Break-even versus actual investment

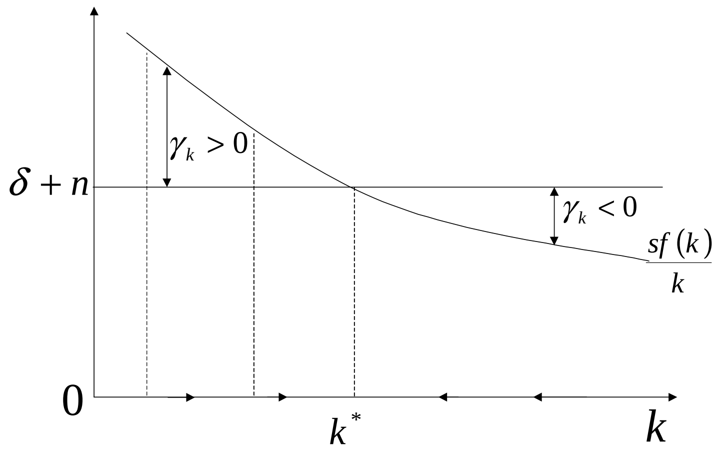

FIGURE 6. Positive and negative growth rates

- For sufficiently large k, actual investment per head is smaller than break-even investment, by the other Inada condition XVI.26. Therefore, capital per head decreases.
- Summarizing, $sf(k) - (n+\delta)k$ is positive for small k and negative for large ones. Therefore, we should find a k^* in between where $sf(k^*) - (n+\delta)k^*$ is zero. This follows from the so-called

intermediate-value theorem which holds for continuous functions. ($sf(k) - (n+\delta)k$ is continuous for $k > 0$.)
- $k = 0$ is a steady state by $f(0) = 0$ (see p. 298).
- Finally, f and hence $sf(k) - (n+\delta)k$ is concave by XVI.25 so that further nulls are excluded.

10. Comparative statics and the golden rule (neoclassical production function)

In general, the comparative-statics results do not change by considering a general neoclassical production function instead of a CD production function. In particular, k^* is a positive function of s ($dk^*/ds > 0$) which is clear from fig. 5.

Starting from

$$\begin{aligned} c^*(s) &= (1-s) f(k^*(s)) \\ &= f(k^*(s)) - (\delta+n) k^*(s) \;\; \text{(eq. XVI.32)} \end{aligned}$$

we maximize consumption per head by

$$\begin{aligned} & f'(k^*(s)) \frac{dk^*}{ds} - (\delta+n) \frac{dk^*}{ds} \overset{!}{=} 0 \\ \Leftrightarrow \quad & f'(k^*(s)) \overset{!}{=} (\delta+n) \;\; \left(\text{note } \frac{dk^*}{ds} > 0\right). \end{aligned}$$

We call

$$f'(k_{gold}) \overset{!}{=} \delta + n$$

the golden rule of capital accumulation, depicted in fig. 7. Indeed, the slope of $f(k^*(s))$ has to be equal to the slope of $(\delta+n) k^*(s)$ because steady-state consumption c^* is the difference of output per head $f(k^*(s))$ and (break-even) investment $(\delta+n) k^*$.

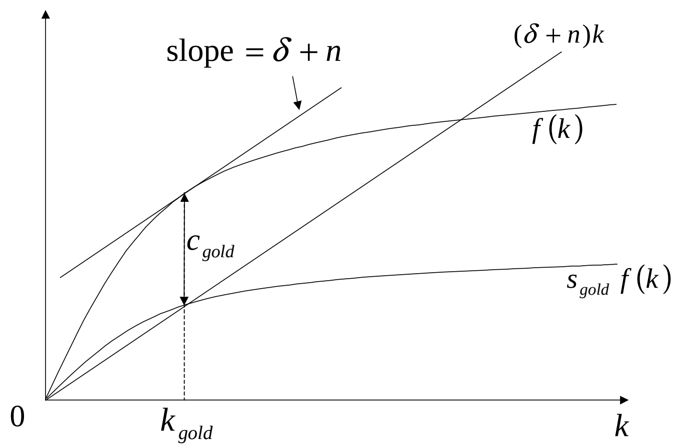

FIGURE 7. The golden rule of capital accumulation

11. Topics and literature

The main topics in this chapter are

- Solow model
- depreciation
- growth rate

-

We recommend the textbook

12. Solutions

Exercise XVI.1

We obtain

$$\begin{aligned} \frac{x_{t+1} - x_t}{x_t} &= \frac{t + 1 - t}{t} = \frac{1}{t} \\ \frac{y_{t+1} - y_t}{y_t} &= \frac{t + 5 - (t + 4)}{t + 4} = \frac{1}{t + 4} < \frac{1}{t} \text{ and} \\ \frac{z_{t+1} - z_t}{y_t} &= \frac{100\,(t + 1) - 100t}{100t} = \frac{1}{t} \end{aligned}$$

Exercise XVI.2

You have found

$$\begin{aligned}
\frac{\frac{dy_t}{dt}}{y_t} &= \frac{\frac{d\left(y_0 e^{gt}\right)}{dt}}{y_0 e^{gt}} \\
&= \frac{y_0 e^{gt} g}{y_0 e^{gt}} \\
&= g.
\end{aligned}$$

If another result were correct, our solution to the differential equation would have been false.

Exercise XVI.3

Using the original definition, we obtain

$$\begin{aligned}
\gamma_Y &= \frac{\dot{Y}_t}{Y_t} = \frac{\frac{d(K_t L_t)}{dt}}{K_t L_t} \\
&= \frac{\frac{dK_t}{dt} L_t + \frac{dL_t}{dt} K_t}{K_t L_t} \\
&= \frac{\frac{dK_t}{dt}}{K_t} + \frac{\frac{dL_t}{dt}}{L_t} \\
&= \gamma_K + \gamma_L.
\end{aligned}$$

Using the logarithm, we have

$$\begin{aligned}
\gamma_Y &= \frac{d \ln Y_t}{dt} \\
&= \frac{d \ln (K_t L_t)}{dt} \\
&= \frac{d (\ln K_t + \ln L_t)}{dt} \\
&= \frac{d \ln K_t}{dt} + \frac{d \ln L_t}{dt} \\
&= \gamma_K + \gamma_L.
\end{aligned}$$

Exercise XVI.4

Applying the natural logarithm on both sides of the equation yields

$$\begin{aligned}
\ln y_t &= \ln y_0 + \ln e^{\gamma_y t} \\
&= \ln y_0 + \gamma_y t
\end{aligned}$$

and hence

$$\gamma_y = \frac{\ln y_t - \ln y_0}{t - 0} = \frac{\ln \frac{y_t}{y_0}}{t - 0}.$$

Exercise XVI.5

Obviously, $y_0 > x_0$. Now,

$$\begin{aligned}
\gamma_y &= \frac{2}{2t+2} \\
&= \frac{1}{t+1} \text{ (multiply by } \frac{1/2}{1/2}) \\
&< \frac{1}{t} \\
&= \gamma_x
\end{aligned}$$

Exercise XVI.6

While x and y converge in a weak sense, they do not in a strong sense:

$$\begin{aligned}
&\lim_{t\to\infty} \frac{2t+2}{t} \\
&= \lim_{t\to\infty} \left(2 + \frac{2}{t}\right) \\
&= 2 + \lim_{t\to\infty} \frac{2}{t} \\
&= 2 > 1
\end{aligned}$$

$\frac{y_t}{x_t}$ decreases (by weak convergence), but $y_t > 2x_t$ for all t.

Exercise XVI.7

Constant returns to scale are easy to show:

$$\begin{aligned}
F(\tau K, \tau L) &= (\tau K)^{\alpha} (\tau L)^{1-\alpha} \\
&= \tau^{\alpha} K^{\alpha} \tau^{1-\alpha} L^{1-\alpha} \\
&= \tau^{\alpha} \tau^{1-\alpha} K^{\alpha} L^{1-\alpha} \\
&= \tau F(K, L).
\end{aligned}$$

Exercise XVI.8

$\frac{\partial Y}{\partial K}$ is just another expression of $\frac{\partial F}{\partial K}$, therefore

$$\begin{aligned}
\varepsilon_{Y,K} &= \frac{\partial F}{\partial K} \frac{K}{Y} \\
&= \alpha \left(\frac{L}{K}\right)^{1-\alpha} \frac{K}{K^{\alpha} L^{1-\alpha}} \\
&= \alpha.
\end{aligned}$$

Exercise XVI.9

You have found (haven't you?)

$$\begin{aligned}
\frac{\partial F}{\partial K}\cdot K+\frac{\partial F}{\partial L}\cdot L &= \alpha\left(\frac{L}{K}\right)^{1-\alpha}\cdot K+(1-\alpha)\left(\frac{K}{L}\right)^{\alpha}\cdot L \\
&= \left[\alpha\frac{L^{1-\alpha}}{K^{1-\alpha}}\cdot K+(1-\alpha)\frac{K^{\alpha}}{L^{\alpha}}\cdot L\right] \\
&= \left[\alpha K^{\alpha}L^{1-\alpha}+(1-\alpha)K^{\alpha}L^{1-\alpha}\right] \\
&= F(K,L).
\end{aligned}$$

Exercise XVI.10

For $Y_t = F(K_t, L_t) = K_t^{\alpha} L_t^{1-\alpha}$, we find

$$\begin{aligned}
\gamma_Y &= \frac{\frac{dY_t}{dt}}{Y_t} \\
&= \frac{\frac{d\left(K_t^{\alpha}L_t^{1-\alpha}\right)}{dt}}{K_t^{\alpha}L_t^{1-\alpha}} \\
&= \frac{\left[\alpha K_t^{\alpha-1}\frac{dK}{dt}L_t^{1-\alpha}+K_t^{\alpha}(1-\alpha)L_t^{-\alpha}\frac{dL}{dt}\right]}{K_t^{\alpha}L_t^{1-\alpha}} \text{ (product rule and chain rule)} \\
&= \alpha\frac{\frac{dK}{dt}}{K_t}+(1-\alpha)\frac{\frac{dL}{dt}}{L_t} \\
&= \alpha\gamma_K+(1-\alpha)\gamma_L.
\end{aligned}$$

Alternatively, we have

$$\begin{aligned}
\gamma_Y &= \frac{d\ln Y_t}{dt} \\
&= \frac{d\ln\left(K_t^{\alpha}L_t^{1-\alpha}\right)}{dt} \\
&= \frac{d\left(\alpha\ln K_t+(1-\alpha)\ln L_t\right)}{dt} \\
&= \frac{d\left(\alpha\ln K_t\right)}{dt}+\frac{d\left((1-\alpha)\ln L_t\right)}{dt} \\
&= \alpha\frac{d\ln K_t}{dt}+(1-\alpha)\frac{d\ln L_t}{dt} \\
&= \alpha\gamma_K+(1-\alpha)\gamma_L.
\end{aligned}$$

Exercise XVI.11

Since K is the product of k and L, we have

$$\gamma_K = \gamma_k + \gamma_L = 0 + n = n.$$

Analogously, we obtain $\gamma_Y = \gamma_C = n$.

Exercise XVI.12

We need to show that

$$k_t = \left[\frac{s}{n+\delta} + e^{(1-\alpha)(-n-\delta)t}\left(k_0^{1-\alpha} - \frac{s}{n+\delta}\right)\right]^{\frac{1}{1-\alpha}} \tag{XVI.33}$$

solves the differential equation

$$\dot{k} = sk^{\alpha} \; - (n+\delta)k. \tag{XVI.34}$$

We define

$$E(t) := \frac{s}{n+\delta} + e^{(1-\alpha)(-n-\delta)t}\left(k_0^{1-\alpha} - \frac{s}{n+\delta}\right).$$

Then, we have

$$\begin{aligned} k_t &= (E(t))^{\frac{1}{1-\alpha}} \text{ and} && \text{(XVI.35)} \\ E(t) &= (k_t)^{1-\alpha} && \text{(XVI.36)} \end{aligned}$$

Before forming the derivative of eq. XVI.33 with respect to t, we take note of the following:

(1) Eq. XVI.33 can be rewritten:

$$k_t^{1-\alpha} = e^{(1-\alpha)(-n-\delta)t}\left(k_0^{1-\alpha} - \frac{s}{n+\delta}\right) + \frac{s}{n+\delta} \tag{XVI.37}$$

and

$$e^{(1-\alpha)(-n-\delta)t} \cdot k_0^{1-\alpha} = k_t^{1-\alpha} + \frac{s}{n+\delta}\left(e^{(1-\alpha)(-n-\delta)t} - 1\right). \tag{XVI.38}$$

(2) It is easy to see

$$\frac{1}{1-\alpha} - 1 = \frac{1}{1-\alpha} - \frac{1-\alpha}{1-\alpha} = \frac{\alpha}{1-\alpha} \tag{XVI.39}$$

(3) Finally, forming the derivative of the term in square brackets of XVI.33 yields

$$\begin{aligned} & \frac{dE(t)}{dt} \\ = \; & (1-\alpha)(-n-\delta)e^{(1-\alpha)(-n-\delta)t}\left(k_0^{1-\alpha} - \frac{s}{n+\delta}\right) \\ = \; & e^{(1-\alpha)(-n-\delta)t}k_0^{1-\alpha}(1-\alpha)(-n-\delta) \\ & +(1-\alpha)e^{(1-\alpha)(-n-\delta)t}s && \text{(XVI.40)} \end{aligned}$$

Now, we form the derivative of k_t with respect to t and obtain the desired result:

$$\begin{aligned}
\frac{dk_t}{dt} &= \frac{1}{1-\alpha}(E(t))^{\frac{1}{1-\alpha}-1}\cdot\frac{dE(t)}{dt} \text{ (chain rule, applied to XVI.35)}\\
&= \frac{1}{1-\alpha}(E(t))^{\frac{\alpha}{1-\alpha}}\frac{dE(t)}{dt} \text{ (XVI.39)}\\
&= k_t^\alpha\cdot\left[e^{(1-\alpha)(-n-\delta)t}k(0)^{1-\alpha}(-n-\delta)+e^{(1-\alpha)(-n-\delta)t}s\right] \text{ (XVI.36,XVI.40)}\\
&= k_t^\alpha\cdot\left[\left(k_t^{1-\alpha}+\frac{s}{n+\delta}\left(e^{(1-\alpha)(-n-\delta)t}-1\right)\right)(-n-\delta)+e^{(1-\alpha)(-n-\delta)t}s\right] \text{ (XVI.38)}\\
&= k_t^\alpha\cdot\left[k_t^{1-\alpha}(-n-\delta)-s\left(e^{(1-\alpha)(-n-\delta)t}-1\right)+e^{(1-\alpha)(-n-\delta)t}s\right]\\
&= k_t^\alpha\cdot\left[k_t^{1-\alpha}(-n-\delta)-se^{(1-\alpha)(s-n-\delta)t}+s+e^{(1-\alpha)(-n-\delta)t}s\right]\\
&= k_t^\alpha\cdot\left[k_t^{1-\alpha}(-n-\delta)+s\right]\\
&= k_t(-n-\delta)+sk(t)^\alpha
\end{aligned}$$

Exercise XVI.13

For any $\tau\geq 0$, we obtain

$$\begin{aligned}
F(\tau K,\tau L) &= \left[\alpha(\tau K)^{-\rho}+(1-\alpha)(\tau L)^{-\rho}\right]^{-1/\rho}\\
&= \left[\alpha\tau^{-\rho}K^{-\rho}+(1-\alpha)\tau^{-\rho}L^{-\rho}\right]^{-1/\rho}\\
&= \left(\tau^{-\rho}\left[\left(\alpha K^{-\rho}+(1-\alpha)L^{-\rho}\right)\right]\right)^{-1/\rho}\\
&= \left(\tau^{-\rho}\right)^{-1/\rho}\left[\left(\alpha K^{-\rho}+(1-\alpha)L^{-\rho}\right)\right]^{-1/\rho}\\
&= \tau^{-\rho\cdot(-1/\rho)}F(K,L)\\
&= \tau F(K,L)
\end{aligned}$$

and confirm that F is constant-returns.

Exercise XVI.14

First, we note $F(0\cdot K,0\cdot L)=0\cdot F(K,L)$, so that the equality holds for $\tau=0$. For $\tau>0$, we have

$$AK\leq BL\Leftrightarrow\tau(AK)\leq\tau(BL)$$

and hence

$$\begin{aligned}
F(\tau K,\tau L) &= \min(A(\tau K),B(\tau L))\\
&= \min(\tau(AK),\tau(BL))\\
&= \tau\min(AK,BL).
\end{aligned}$$

Exercise XVI.15

For $\tau:=0$, the desired equation follows easily:

$$F(0,0)=F(0\cdot K,0\cdot L)=0\cdot F(K,L)=0$$

Exercise XVI.16

The intensive form of the CES production function is given by

$$\begin{aligned}
f(k) &= F\left(\frac{K}{L},1\right) \\
&= \left[\alpha\left(\frac{K}{L}\right)^{-\rho} + (1-\alpha)\cdot 1^{-\rho}\right]^{-1/\rho} \\
&= \left[\alpha k^{-\rho} + (1-\alpha)\right]^{-1/\rho}
\end{aligned}$$

Exercise XVI.17

The derivative of $F(\tau K, \tau L) = \tau F(K, L)$ with respect to K yields

$$\begin{aligned}
& \frac{\partial F(\tau K,\tau L)}{\partial K} = \frac{\partial\left[\tau F(K,L)\right]}{\partial K} \\
\Leftrightarrow\ & \frac{\partial F(\tau K,\tau L)}{\partial(\tau K)}\frac{d(\tau K)}{dK} = \tau\frac{\partial\left[F(K,L)\right]}{\partial K} \\
\Leftrightarrow\ & \frac{\partial F(\tau K,\tau L)}{\partial(\tau K)} = \frac{\partial\left[F(K,L)\right]}{\partial K} \\
\Leftrightarrow\ & \left.\frac{\partial F}{\partial K}\right|_{(\tau K,\tau L)} = \left.\frac{\partial F}{\partial K}\right|_{(K,L)}.
\end{aligned}$$

Analogously, forming the derivative with respect to L leads to

$$\begin{aligned}
& \frac{\partial F(\tau K,\tau L)}{\partial L} = \frac{\partial\left[\tau F(K,L)\right]}{\partial L} \\
\Leftrightarrow\ & \left.\frac{\partial F}{\partial L}\right|_{(\tau K,\tau L)} = \left.\frac{\partial F}{\partial L}\right|_{(K,L)}.
\end{aligned}$$

Exercise XVI.18

You have found

$$\begin{aligned}
\frac{\partial F(K,L)}{\partial L} &= \frac{\partial\left(Lf\left(KL^{-1}\right)\right)}{\partial L} \\
&= f\left(KL^{-1}\right) + L\frac{\partial f}{\partial\left(KL^{-1}\right)}\frac{d\left(KL^{-1}\right)}{dL} \\
&= f(k) + L\frac{\partial f}{\partial k}(-1)KL^{-2} \\
&= f(k) - \frac{df}{dk}k.
\end{aligned}$$

Exercise XVI.19

Euler's theorem:

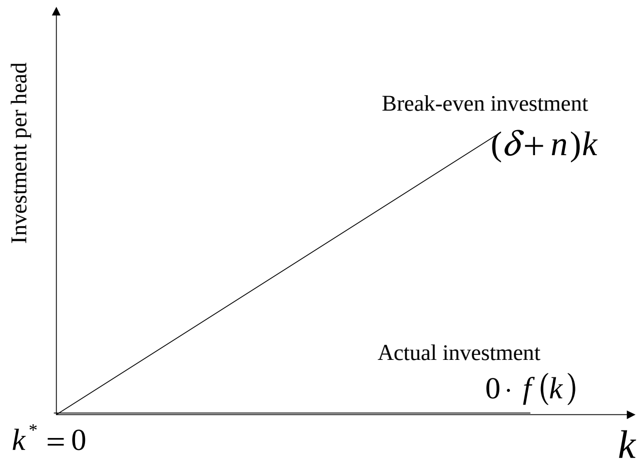

FIGURE 8. Break-even versus actual investment for zero savings

$$\begin{aligned}
\frac{\partial F}{\partial K}K + \frac{\partial F}{\partial L}L &= \frac{df}{dk}K + \left(f(k) - k\frac{df}{dk} \right) L \\
&= \frac{df}{dk}K + f(k)L - \frac{K}{L}\frac{df}{dk}L \\
&= Lf(k) \\
&= F(K, L).
\end{aligned}$$

Exercise XVI.20

Compare fig. 8 and 9.

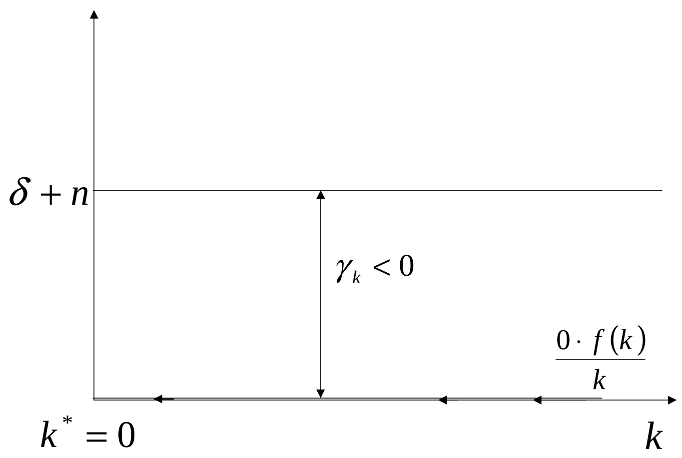

FIGURE 9. Positive and negative growth rates

13. Further exercises without solutions

CHAPTER XVII

Growth theory without constant returns

1. Introduction

[1]The standard Solow (1956) model uses a constant-returns production function in order to trace the capital-per-head trajectory in terms of the rate of saving (s), the depreciation rate (δ), the growth rate of the (working) population (n) and the initial capital per head (k_0). We present a Solow-type model where the population is divided in m groups which differ in s, δ, n, and k_0. It is an easy exercise to present a suchlike model if returns to scale are constant. In that case, paying factors their marginal products with respect to both labor and capital exhausts the total product (Euler's theorem). In our paper, we allow for non-constant returns to scale. In order to solve the exhaustion problem, we apply the (Pareto efficient!) Shapley value.

This chapter has a static and a dynamic part. In the first, static, part, we consider a population of worker-capitalists. They contribute one unit of labour and a given amount of capital and get paid their Shapley value. The Shapley value makes use of so-called marginal contributions which are somewhat similar to marginal products. For tractability reasons, we deal with atomless populations and need to introduce the continuous Shapley value, proposed by Aumann & Shapley (1974). We find that the continuous Shapley value is equal to marginal-product payment in the special case of constant returns.

We apply the Shapley outcome to the specific question of when immigration into an economy is welcomed by the incumbent groups. We have two results. First, immigration of both labor and capital (capital imports) is welcomed only in case of increasing returns to scale. Second, the relative capital richness of the incumbent groups are decisive for their attitude towards immigration. In particular, capital-rich groups are more welcoming towards labor immigration and capital-poor groups are more welcoming towards capital immigration.

In the dynamic, growth theory, part of our paper, we use the static results to derive the m-group dynamics for production function which is Cobb-Douglas but not (in general) constant-returns. In the case of one

[1]This chapter is part of a joint project together with Andre Casajus and Thomas Steger.

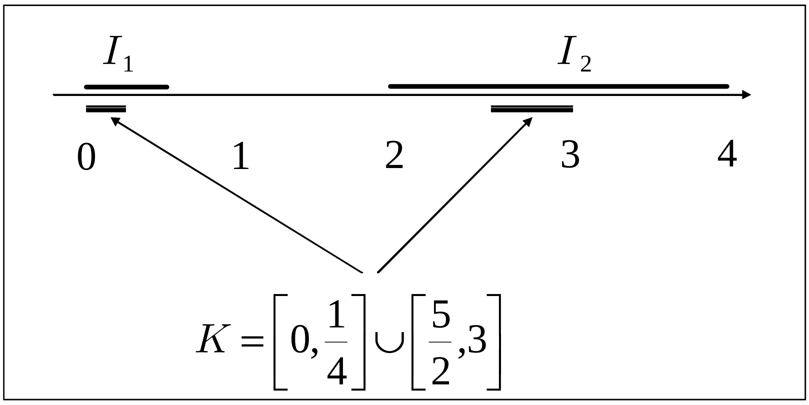

FIGURE 1. Attributing measures to players

group only, we can calculate the steady state. In the general case with more than one group, a steady state does not exists. However, we are able to characterize an approximate steady state.

2. The static case

2.1. The population and its structure. Our economy is not populated by the n players but by m intervals of workers. Assume any vector $\vec{L} = (L_1, ..., L_m) \in \mathbb{R}_+^m$ where L_i is the number of workers belonging to group i. Let λ be the Lebesgues-Borel measure on $\mathbb{R}$. We now choose m intervals $I_i \subseteq \mathbb{R}$, such that $\lambda(I_i) = L_i$ holds for every $i = 1, ..., m$ and such that the intervals do not intersect. Thus, I_i stands for the workers of group i and $I := \cup_{i=1}^m I_i$ is the set of all workers with cardinality $L := \sum_{i=1}^m L_i = \lambda(I)$. By $\mathcal{B}$ we mean the set of Borel sets of I. We now define $\mu_i^{\vec{L}}$ by

$$\mu_i^{\vec{L}}(K) := \lambda(K \cap I_i), K \in \mathcal{B}.$$

It is easy to show that $\mu_i^{\vec{L}}$ is a measure on $(I, \mathcal{B})$. Let $\mu^{\vec{L}} = \prod_{i \in N} \mu_i^{\vec{L}} :$ $\mathcal{B} \to \mathbb{R}^N$, $K \mapsto \left(\mu_i^{\vec{L}}(K)\right)_{i \in N}$ be the Cartesian product of these measures. $\mu(K)$ distributes the agents in K among the m groups and attributes a size to each group.

EXAMPLE XVII.1. *Consider fig. 1. We have* $N = \{1, 2\}$, $\vec{L} = \left(\frac{1}{2}, 2\right)$ *and intervals* $I_1 = \left[0, \frac{1}{2}\right]$ *and* $I_2 = [2, 4]$. *For* $K := \left[0, \frac{1}{4}\right] \cup \left[\frac{5}{2}, 3\right] \in \mathcal{B}$ *we obtain*

$$\begin{aligned} \mu_1(K) &= \lambda(K \cap I_1) = \lambda\left(\left[0, \frac{1}{4}\right]\right) = \frac{1}{4} \text{ and} \\ \mu_2(K) &= \lambda(K \cap I_2) = \lambda\left(\left[\frac{5}{2}, 3\right]\right) = \frac{1}{2}. \end{aligned}$$

2.2. Factors of production and production function. The m groups differ with respect to the capital they own. Let k_i be the amount of capital owned by a worker of group i. The amount of capital is equal to

$$\begin{aligned} K_i &: = k_i L_i \text{ for group } i, \text{ and} \\ K &: = \sum_{i=1}^{m} k_i L_i \text{ for all groups together.} \end{aligned}$$

We now define

$$\begin{aligned} g &= (g_K, g_L) : \mathbb{R}_+^m \to \mathbb{R}^2, \\ & (L_1, ..., L_m) \mapsto \left(\sum_{i=1}^{m} k_i L_i, \sum_{i=1}^{m} L_i \right). \end{aligned}$$

$(k_i L_i, L_i)$ is the capital and labor available to group i of size L_i.

For a production function $Y = F(K, L)$,

$$\begin{aligned} F \circ g &: \mathbb{R}_+^m \to \mathbb{R}, \\ & (L_1, ..., L_m) \mapsto F(g_K(L_1, ..., L_m), g_L(L_1, ..., L_m)) \end{aligned}$$

yields the output producible by the m groups of size $L_1, ..., L_m$.

2.3. Vector measure game and Shapley value. Before we can apply the continuous Shapley value, we need to define the appropriate vector measure game. We choose

$$v := F \circ g \circ \mu : \mathcal{B} \to \mathbb{R}$$

Given a coalition $S \in \mathcal{B}$, $\mu(S)$ specifies how to devide S among the m groups, g shows the labor and capital available to all those groups and F yields the product. If F is continuously differentiable, so is $F \circ g$. If $F(0,0) = 0$, we also have $(F \circ g)(0) = 0$. From now on, we assume that F has these two properties. We can then apply the continuous Shapley value (the so-called diagonal formula) as proposed by Aumann & Shapley (1974, p. 23) (see also Neyman 2002, pp. 2141). For $S \in \mathcal{B}$, it is given by

$$\begin{aligned} Sh(v)(S) &= \sum_{j=1}^{m} \mu_j(S) \int_0^1 \left. \frac{\partial (F \circ g)}{\partial L_j} \right|_{(\tau\mu_1(I), ..., \tau\mu_m(I))} d\tau \\ &= \sum_{j=1}^{m} \lambda(S \cap I_j) \int_0^1 \left. \frac{\partial (F \circ g)}{\partial L_j} \right|_{(\tau L_1, ..., \tau L_m)} d\tau. \end{aligned}$$

The analogue of a player j's marginal contribution in the discrete Shapley formula is the derivative of the coalition's worth with respect to the measure of agents of player j. This derivative is evaluated at $\tau\vec{L} = (\tau L_1, ..., \tau L_m)$. Thus, the formula looks at coalitions on the diagonal only. Remember that we have a continuum of agents. If we take a subset of agents by chance, it is likely that the composition in this subset (how many agents of player 1, player 2 etc.) does not deviate much from the composition in the overall population (see Aumann & Shapley 1974, pp. 23).

We now show that the Shapley value coincides with payments according to the marginal products if F is of constant returns. We prepare the following lemma by three observations.

(1) As is known from the previous chapter, lemma XVI.4 on p. 296, homogeneity of degree d (constant returns remain degree 1) of a production function implies homogeneity of degree $d-1$ of the marginal productivities. Formally:

$$\left.\frac{\partial F}{\partial g_K}\right|_{(\tau K,\tau L)} = \tau^{d-1} \left.\frac{\partial F}{\partial g_K}\right|_{(K,L)} \tag{XVII.1}$$

(2) Therefore, we can write the derivative $\left.\frac{\partial(F\circ g)}{\partial L_j}\right|_{(\tau L_1,\ldots,\tau L_m)} d\tau$ featuring in the diagonal formula as

$$\begin{aligned}
& \left.\frac{\partial F}{\partial g_K}\right|_{(g_K(\tau L_1,\ldots,\tau L_m),g_L(\tau L_1,\ldots,\tau L_m))} \left.\frac{\partial g_K}{\partial L_i}\right|_{(\tau L_1,\ldots,\tau L_m)} + \left.\frac{\partial F}{\partial g_L}\right|_{(g_K(\tau L_1,\ldots,\tau L_m),g_L(\tau L_1,\ldots,\tau L_m))} \left.\frac{\partial g_L}{\partial L_i}\right|_{(\tau L} \\
= & \left.\frac{\partial F}{\partial g_K}\right|_{\left(\tau\sum_{j=1}^m k_j L_j,\tau\sum_{j=1}^m L_j\right)} k_i + \left.\frac{\partial F}{\partial g_L}\right|_{\left(\tau\sum_{j=1}^m k_j L_j,\tau\sum_{j=1}^m L_j\right)} \quad \text{(eq. XVII.1)} \\
= & \ \tau^{d-1} \left.\frac{\partial F}{\partial g_K}\right|_{\left(\sum_{j=1}^m k_j L_j,\sum_{j=1}^m L_j\right)} k_i + \tau^{d-1} \left.\frac{\partial F}{\partial g_L}\right|_{\left(\sum_{j=1}^m k_j L_j,\sum_{j=1}^m L_j\right)} \\
= & \ \tau^{d-1} \left[\left.\frac{\partial F}{\partial g_K}\right|_{(K,L)} k_i + \left.\frac{\partial F}{\partial g_L}\right|_{(K,L)} \right].
\end{aligned}$$

(3) Finally, we calculate the integral

$$\int_0^1 \tau^{d-1} d\tau = \left.\frac{1}{d}\tau^d\right|_0^1 = \frac{1}{d}. \tag{XVII.2}$$

Now, for $S := I_i$, we find

$$\begin{aligned}
\frac{Sh(v)(I_i)}{L_i} &= \frac{1}{L_i}\sum_{j=1}^m \lambda(I_i\cap I_j) \int_0^1 \left.\frac{\partial(F\circ g)}{\partial L_j}\right|_{(\tau L_1,\ldots,\tau L_m)} d\tau \\
&= \int_0^1 \tau^{d-1} \left[\left.\frac{\partial F}{\partial g_K}\right|_{(K,L)} k_i + \left.\frac{\partial F}{\partial g_L}\right|_{(K,L)} \right] d\tau \ \text{(step 2)} \\
&= \left[\left.\frac{\partial F}{\partial g_K}\right|_{(K,L)} k_i + \left.\frac{\partial F}{\partial g_L}\right|_{(K,L)} \right] \int_0^1 \tau^{d-1} d\tau \\
&= \left[\left.\frac{\partial F}{\partial g_K}\right|_{(K,L)} k_i + \left.\frac{\partial F}{\partial g_L}\right|_{(K,L)} \right] \frac{1}{d} \ \text{(eq. XVII.2)}
\end{aligned}$$

so that, indeed, the Shapley value of an average agent in I_i is equal to his marginal product payment in case of constant returns. In general, if the production function is homogeneous of degree d, the Shapley value accruing to an agent in group i is equal to the marginal product payment devided by d. In this sense, the Shapley value provides an argument to renumerate workers and capital proportional to the marginal products.

We now turn to the production function given by

$$Y = F(K, L) = K^{\alpha} L^{\beta},\ 0 < \alpha, \beta$$

which is homogeneous of degree $d := \alpha + \beta$. We obtain the vector measure game given by

$$v(S) = \left(\sum_{i=1}^{m} k_i \lambda (S \cap I_i) \right)^{\alpha} (\lambda (S))^{\beta},$$

and the Shapley value for group i

$$\begin{aligned} Y_i &:= Sh(v)(I_i) \\ &= L_i \left[\left. \frac{\partial F}{\partial g_K} \right|_{(K,L)} k_i + \left. \frac{\partial F}{\partial g_L} \right|_{(K,L)} \right] \frac{1}{\alpha + \beta} \\ &= L_i \left[\frac{\alpha}{\alpha + \beta} K^{\alpha - 1} L^{\beta} k_i + \frac{\beta}{\alpha + \beta} K^{\alpha} L^{\beta - 1} \right] \qquad \text{(XVII.3)} \end{aligned}$$

EXERCISE XVII.1. *Simplify*

$$Y_i = L_i \left[\frac{\alpha}{\alpha + \beta} K^{\alpha - 1} L^{\beta} k_i + \frac{\beta}{\alpha + \beta} K^{\alpha} L^{\beta - 1} \right]$$

for the one-group case ($m = 1$, *dropping the* i*-index) and comment!*

3. Labor and capital immigration

Let us now examine the question of whether incumbent agents (of group i) welcome additional groups, via immigration. To fix ideas, we assume that a "small" group $m + 1$ joins the economy.

DEFINITION XVII.1. *We have labor immigration into an economy if* $L_{m+1} > 0$ *and* $k_{m+1} = 0$ *hold. We have capital immigration (capital imports) into an economy if* $L_{m+1} = 0$, $k_{m+1} > 0$ *hold.*

Note that capital imports imply that Y_{m+1} is paid to agents outside our economy. For practical purposes, however, there is no harm in assuming some very small population size $L_{m+1} > 0$ earning Y_{m+1}.

DEFINITION XVII.2. *Group* i *is said to be welcoming towards labor (capital) immigration if* $\frac{dY_i}{dL} > 0$ ($\frac{dY_i}{dK} > 0$) *holds.*

Note that consumption is defined by $C_i := (1 - s_i) Y_i$ so that "welcoming" could equivalently be defined with respect to consumption. We find

$$\begin{aligned} \frac{dY_i}{dK} &> 0 \text{ in case of } \frac{K}{L} > \frac{1 - \alpha}{\beta} k_i \text{ and} \\ \frac{dY_i}{dL} &> 0 \text{ in case of } k_i > \frac{1 - \beta}{\alpha} \frac{K}{L}. \end{aligned}$$

PROPOSITION XVII.1. *Considering immigration at time into an economy, we find:*

- *Group i benefits from labor immigration if $k_i > \frac{1-\beta}{\alpha}\frac{K}{L}$ holds. In particular, if group i is as capital-rich as the economy as a whole ($k_i = \frac{K}{L}$), group i is welcoming towards labor immigration iff increasing returns to scale hold. The more capital-rich a group is, the more welcoming towards labor immigration it is. In case of constant returns to scale, the more-than-average capital-rich groups welcome labor immigration while the less-than-average capital-rich groups oppose labor immigration.*
- *Group i benefits from capital immigration if $\frac{K}{L} > \frac{1-\alpha}{\beta}k_i$ holds. In particular, if group i is as capital-rich as the economy as a whole ($k_i = \frac{K}{L}$), group i is welcoming towards capital immigration iff increasing returns to scale hold. The more capital-rich a group is, the less welcoming towards capital immigration it is. In case of constant returns to scale, the more-than-average capital-rich groups oppose capital immigration while the less-than-average capital-rich groups welcome capital immigration.*
- *Increasing returns to scale are a necessary condition for any group to be welcoming to both capital and labor.*

It may be interesting to speculate about whether some group i benefitting from immigration will later come to regret it. Consider, for example, a capital-rich group that welcomes labor immigration but opposes capital immigration. If an initially capital-poor group has a high rate of saving, group i may finally be harmed.

4. Dynamics of per-head capital endowment

In order to treat the dynamics, we add the time index and denote by

- I_{it} the set of group-i workers at time t,
- $I_t : \cup_{i=1}^m I_{it}$ the set of all workers at time t,
- $L_{it} := \lambda(I_{it})$ the size of group i at time t,
- $L_t := \sum_{i=1}^m L_{it}$ the size of the overall working population at time t,
- $\vec{L}_t = (L_{1t}, ..., L_{mt}) \in \mathbb{R}_+^m$ the population size vector at time t,
- $\mathcal{B}_t$ the set of Borel sets of I_t, and
- $\mu_{it}^{\vec{L}_t} : \mathcal{B}_t \to \mathbb{R}_+^m$ the measure on $(I_t, \mathcal{B}_t)$ defined by $\mu_{it}^{\vec{L}_t}(S) := \lambda(S \cap I_{it}), S \in \mathcal{B}_t$.

The m groups grow at rates $n_1, ..., n_m$, respectively.

Taking depreciation into account, the capital stock of group i develops in accordance with

$$\begin{aligned}\frac{d\left(L_{it}k_{it}\right)}{dt} &= \frac{dK_{it}}{dt} = s_i Y_{it} - \delta_i K_{it} \\ &= s_i L_{it}\left[\frac{\alpha}{\alpha+\beta}K_t^{\alpha-1}L_t^{\beta}k_{it} + \frac{\beta}{\alpha+\beta}K_t^{\alpha}L_t^{\beta-1}\right] - \delta_i L_{it}k_{it}.\end{aligned}$$

Now, because of

$$\begin{aligned}\frac{\frac{d(L_{it}k_{it})}{dt}}{L_{it}} &= \frac{dk_{it}}{dt} + k_{it}\frac{\frac{dL_{it}}{dt}}{L_{it}} \\ &= \frac{dk_{it}}{dt} + k_{it}n_i\end{aligned}$$

we obtain

$$\frac{dk_{it}}{dt} = \frac{\frac{d(L_{it}k_{it})}{dt}}{L_{it}} - k_{it}n_i = s_i\left[\frac{\alpha}{\alpha+\beta}K_t^{\alpha-1}L_t^{\beta}k_{it} + \frac{\beta}{\alpha+\beta}K_t^{\alpha}L_t^{\beta-1}\right] - \left(\delta_i + n_i\right)k_{it}$$

and hence

$$\frac{\dot{k}_{it}}{k_{it}} = s_i\left[\frac{\alpha}{\alpha+\beta}\frac{L_t^{\beta}}{K_t^{1-\alpha}} + \frac{\beta}{\alpha+\beta}\frac{K_t^{\alpha}}{L_t^{1-\beta}}\cdot\frac{1}{k_{it}}\right] - \left(\delta_i + n_i\right). \qquad \text{(XVII.4)}$$

In the one-group case ($m = 1$, dropping the i-index), we obtain

$$\frac{\dot{k}_t}{k_t} = s\frac{L_t^{\beta}}{K_t^{1-\alpha}} - (\delta + n) \qquad \text{(XVII.5)}$$

If, on top, we assume $\alpha+\beta = 1$ (neoclassical production function), we obtain the well-known Solow equation

$$\frac{\dot{k}_t}{k_t} = \frac{s}{k_t^{1-\alpha}} - (\delta + n)$$

For two groups i and j with $k_{it} > k_{jt}$, we obtain the difference in growth rates of capital per head

$$\begin{aligned}\frac{\dot{k}_{it}}{k_{it}} - \frac{\dot{k}_{jt}}{k_{jt}} &= \left(s_i - s_j\right)\frac{\alpha}{\alpha+\beta}\frac{L_t^{\beta}}{K_t^{1-\alpha}} + \left(\frac{s_i}{k_{it}} - \frac{s_j}{k_{jt}}\right)\frac{\beta}{\alpha+\beta}\frac{K_t^{\alpha}}{L_t^{1-\beta}} \\ &\quad - \left(\delta_i + n_i\right) + \left(\delta_j + n_j\right)\end{aligned}$$

We now turn to the special case of equal rates of growth for both sectors of the population, $n := n_i = n_j$, equal depreciation rates, $\delta := \delta_i = \delta_j$, and equal rates of saving, $s := s_i = s_j$. Then, we have convergence of growth rates in capital per head.

5. The steady state for a one-group economy

Our model admits a steady state for one group. In order to trace out the capital-per-head trajectory, we write L as $L_0 e^{nt}$. We then find

$$\frac{L^\beta}{K^{1-\alpha}} = \frac{\left(L_0 e^{nt}\right)^\beta}{\left(L_0 e^{nt} k\right)^{1-\alpha}} = \frac{\left(L_0 e^{nt}\right)^\beta}{\left(L_0 e^{nt}\right)^{1-\alpha} k^{1-\alpha}} = L_0^{\alpha+b-1} \frac{\left(e^{nt}\right)^{\alpha+\beta-1}}{k^{1-\alpha}} = L_0^{\alpha+b-1} \frac{e^{(\alpha+\beta-1)nt}}{k^{1-\alpha}}. \tag{XVII.6}$$

Omitting the time as well as the group index, the differential equation XVII.5 can be written as

$$\begin{aligned} \frac{\dot{k}_t}{k_t} &= s\frac{L_t^\beta}{K_t^{1-\alpha}} - (\delta+n) \\ &= sL_0^{\alpha+b-1}\frac{e^{(\alpha+\beta-1)nt}}{k^{1-\alpha}} - (\delta+n) \text{ (eq. XVII.6)} \end{aligned}$$

Normalizing the population at time 0, we let $L_0 = 1$ and can work with

$$\frac{\dot{k}}{k} = s\frac{e^{(\alpha+\beta-1)nt}}{k^{1-\alpha}} - (\delta+n)$$

If (!) a steady state exists, the growth rate of capital per head has to be zero and we find

$$\begin{aligned} 0 &= \frac{\partial \frac{\dot{k}}{k}}{\partial t} = \frac{\partial\left(\frac{se^{(\alpha+\beta-1)nt}}{k^{1-\alpha}} - (\delta+n)\right)}{\partial t} \\ &= \frac{se^{(\alpha+\beta-1)nt}(\alpha+\beta-1)n\cdot k^{1-\alpha} - (1-\alpha)k^{-\alpha}\dot{k}\cdot se^{(\alpha+\beta-1)nt}}{k^{2(1-\alpha)}}, \end{aligned}$$

hence

$$se^{(\alpha+\beta-1)nt}(\alpha+\beta-1)n\cdot k^{1-\alpha} - (1-\alpha)k^{-\alpha}\dot{k}\cdot se^{(\alpha+\beta-1)nt} = 0$$

and finally

$$\frac{\dot{k^c}}{k^c} = \frac{\alpha+\beta-1}{1-\alpha}n,$$

where c denotes "candidate". From

$$\frac{se^{(\alpha+\beta-1)nt}}{k^{1-\alpha}} - (\delta+n) = \frac{\alpha+\beta-1}{1-\alpha}n$$

we obtain the candidate equilibrium path

$$k^c = \left(\frac{se^{(\alpha+\beta-1)nt}}{\frac{(\alpha+\beta-1)n}{1-\alpha} + (\delta+n)}\right)^{\frac{1}{1-\alpha}}.$$

Calculating the growth rate on this path, we obtain

$$\frac{\frac{d\left(\frac{se^{(\alpha+\beta-1)nt}}{\frac{(\alpha+\beta-1)n}{1-\alpha}+(\delta+n)}\right)^{\frac{1}{1-\alpha}}}{dt}}{\left(\frac{se^{(\alpha+\beta-1)nt}}{\frac{(\alpha+\beta-1)n}{1-\alpha}+(\delta+n)}\right)^{\frac{1}{1-\alpha}}} = \frac{\alpha+\beta-1}{1-\alpha}n$$

so that

$$k^* = k^c$$

is, indeed, the equilibrium path of per-head capital. In order to check on stability, we note that

$$\frac{\dot{k}}{k} > \frac{\dot{k^*}}{k^*}$$

implies, and is implied by,

$$k < k^*.???$$

In case of constant returns to scale, we obtain (of course) growth rate 0. For increasing returns to scale the per-head capital endowment grows at a constant rate, for decreasing returns it shrinks. Note that both growth and shrinkage is leveraged by the growth of the population. In case of one group only, steady-state per-head consumption is given by

$$\begin{aligned} c \quad &: \quad = (1-s)\, K^{\alpha} L^{\beta-1} \\ &= \quad (1-s)\left(e^{nt}k\right)^{\alpha}\left(e^{nt}\right)^{\beta-1} \;\; (L_0 = 1) \\ &= \quad (1-s)\, e^{nt(\alpha+\beta-1)} k^{\alpha}. \end{aligned}$$

In case of decreasing returns to scale (possibly due to shortage of land), consumption tends to zero which makes a positive growth rate of the population unsustainable and gives rise to a Malthusian interpretation.

6. Conclusion

The continuous Shapley value seems to be a potent instrument to tackle distributive questions in growth theory in the absence of constant returns. We could show how to incorporate the continuous Shapley value into a Solow-type model with non-constant returns. Our model allows to shed some light on hotly debated immigration policy. In particular, we could identify the circumstances under which incumbent agents welcome immigration.

7. Topics and literature

The main topics in this chapter are

- continuous Shapley value
- immigration

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We recommend the textbook

8. Solutions

Exercise XVII.1

In the one-group case, we obtain

$$Y = K^\alpha L^\beta$$

which is also a consequence of Pareto efficiency.

9. Further exercises without solutions

CHAPTER XVIII

Evolutionary cooperative game theory

1. Introduction

[1]Evolutionary models of various forms have been part and parcel of economics for a long time (see, for example, the articles collected by Witt 1993). A specific class of models have been developed within game theory. In usual parlance, evolutionary game theory (see, for example, Weibull (1995) or Samuelson (1997)) means evolutionary theory applied to non-cooperative games. The aim of this paper is to develop an evolutionary cooperative game theory where we concentrate on the transferable-utility case. Apparently, ideas in this direction have been around for some time. Nasar (2002, p. xxiv) reports that John Nash, picking up his old interest in game theory, "received a grant from the National Science Foundation to develop a new 'evolutionary' solution concept for cooperative games".

Let us reconsider the apex game h for $N = \{1, ..., 4\}$. It is defined by

$$h(K) = \begin{cases} 1, & 1 \in K \text{ and } K \backslash \{1\} \neq \emptyset \\ 1, & K = N \backslash \{1\} \\ 0, & \text{otherwise} \end{cases}$$

The Shapley payoff vector is

$$Sh(h) = \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right).$$

Imbedding games like the apex game into an evolutionary setting, we interpret the payoffs as fitness. A player's success feeds into his proliferation. In order to model reproductive differences between players, we distinguish between players (like the four players in the apex game) and agents who take up the roles (or types) of these n players. If a role is particularly fruitful (in producing relatively high payoffs for the agents assuming that role), the relative number of agents assuming that role increases.

While the number of players is a natural number, we deal with a continuum of agents for each player. Therefore, we need an extended coalition function that is capable of dealing with non-integer players (agents). The Lovasz extension v^ℓ or the multi-linear (Owen) extension v^{MLE} are suitable candidates (see chapter XVII). For reasons explained in the conclusions, we prefer the Lovasz extensions over the multi-linear extension. The use of the

[1]This chapter is part of a joint project together with Andre Casajus.

Lovasz extension has important repercussions for our model. Indeed, we will find that the scarce players get all the payoff.

Extensions of coalition functions cannot be an input for the (standard) Shapley value. Therefore, we use the continuous Shapley value that we introduced in the previous chapter on growth theory. We thus obtain the payoff information seen as a fitness variable and can then define the replicator dynamics. Whenever an agent receives an above-average payoff, the population share of this agent (and of all other agents playing the same role) will increase – a standard result in replicator-dynamics models (for example Weibull 1995, chapter 3).

For the apex game and the Lovasz extension, we find two (up to symmetry) asymptotically stable population configurations:

- $\hat{x} = \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right)$ and
- $\hat{x} = \left(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

The first configuration obtains whenever the apex player's initial share is at least as high as the share of any of the weak players. In that case, the apex player teams up with the weak player who has the largest initial share. If, however, the apex player's initial share is lower than the shares of all the weak players, our replicator dynamics yields the second asymptotically stable population configuration – the apex player's share tends to zero.

In terms of interpretation, evolutionary noncooperative game theory (ENGT) differs from evolutionary cooperative game theory (ECGT). ENGT builds on the idea that two players are drawn at random from a large population. They are programmed to play a certain (mixed) strategy and the strategy that does better than other strategies grows faster. In contrast, ECGT concern all the agents of all players at the same time – the whole economy, so to speak.

We suggest to use the term ECGT for the non-atomic (or at least many-agents) setup. It is the agents whose shares change. In contrast, one might envision a model where the players themselves grow or shrink. A suitable example is provided by firms. Depending on their profits they will grow in an organic fashion (rather than grow by mergers and acquisitions).

In the following section, we will formally introduce agents, the extended coalition functions, and the continuous Shapley value. We present the replicator dynamics in section 5 and some general results in section 6. We mention two of these results here: 1. Dominated players (with lower marginal contributions) may survive in the long run. 2. For simple games, asymptotically stable population configurations involve minimal winning coalitions. The organizational dynamics is derived in section 7. Section 8 concludes the paper.

2. Some formal definitions

2.1. Payoff vectors. A payoff vector x for N is an element of $\mathbb{R}^N$ or a function $N \to \mathbb{R}$. For future reference, we define

- $\mathbb{R}^N_+ := \left\{x \in \mathbb{R}^N : x_i \geq 0 \text{ for all } i \in N\right\}$,
- $\mathbb{R}^N_{++} := \left\{x \in \mathbb{R}^N : x_i > 0 \text{ for all } i \in N\right\}$,
- $\Delta := \Delta(N) := \left\{x \in \mathbb{R}^N_+ : \sum x_i = 1\right\}$ and
- $int(\Delta) := int(\Delta(N)) = \left\{x \in \mathbb{R}^N_{++} : \sum x_i = 1\right\}$.

2.2. Agents and vector measure games. We work with continua of players who are called agents in this chapter. We have n players in a cooperative game v. Each player is associated with an interval of agents. Assume any vector $s = (s_1, ..., s_n) \in \mathbb{R}^N_+$ where s_i is the number of agents taking on player i's role.

Similar to the previous chapter XVII, we denote by

- I_i the interval of agents representing player i, $i \in N$,
- $I := \cup_{i \in N} I_i$ the union of the non-intersecting intervals, i.e., the set of all agents,
- λ the Lebesgues-Borel measure λ on $\mathbb{R}$,
- $\mathcal{B}$ the set of Borel sets of I,
- μ^s_i, $i \in N$, a measure on $\mathcal{B}$ defined by
$$\mu^s_i(K) := \lambda(K \cap I_i), K \in \mathcal{B}$$
and
- $\mu^s = \prod_{i \in N} \mu^s_i : \mathcal{B} \to \mathbb{R}^N$, $K \mapsto (\mu^s_i(K))_{i \in N}$ the Cartesian product of these measures.

3. The Lovasz extension

We use the Lovasz extension of a coalition functio v as introduced in chapter XIII. It is denoted by v^ℓ and given by
$$u^\ell_T(s) := \min_{i \in T} s_i, T \subseteq N, T \neq \emptyset$$
and
$$v^\ell(s) := \sum_{T \in 2^N \setminus \{\emptyset\}} d^v(T) \cdot \min_{i \in T} s_i$$

Lemma XVIII.1. *Let $v \in V(N)$ be a simple game. For $s \in \mathbb{R}^N_+$, we have*
$$v^\ell(s) = \max_{K \in \mathbb{M}} \min_{i \in K} s_i.$$

The proof of this lemma ...

The arguments solving the max min problem are denoted by the set of minimal winning coalitions $\mathbb{M}^{\max\min} \subseteq \mathbb{M}$ and the set of players $N^{\max\min} \subseteq N$.

Definition XVIII.1. *For future reference, we define a simple game* $v(\mathbb{M}, s)$ *on the player set* $N^{\max\min}$ *by specifying its set of minimal winning coalitions by*

$$\left\{W \subseteq N^{\max\min} : \textit{there exists a coalition } K \in \mathbb{M}^{\max\min} \textit{ s.t. } W = K \cap N^{\max\min}\right\}. \tag{XVIII.1}$$

$v(\mathbb{M}, s)$ depends directly on $\mathbb{M}^{\max\min}$ and $N^{\max\min}$ and on s only insofar as they are determined by s.

Thus, we employ the following algorithm:

- We start with a set of minimal winning coalitions $\mathbb{M}$ on some set N.
- We then delete all those minimal winning sets whose minimal size is smaller than the minimal size of any other minimal winning set, thereby obtaining $\mathbb{M}^{\max\min}$.
- The set of players in winning coalitions $\mathbb{M}^{\max\min}$ with minimal sizes (all of them identical) are denoted $N^{\max\min}$.
- We intersect all the winning coalitions from $\mathbb{M}^{\max\min}$ with $N^{\max\min}$ so as to obtain a new minimal winning set XVIII.1.

3.0.1. *Differentiability.* Alas, the Lovasz extension is not differentiable. Consider a unanimity game u_T and its Lovasz extension that is given by

$$u_T^\ell(s) := \min_{i \in T} s_i.$$

Let $s_- := \min_T(s) := \min_{i\in T} s_i$ be the minimum player size of the T-players and let $T_- := \{j | s_j = s_-\}$ be the set of T-players with minimal size. Then, for any unanimity game u_T and any player $i \in T$ we find

$$\frac{\partial u_T^\ell(s)}{\partial s_i} = \begin{cases} 1, & T_- = \{i\} \\ 0, & i \notin T_- \end{cases}$$

but u_T^ℓ is not partially differentiable at s with respect to s_i in case of $i \in T_- \neq \{i\}$ (i is one of several players with minimal size).

For later purposes, we consider the following approximation by partially differentiable functions $u_T^{\ell,m} = \min_T^m : \mathbb{R}_+^N \to \mathbb{R}$ which are defined, for all $\emptyset \neq T \subseteq N$ and $m \in \mathbb{N}$, by

$$\min_T^m(s) := \begin{cases} 0, & s_- = 0 \\ \dfrac{|T|^{\frac{1}{m}}}{\left(\sum_{i\in T} \frac{1}{s_i^m}\right)^{\frac{1}{m}}} = |T|^{\frac{1}{m}} \left(\sum_{i\in T} s_i^{-m}\right)^{-\frac{1}{m}}, & \text{else.} \end{cases} \tag{XVIII.2}$$

By standard rules for limites, we can confirm $\lim_{m\to\infty} \min_T^m(s) = \min_T(s)$ for all $\emptyset \neq T \subseteq N$ and all $s \in \mathbb{R}_+^N$.

We denote by $v^{\ell,m}$ the m-th approximation of v^ℓ given by

$$v^{\ell,m}(s) := \sum_{\emptyset \neq T \subseteq N} d^v(T) \cdot \min_T^m(s), \qquad m \in \mathbb{N},$$

which also is linear in $v \in V(N)$ and non-negatively homogenous in $s \in \mathbb{R}^N_+$.

4. Vector measure games and Shapley value

Before we can apply the continuous Shapley value, we need to define the appropriate vector measure games

$$\begin{aligned} v^{\ell,s} &: = v^{\ell} \circ \mu^s : \mathcal{B} \to \mathbb{R} \text{ and} \\ v^{\ell,m,s} &: = v^{\ell,m} \circ \mu^s : \mathcal{B} \to \mathbb{R} \end{aligned}$$

Given a coalition $K \in \mathcal{B}$, $\mu^s(K)$ specifies how to devide K among the n groups and how to measure these subgroups. v^{ℓ} or $v^{\ell,m}$ then yield the worth in accordance with the underlying TU game v.

We now apply the Aumann & Shapley (1974, Theorem B) diagonal formula (see also Neyman 2002, pp. 2141) to $v^{\ell,m,s}$. Taking any coalition $K \in \mathcal{B}$ and any coalition function v, this formula yields

$$\mathrm{Sh}\, v^{\ell,m,s}(K) = \sum_{j=1}^{n} \mu_j^s(K) \int_0^1 \left.\frac{\partial v^{\ell,m}}{\partial s_j}\right|_{\tau s} d\tau,$$

respectively. $\mathrm{Sh}\, u_T^{\ell,m,s}(K)$ are the payoffs accruing to coalition K. The analogue of player j's marginal contribution in the discrete Shapley formula is the derivative of the coalition's worth with respect to the measure of agents of player j. This derivative is evaluated at $\tau s = (\tau s_1, ..., \tau s_n)$. Thus, the formula looks at coalitions on the diagonal only. Remember that we have a continuum of agents. If we take a subset of agents by chance, it is likely that the composition in this subset (how many agents of player 1, player 2 etc.) will not deviate much from the composition in the overall population (see Aumann & Shapley 1974, pp. 23).

Lemma XVIII.2. *The Lovasz extension leads to*

$$\mathrm{Sh}\, u_T^{\ell,m,s}(I_i) = \begin{cases} 0, & i \notin T \\ 0, & s_- = 0 \\ |T|^{\frac{1}{m}} s_i^{-m} \left(\sum_{j\in T} s_j^{-m}\right)^{-\frac{m+1}{m}}, & i \in T \text{ and } s_- \neq 0 \end{cases}$$

and

$$\mathrm{Sh}\, u_T^{\ell,s}(I_i) := \lim_{m\to\infty} \mathrm{Sh}\, u_T^{\ell,m,s}(I_i) = \begin{cases} \frac{s_-}{|T_-|}, & i \in T_-, s_- \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

Commenting on the Lovasz extension, consider $N = \{1, 2\}$ and the unanimity game $u_{\{1,2\}}$. Assume $s_1 < s_2$ so that player 1 is the scarce player. The diagonal formula implies that we look at marginal contributions (partial derivatives) along a line through the origin. By $\tau s_1 < \tau s_2$, player 2's Shapley payoff approaches 0 as m approaches infinity.

Using the additivity of the value, one also obtains the payoffs $\mathrm{Sh}\, v^{\ell,s}(I_i)$ and $\mathrm{Sh}\, v^{\ell,m,s}(I_i)$, $i \in N$, $m \in \mathbb{N}$. One easily checks that $\mathrm{Sh}\, v^{\ell,s}(I_i)$ and $\mathrm{Sh}\, v^{\ell,m,s}(I_i)$ are homogenous of degree 1 with respect to s.

Definition XVIII.2. *Consider a coalition function $v \in V(N)$ and its extension v^{ext}, a player $i \in N$ and a population configuration $s \in \mathbb{R}_+^N$ with $s_i > 0$. The averge payoff accruing to agents from I_i is also called agent i's payoff and is given by*

$$\mathrm{Sh}_i\left(v^{ext,s}\right) := \frac{\mathrm{Sh}\, v^{ext,s}(I_i)}{s_i}.$$

Example XVIII.1. *For any unanimity game u_T, we find*

$$\mathrm{Sh}_i\left(u_T^{\ell,s}\right) = \begin{cases} \frac{1}{|T_-|}, & i \in T_-, s_- \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

so that $N = T = \{1,2,3\}$ yields

$$\mathrm{Sh}_1\left(u_T^{\ell,s}\right) = \begin{cases} 1, & s_1 < \min(s_2, s_3) \\ \frac{1}{2}, & s_1 = s_2 < s_3 \\ \frac{1}{2}, & s_1 = s_3 < s_2 \\ \frac{1}{3}, & s_1 = s_2 = s_3 \\ 0, & s_1 > s_2 \text{ or } s_1 > s_3 \end{cases}$$

and, using example XIII.1, the apex payoffs for the Lovasz extension are given by

$$\left(\mathrm{Sh}_1\left(h^{\ell,s}\right),\ \mathrm{Sh}_2\left(h^{\ell,s}\right),\ \mathrm{Sh}_3\left(h^{\ell,s}\right),\ \mathrm{Sh}_4\left(h^{\ell,s}\right)\right)$$

$$= \begin{cases} (0,1,0,0), & s_1 < s_2 < s_3 < s_4 \\ \left(0,\frac{1}{2},\frac{1}{2},0\right) & s_1 < s_2 = s_3 < s_4 \\ \left(0,\frac{1}{3},\frac{1}{3},\frac{1}{3}\right) & s_1 < s_2 = s_3 = s_4 \\ \left(\frac{1}{2},\frac{1}{2},0,0\right) & s_1 = s_2 < s_3 < s_4 \\ \left(\frac{2}{3},\frac{1}{6},\frac{1}{6},0\right) & s_1 = s_2 = s_3 < s_4 \\ \left(\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6}\right) & s_1 = s_2 = s_3 = s_4 \\ (1,0,0,0), & s_2 < s_1 < s_3 < s_4 \\ (1,0,0,0) & s_2 < s_1 = s_3 < s_4 \\ \left(\frac{2}{3},0,\frac{1}{6},\frac{1}{6}\right) & s_2 < s_1 = s_3 = s_4 \\ (1,0,0,0) & s_2 < s_1 < s_3 = s_4 \\ (1,0,0,0), & s_2 < s_3 < s_1 < s_4 \\ (1,0,0,0) & s_2 = s_3 < s_1 < s_4 \\ \left(\frac{1}{2},0,0,\frac{1}{2}\right) & s_2 \leq s_3 < s_1 = s_4 \\ (0,0,0,1) & s_2 < s_3 < s_4 < s_1 \\ \left(0,0,\frac{1}{2},\frac{1}{2}\right) & s_2 < s_3 = s_4 < s_1 \\ \left(0,\frac{1}{3},\frac{1}{3},\frac{1}{3}\right) & s_2 = s_3 = s_4 < s_1 \\ (0,0,0,1) & s_2 = s_3 < s_4 < s_1 \end{cases}$$

LEMMA XVIII.3. *Let $v \in V(N)$ be a simple game with the set of minimal winning coalitions $\mathbb{M}$ and let $v(\mathbb{M}, s)$ be the game defined in definition XVIII.1. The agents' Shapley values for players $i \in N$ are given by*

$$\mathrm{Sh}_i\left(v^{\ell,s}\right) = \begin{cases} \mathrm{Sh}_i\left(v\left(\mathbb{M}, s\right)\right), & i \in N^{\max\min} \\ 0, & \textit{otherwise} \end{cases},$$

Thus, in a simple game, a player obtains a non-zero payoff zero only if

- he belongs to minimal winning coalition,
- his size is minimal within at least one minimal winning coalition, and
- this minimal size is at least as large as the minimal sizes found in any other winning coalition.

The lemma implies that, generically, small changes in the players' sizes do not affect the agents' Shapley values.

LEMMA XVIII.4. *Assume a set of minimal winning coalitions $\mathbb{M} = \{W_1, W_2\}$ and a nonempty intersection $K := W_1 \cap W_2 \neq \emptyset$.*

- *Assume $\min_{j\in W_2\backslash K} s_j > \min_{j\in W_1\backslash K} s_j > \min_{j\in K} s_j$. Then, positive payoffs accrue to players from $\{i \in K : s_i = \min_{j\in K} s_j\}$ while all other players have zero payoff.*
- *Assume $\min_{j\in W_2\backslash K} s_j > s_i \geq \min_{j\in W_1\backslash K} s_j$ for all $i \in K$. Then, positive payoffs accrue to players from $\{i \in K : s_i = \min_{j\in K} s_j\}$ while all other players have zero payoff.*
- *Assume $\min_{j\in W_2\backslash K} s_j = s_i > \min_{j\in W_1\backslash K} s_j$ for all $i \in K$. Then, positive payoffs accrue to players from $\{i \in W_2 : s_i = \min_{j\in W_2} s_j\}$ while all other players have zero payoff.*

THEOREM XVIII.1. *If $i, j \in N$ are symmetric in (N, v) and $s_i = s_j$ then $\mathrm{Sh}\, v^{\ell,s}(I_i) = \mathrm{Sh}\, v^{\ell,s}(I_j)$ and $\mathrm{Sh}\, v^{\ell,m,s}(I_i) = \mathrm{Sh}\, v^{\ell,m,s}(I_j)$.*

5. Replicator dynamics

5.1. Formula. Interpreting the agents' Shapley payoffs as fitness and assuming a constant birthrate β and a constant death rate δ, the evolution of s_i is defined by

$$\dot{s}_i = \left[\beta + \mathrm{Sh}_i\left(v^{\ell,s}\right) - \delta\right] s_i.$$

In terms of population shares

$$x_i := \frac{s_i}{\sum_{j=1}^n s_j}$$

we obtain the replicator dynamics

$$\dot{x}_i = \left(\mathrm{Sh}_i\left(v^{ext,s}\right) - \sum_{j=1}^n \mathrm{Sh}_j\left(v^{ext,s}\right) x_j\right) x_i \tag{XVIII.3}$$

where the growth rate of a player's population share equals the difference of his agents' fitness and the average fitness of all agents. Of course, we have $x \in \Delta$ and this procedure does not work for $s = 0$.

5.2. Differential equation and discrete replicator dynamics. Standard methods do not guarantee the existence of a solution to our replicator equation. Therefore, we resort to discrete replicator dynamics which obviously exist. Assuming a starting point at time $t = 0$ and a population share vector $x(0) = (x_1(0), ..., x_n(0)) \in \Delta$, we define the discrete replicator dynamics by

$$x_i(t) = x_i(t-1) + x_i(t-1)\left[\mathrm{Sh}_i\left(v^{\ell,x(t-1)}\right) - \sum_{j=1}^{n} \mathrm{Sh}_j\left(v^{\ell,x(t-1)}\right) x_j\right], t \geq 1$$

While it is easy to check that $\sum x_i(t) = 1$ follows from $\sum x_i(t-1) = 1, t \geq 1$, we cannot, in general, exclude a negative population share. In order to avoid this problem and in order to smooth out the solution orbit, we introduce a (very small) step length $\sigma > 0$ and work with the replicator dynamics

$$x_i(t) = x_i(t-1) + x_i(t-1)\,\sigma\left[\mathrm{Sh}_i\left(v^{\ell,x(t-1)}\right) - \sum_{j=1}^{n} \mathrm{Sh}_j\left(v^{\ell,x(t-1)}\right) x_j\right], t \geq 1 \tag{XVIII.4}$$

In a continuous case, σ would affect the velocity of change but not the solution orbit.

We now revisit the apex game. The initial population share vector $x(0) = \left(\frac{2}{10}, \frac{1}{10}, \frac{3}{10}, \frac{4}{10}\right)$ is used in fig. 1 where the plot builds on S =1200 steps with step length $\sigma = \frac{1}{600}$. In the beginning, only player 1's agent set grows. As soon as the sizes of player 1's agent set and player 4's agent set equal, both agent sets grow while the agent sets of players 2 and 3 tend towards zero.

Fig. 2 starts with the population configuration $\left(\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}\right)$. Finally, the three unimportant players grow until each reaches a population share of $\frac{1}{3}$.

Applying the formula

number of time periods = number of steps times step length,

the above examples rest on 2 (physical) time periods, $2 = T = S \cdot \sigma = 1200 \cdot \frac{1}{600}$.

DEFINITION XVIII.3. *Consider a coalition function $v \in V(N)$ and a starting population share vector $x(0) = (x_1(0), ..., x_n(0)) \in \Delta(N)$. The Euler replicator dynamic for T time periods is defined by the discrete replicator dynamics XVIII.4 obeying $0 \leq t \leq S$, $\sigma = \frac{T}{S}$ and $S \to \infty$.*

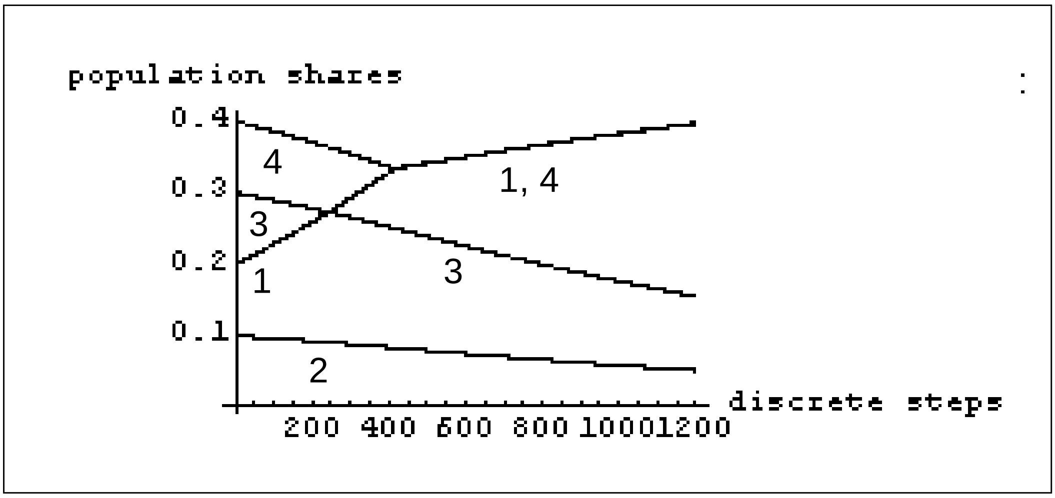

FIGURE 1. The apex player teams up with player 4

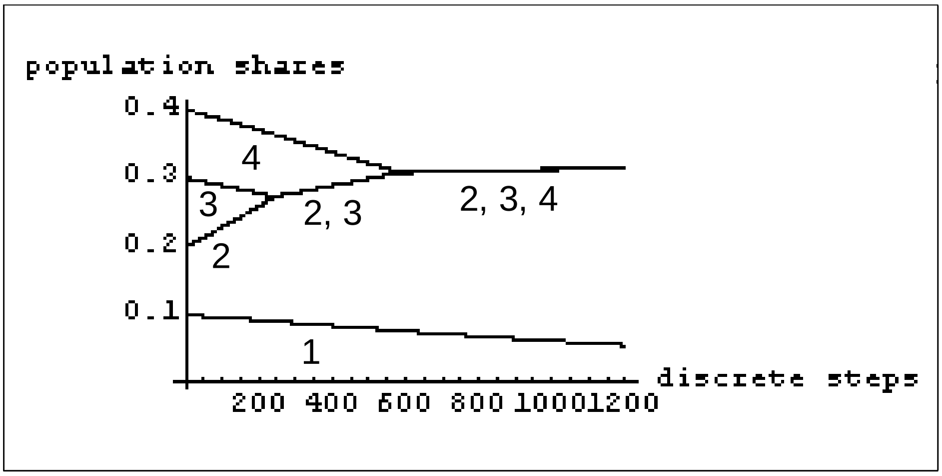

FIGURE 2. The three unimportant players trump the apex player

DEFINITION XVIII.4. *A vector of population shares* $\hat{x} = (\hat{x}_1, ..., \hat{x}_n) \in \Delta$ *is a steady state if there exists a population share vector* $x(0) = (x_1(0), ..., x_n(0)) \in \Delta$ *such that the Euler replicator dynamics yields*

$$\lim_{T \to \infty} x_i(t) = \hat{x}_i$$

for all $i = 1, ..., n$.

DEFINITION XVIII.5. *A steady state* $\hat{x} = (\hat{x}_1, ..., \hat{x}_n) \in \Delta$ *is called asymptotically stable if there exists some* $\varepsilon > 0$ *such that for all population vectors* $x(0)$ *obeying* $\|x(0) - \hat{x}\|_2 < \varepsilon$ *we have*

$$\lim_{T \to \infty} x(t) = \hat{x}. \qquad \text{(XVIII.5)}$$

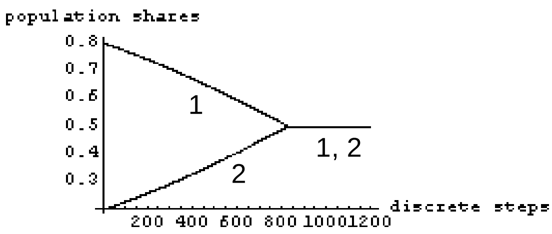

FIGURE 3. Player 2 is dominated but does not vanish.

6. General results

In ENGT, strictly dominated strategies are wielded out. We present a dominance definition and show that we do not have a similar result in ECGT.

DEFINITION XVIII.6. *Let $v \in V(N)$. Player $i \in N$ strictly dominates player $j \in N$ if $v(K \cup \{i\}) > v(K \cup \{j\})$ holds for all $K \subseteq N \backslash \{i,j\}$. Player $i \in N$ weakly dominates player $j \in N$ if $v(K \cup \{i\}) \geq v(K \cup \{j\})$ holds for all $K \subseteq N \backslash \{i,j\}$ and if there is a coalition $\hat{K} \in N \backslash \{i,j\}$ such that $v\left(\hat{K} \cup \{i\}\right) > v\left(\hat{K} \cup \{j\}\right)$ is true. In that case, we also say that i weakly dominates j with strong $\hat{K}$-dominance.*

Note that strict and weak dominance are equivalent for $n = 2$. It is not difficult to show that weak and strict dominance are transitive relations on N.

We begin with two negative results. A strictly dominated player does not need to vanish nor does a null player. The first assertion follows from the game given by $N = \{1,2\}$, $v(1) = 1, v(2) = 0$ and $v(1,2) = 3$. Assume the initial population share vector $x(0) = \left(\frac{4}{5}, \frac{1}{5}\right)$. Player 2 is strictly dominated but holds his ground as can be seen in fig. 3. Also, a weakly dominating player (as the apex player) can vanish while the player dominated by him does not, as we have seen above (fig. 2).

Consider, now, the game for two players given by $v(1) = 0$, $v(2) = -1 = v(1,2)$. Fig. 4 (with $x(0) = \left(\frac{1}{5}, \frac{4}{5}\right)$) proves that a null player (player 1 in our case) does not need to vanish.

However, a dominated null player vanishes:

LEMMA XVIII.5. *Let $v \in V(N)$ where $j \in N$ is strictly dominated by player i, j is a null player and $x_i(0) > 0$ and there exists a player $k \in N \backslash \{i,j\}$ with $x_k > 0$. Then,* $\lim_{T \to \infty} x_j(t) = 0$.

In contrast to the examples considered so far, we can have different non-zero shares in the long run. Consider $N = \{1,2,3\}$, $v \in V(N)$ given by

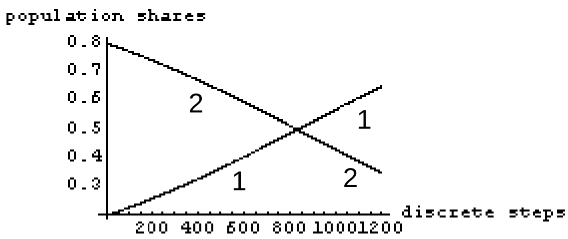

FIGURE 4. Player 1 is a dominating null player.

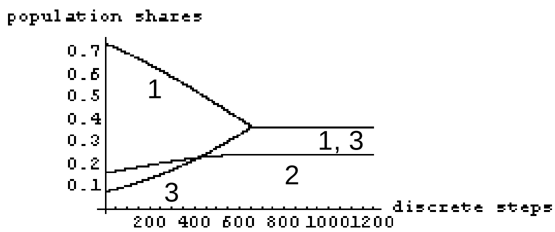

FIGURE 5. Different non-zero shares in the long run

$v(1) = v(2) = v(3) = 0$, $v(1,3) = 2$, $v(1,2) = v(2,3) = 1$ and $v(1,2,3) = 3$. The initial population share vector $x(0) = \left(\frac{3}{4}, \frac{1}{6}, \frac{1}{12}\right)$ yields fig. 5.

PROPOSITION XVIII.1. *Let $v \in V(N)$ be a simple game with the set of minimal winning coalitions $\mathbb{M}$. Then, the asymptotically stable states $\hat{x} = (\hat{x}_1, ..., \hat{x}_n)$ are characterized by minimal wining coalitions $W \in \mathbb{M}$ and*

$$\hat{x}_i = \begin{cases} \frac{1}{|W|}, & i \in W \\ 0, & \textit{otherwise} \end{cases} \tag{XVIII.6}$$

Building on the two lemmata XVIII.3 and XVIII.4, the proof is not difficult. Assume a minimal winning coalition W and construct a size vector x such that $N^{\max\min} \subseteq W$. ($x_i = \hat{x}_i$ provides the simplest example.) Then, all the agents outside W obtain zero payoff while all the (symmetric!) agents within $N^{\max\min}$ obtain $\frac{1}{|N^{\max\min}|}$. Their shares grow until, finally, the shares for the W-agents are the same. Thus, we have $\lim_{T\to\infty} x(t) = \hat{x}$.

Inversely, take any asymptotically stable state $\hat{x} = (\hat{x}_1, ..., \hat{x}_n)$ not obeying eq. XVIII.6. There must be at least one minimal winning coalition W' such that $\hat{x}_i > 0$ for all $i \in W'$. Otherwise, all the payoffs are zero and we can identify a minimal winning coalition W, change the relative sizes minimally such that $\hat{x}_i > 0$ for all $i \in W$ holds and such that this minimal winning coalition wins in the sense of equations XVIII.5 and XVIII.6. (The trick is to choose a minimal winning coalition where the players minimal sizes (except those with zero size) are maximal.)

COROLLARY XVIII.1. *Consider the apex game for four players and* $x_2(0) \leq x_3(0) \leq x_4(0)$ *without loss of generality. The dynamics of the apex game admit four steady states* $\hat{x} = (\hat{x}_1, ..., \hat{x}_4)$*:*

(1) $x_1(0) < x_2(0) : \hat{x}_1 = 0, \hat{x}_2 = \hat{x}_3 = \hat{x}_4 = \frac{1}{3}$ *(steady state WWW: coalition of the weak)*
(2) $x_1(0) \geq x_2(0)$ *and*
   - $x_3(0) < x_4(0) : \hat{x}_1 = \hat{x}_4 = \frac{1}{2}, \hat{x}_2 = \hat{x}_3 = 0$ *(steady state SW: the strong with one weak agent)*
   - $x_3(0) = x_4(0)$ *and*
     - $x_2(0) < x_3(0)$ *and* $x_1(0) > x_2(0) : \hat{x}_1 = \hat{x}_3 = \hat{x}_4 = \frac{1}{3}, \hat{x}_2 = 0$ *(steady state SWW: the strong with two weak agents)*
     - $x_2(0) = x_3(0)$ *or* $x_1(0) = x_2(0) : \hat{x}_1 = \hat{x}_2 = \hat{x}_3 = \hat{x}_4 = \frac{1}{4}$ *(steady state SWWW: grand coalition)*

The steady states WWW and SW are asymptotically stable while SWW and SWWW are not.

7. Organizational model

So far, we differentiate between players and agents who assume the role of players. We now turn to the organizational model where the sizes of the players (organizations such as firms) change.

Consider a TU game $v \in V(N)$ and its extension v^{ext}. For a given size vector $s \in \mathbb{R}^n_+$, we can define a TU game $\mathring{v}^{ext,s} \in V(N)$ by

$$\mathring{v}^{ext,s}(K) = v^{ext}(s_K)$$

where $s_K \in \mathbb{R}^n_+$ is given by

$$(s_K)_i = \begin{cases} s_i, & i \in K \\ 0, & i \notin K \end{cases}$$

Interpreting the Shapley payoffs for $\mathring{v}^{ext,s}$ as profits which are reinvested in the firms, firm i's share develops according to

$$\dot{s}_i = \beta \operatorname{Sh}_i\left(\mathring{v}^{ext,s}\right)$$

where $\beta > 0$ translates profits into size changes. Rewriting this equality in terms of population shares yields the organizational dynamics

$$\dot{x}_i = \beta\left(\frac{\operatorname{Sh}_i\left(\mathring{v}^{ext,s}\right)}{x_i} - \sum_{j=1}^n \operatorname{Sh}_j\left(\mathring{v}^{ext,s}\right)\right)x_i.$$

For the Lovasz extension, we obtain

$$Sh_i\left(\overset{\circ}{u}_T^{\ell,s}\right) = \begin{cases} \frac{\min_{j\in T} s_j}{|T|}, & i \in T \\ 0, & \text{otherwise} \end{cases}$$

8. Conclusion

In conclusion, we offer some remarks and point to future research. First, we find that evolutionary cooperative game theory (ECGT) is very attractive from an interpretational point of view. First of all, even the most basic models as they are presented in this paper belong to the (a) "playing the field" and the (b) polymorphic variety. (a) just results from the way the Shapley value is calculated and interpreted – an agent's payoff depends on the set-up of the economy as a whole. (b) is also the natural outflow from different players roles. In contrast, the most basic model of evolutionary noncooperative game theory (ENGT) builds on "pairwise contests" and a monomorphic population playing a symmetric game. Of course, more advanced ENGT models also deal with polymorphic playing-the-field situations.

Second, the specific extension (the Lovasz extension in this paper) is very critical for the results obtained. We prefer the intuition underlying the Lovasz extension of that for the multi-linear extension. If players (or agents) work together (in the framework of a unanimity or an apex game) and if the size of the agents is below 1, the multilinear extension, v^{MLE}, has a probabilistic interpretation (as noted by Owen 1972, p. 64) – the players work together only if their time schedules happen to coincide. For example, two productive players in the unanimity game $u_{\{1,2\}}$ with $s = \left(\frac{1}{2}, \frac{1}{3}\right)$ can produce $\frac{1}{2} \cdot \frac{1}{3}$, only. It seems to us that (by appropriate coordination), the two agents should be able to produce the minimum of these two figures, $\frac{1}{3}$,. which is exactly what the Lovasz extension does. Also, consider $s = (2, 3)$. The multi-linear extension yields $2 \cdot 3 = 6$ whereas the minimum extension leads to 2. Also, an extension's worth may turn out to be negative even if the underlying coalition function itself is positive. In fact, we find $h^{MLE}(2, 1, 3, 4) = -10$.

Third, a major application of ENGT is equilibrium selection (see the titel of the book by Samuelson 1997). In contrast, ECGT focuses on the evolutionary pressure against players. Productive players survive where the productivity depends on the size distribution. Alternatively, one might view ECGT as a contribution to the vast field of coalition formation.

Fourth, Filar & Petrosjan (2000) have published a related paper where they introduce dynamic cooperative games. The idea is to define a sequence of games (in discrete or in continuous time) so that one TU game is determined by the previous one and by the payoffs achieved under some solution concept. While our organizational model also produces a sequence of TU games, it cannot be subsumed under the heading of dynamic cooperative games as defined by these authors. The technical reason is that the size vector s cannot be derived from $\mathring{v}^{\ell,s}$. More importantly, the focus of Filar & Petrosjan's (2000) approach is quite different. The players in that paper obtain the sum of payoffs for a sequence of coalition functions. The authors deal with the problem of whether these payoffs obey some consistency criterion.

We now turn to future work in ECGT and note that the replicator dynamics are concerned with selection. Of course, mutation is the other evolutionary force to be reckoned with. It is concerned with the change of parameters rather than the selection pressures for a given set of parameters. Within our framework, mutation can take different forms:

(1) We may consider small changes of the coalition function v.
(2) Other players could be added with very small sizes such that the worths for the other players stays the same for a zero size of the new arrival.

9. Appendix

9.1. Proof of lemma XVIII.2. The 0 payoffs follow from eq. XVIII.2. In case of $s_- > 0$ and $i \in T$, the diagonal formula yields

$$
\begin{aligned}
\operatorname{Sh} u_T^{\ell,m,s}(I_i) &= \mu_i^s(I_i) \int_0^1 \left.\frac{\partial u_T^{\ell,m}}{\partial s_i}\right|_{ts} dt \\
&= s_i |T|^{\frac{1}{m}} \int_0^1 \left(-\frac{1}{m}\right) \left(\sum_{j\in T} s_j^{-m}\right)^{-\frac{1}{m}-1} \cdot (-m)\, s_i^{-m-1} \Bigg|_{ts} dt \\
&= s_i |T|^{\frac{1}{m}} \int_0^1 \left(\sum_{j\in T} s_j^{-m}\right)^{-\frac{m+1}{m}} \cdot s_i^{-(m+1)} \Bigg|_{ts} dt \\
&= s_i |T|^{\frac{1}{m}} \int_0^1 \left(\sum_{j\in T} (ts_j)^{-m}\right)^{-\frac{m+1}{m}} \cdot (ts_i)^{-(m+1)}\, dt \\
&= s_i s_i^{-(m+1)} |T|^{\frac{1}{m}} \int_0^1 \left(\sum_{j\in T} t^{-m} s_j^{-m}\right)^{-\frac{m+1}{m}} \cdot (t^m)^{-\frac{m+1}{m}}\, dt \\
&= s_i^{-m} |T|^{\frac{1}{m}} \int_0^1 \left(\sum_{j\in T} s_j^{-m}\right)^{-\frac{m+1}{m}} dt \\
&= |T|^{\frac{1}{m}} s_i^{-m} \left(\sum_{j\in T} s_j^{-m}\right)^{-\frac{m+1}{m}}
\end{aligned}
$$

We rewrite $\operatorname{Sh} u_T^{\ell,m,s}(I_i)$ for the third case to find

$$
\begin{aligned}
|T|^{\frac{1}{m}} s_i^{-m} \left(\sum_{j\in T} s_j^{-m}\right)^{-\frac{m+1}{m}} &= s_i |T|^{\frac{1}{m}} (s_i^m)^{-\frac{m+1}{m}} \left(|T_-|\, s_-^{-m} + \sum_{j\in T\setminus T^-} s_j^{-m}\right)^{-\frac{m+1}{m}} \\
&= s_i |T|^{\frac{1}{m}} \left(|T_-| \frac{s_i^m}{s_-^m} + \sum_{j\in T\setminus T^-} \frac{s_i^m}{s_j^m}\right)^{-\frac{m+1}{m}} \\
&= \begin{cases}
s_i |T|^{\frac{1}{m}} \left(|T_-| \underbrace{\frac{s_i^m}{s_-^m}}_{1} + \sum_{j\in T\setminus T^-} \left(\underbrace{\frac{s_i}{s_j}}_{<1}\right)^m\right)^{-\frac{m+1}{m}}, & i \in T_- \\
s_i |T|^{\frac{1}{m}} \left(|T_-| \left(\underbrace{\frac{s_i}{s_-}}_{>1}\right)^m + \sum_{j\in T\setminus T^-} \frac{s_i^m}{s_j^m}\right)^{-\frac{m+1}{m}}, & i \notin T_-
\end{cases}
\end{aligned}
$$

Standard rules for limites imply the desired results.

In the previous chapter, we focus on a specific class of games, the gloves games. In this chapter, we aim to familiarize the reader with many other interesting games.

10. Topics and literature

The main topics in this chapter are

- simple game
- winning coalition
- veto player
- dictator
- null player
- unanimity game
- apex game
- weighted voting game
- buying-a-car game
- Maschler-Spiel
- endowment game
- superadditivity
- convexity
- monotonicity

We introduce the following mathematical concepts and theorems:

- linear independence
- span
- basis
- coefficients

We recommend the textbook by Wiese (2005c).

11. Solutions

Exercise ??

There is only one such coalition function, the zero coalition function (that fulfills $v(K) = 0$ for all $K \subseteq N$).

Exercise ??

12. Further exercises without solutions

Non-transferable utility

Part G

Non-transferable utility

In this last part of the course, we do not consider transferable-utility coalition functions any more. Instead, we deal with coalition functions without transferable utility. Exchange economies provide an important application that we present in chapter XIX. Also, the bargaining theory due to John Nash (chapter XX) is best understood as building on non-transferability.

CHAPTER XIX

Exchange economies

1. Introduction

Transferable utility is underlying the coalition functions considered so far. To every coalition $K \subseteq N$, a real number $v(K)$ is attributed with the understanding that the players from K produce $v(K)$ together. All the solution concepts so far assume efficiency (Pareto efficiency or component efficiency) where all the players (all the players in a component C) share the worth $v(N)$ (or $v(C)$). Thus, utility can be transferred from one player to another one.

Non-transferable utility does away with this simplification. Instead of attributing a real number to a coalition, a set of payoff vectors is specified. For example, two players that exchange goods can achieve any payoff vector that is linked to a feasible allocation.

In order to convey a first impression of what a coalition function without transferable utility looks like, consider fig. 1. We have three players and eight coalitions (left-hand side of the figure). Take coalition {Peter, Otto}. The coalition function V (capital V rather than small v in case of transferable utility) attributes $V(\{\text{Peter, Otto}\}) \subseteq \mathbb{R}^2$ to that coalition (right-hand side). Similarly, $V(\{\text{Peter, Otto, Carl}\})$ is a subset of $\mathbb{R}^3$ and $V(\emptyset) = \emptyset$.

In the following section, we formally define coalition functions without transferable utility (for short, NTU coalition functions). We then consider

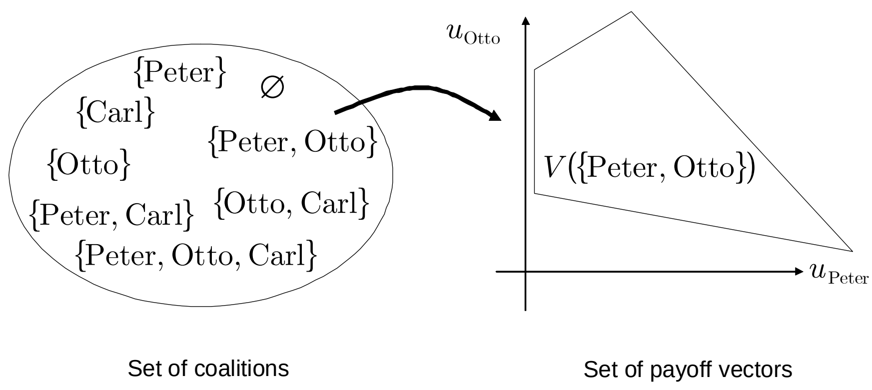

FIGURE 1. Part of a coalition function with non-transferable utility

two different models. The first (sections through) deals with General Equilibrium Theory (GET). Here, agents observe prices and choose their good bundles accordingly. GET envisions a market system with perfect competition. This means that all agents (households and firms) are price takers. The aim is to find prices such that

- all actors behave in a utility, or profit, maximizing way and
- the demand and supply schedules can be fulfilled simultaneously.

In that case, we have found a Walras equilibrium. Note that the price-finding process is not addressed in GET. Walras suggests that an auctioneer might try to inch towards an equilibrium price vector. This is the so-called tâtonnement.

Our purpose is to formulate the exchange economy by way of a NTU coalition function. We then define the core concept and show that the Walras equilibrium belongs to the core. In so doing, we follow a long tradition and differentiate between

- the implications of Pareto efficiency on the one hand (this is the Edgeworthian theme of cooperation, see chapter II, pp. 16) and
- the implications of individual utility and profit maximization for markets (the Walrasian theme of decentralization).

The second model is concerned with matching (of spouses, for example) and is presented in sections through. Again, we consider the implications of core allocations.

2. Household theory

We now consider decisions in the face of prices. Assuming price takership, the households buy a best bundle within their budget. Therefore, we analyze the budget first and then derive a best bundle on the basis of budget and preferences.

2.1. Notation. In general (for an arbitrary dimension $\ell \in \mathbb{N}$) we write

$$\mathbb{R}^\ell := \{(x_1, ..., x_\ell) : x_g \in \mathbb{R}, g = 1, ..., \ell\}.$$

$0 \in \mathbb{R}^\ell$ is the null vector $(0, 0, ..., 0)$. The vectors are often called points (in $\mathbb{R}^\ell$).

Remark XIX.1. *For vectors x and y with ℓ entries, we define*

- $x \geq y$ *by* $x_g \geq y_g$ *for all* g *from* $\{1, 2, ..., \ell\}$,
- $x > y$ *by* $x \geq y$ *and* $x \neq y$,
- $x \gg y$ *by* $x_g > y_g$ *for all* g *from* $\{1, 2, ..., \ell\}$.

In household theory, we will work with the goods space

$$\mathbb{R}^\ell_+ := \left\{x \in \mathbb{R}^\ell : x \geq 0\right\}$$

rather than $\mathbb{R}^\ell$ where negative amounts of goods are allowed.

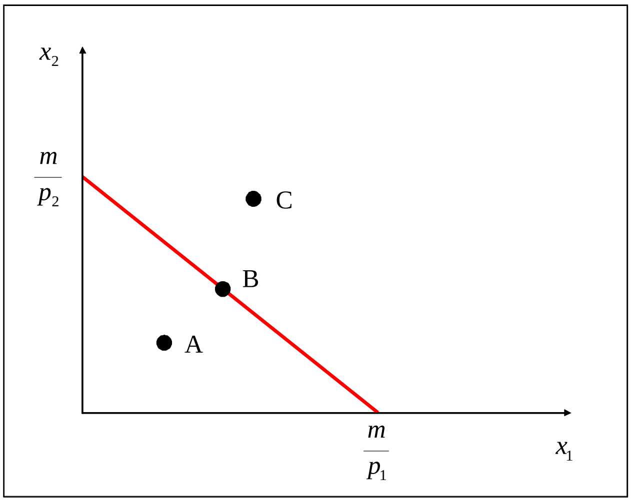

FIGURE 2. The budget for two goods

2.2. Budget.

2.2.1. *Money budget.* We first assume that the household has some monetary amount m at his disposal. The budget is the set of good bundles that the household can afford, i.e., the set of bundles whose expenditure is not above m. The expenditure for a bundle of goods $x = (x_1, x_2, ..., x_\ell)$ at a vector of prices $p = (p_1, p_2, ..., p_\ell)$ is the dot product (or the scalar product), of the two vectors:

$$p \cdot x := \sum_{g=1}^{\ell} p_g x_g.$$

DEFINITION XIX.1 (money budget). *For a price vector $p \in \mathbb{R}^\ell$ and monetary income $m \in \mathbb{R}_+$, the money budget is defined by*

$$B(p, m) := \left\{ x \in \mathbb{R}_+^\ell : p \cdot x \leq m \right\}$$

where

$$\left\{ x \in \mathbb{R}_+^\ell : p \cdot x = m \right\}$$

is called the budget line.

For example, in case of two goods, the budget is the set of bundles fulfilling $p_1 x_1 + p_2 x_2 \leq m$. If the household does not consume good 1 ($x_1 = 0$), he can consume up to m/p_2 units of good 2. (Just solve the inequality for x_2.). In fig. 2, the household can afford bundles A and B, but not C.

The following theorem should be no surprise to you. If you double all prices and income, your budget remains unchanged:

LEMMA XIX.1. *For any number $\alpha > 0$, we have $B(\alpha p, \alpha m) = B(p, m)$.*

EXERCISE XIX.1. *Fill in: For any number* $\alpha > 0$*, we have* $B(\alpha p, m) = B(p, ?)$.

EXERCISE XIX.2. *Assume that the household consumes bundle A in fig. 2. Identify the "left-over" in terms of good* 1*, in terms of good* 2 *and in money terms.*

LEMMA XIX.2. *The money budget is nonempty, closed and convex. If* $p >> 0$ *holds, the budget is bounded.*

PROOF. The budget is nonempty because we have $(0, ..., 0) \in \mathbb{R}^\ell_+$ and $0 \cdot p = 0 \leq m$. It is closed because it is defined by way of weak inequalities ($x_g \geq 0$, $g = 1, ..., \ell$, $x \cdot p \leq m$). We now show convexity. Consider two bundles x and x' and a number $k \in [0, 1]$. Then $x \cdot p \leq m$ and $x' \cdot p \leq m$ imply $(kx + (1-k)x') \cdot p = kx \cdot p + (1-k)x' \cdot p \leq km + (1-k)m = m$. Therefore, the budget is convex. Finally, the budget is bounded in case of $p >> 0$ because every bundle x in the budget fulfils $0 \leq x \leq \left(\frac{m}{p_1}, ..., \frac{m}{p_\ell}\right)$. □

EXERCISE XIX.3. *Verify that the budget line's slope is given by* $-\frac{p_1}{p_2}$ *(in case of* $p_2 \neq 0$*).*

If both prices are positive, the budget line is negatively sloped.

DEFINITION XIX.2. *If prices are non-negative and the price of good* 2 *is positive, the marginal opportunity cost of consuming one unit of good* 1 *in terms of good* 2 *is denoted by* $MOC(x_1)$ *and given by*

$$MOC(x_1) = \left|\frac{dx_2}{dx_1}\right| = \frac{p_1}{p_2}.$$

Thus, if the household wants to consume one additional unit of good 1, he needs to forgo MOC units of good 2 (see also fig. 3). Note that we use the absolute value of the budget line's slope – very similar to the definition of the marginal rate of substitution on p. **??**.

2.2.2. *Endowment budget.*

Definition. In the previous section, the budget is defined by some monetary income m. We now assume that the household has some endowment $\omega \in \mathbb{R}^\ell_+$ which it can consume or, at the prevailing prices, use to buy another bundle. In any case, we obtain the following definition:

DEFINITION XIX.3. *For a price vector* $p \in \mathbb{R}^\ell$ *and an endowment* $\omega \in \mathbb{R}^\ell_+$*, the endowment budget is defined by*

$$B(p, \omega) := \left\{x \in \mathbb{R}^\ell_+ : p \cdot x \leq p \cdot \omega\right\}.$$

Again, equality defines the budget line.

By letting $m := \omega \cdot p$, the endowment budget turns into a money budget. Therefore, lemma XIX.2 holds for an endowment budget as well.

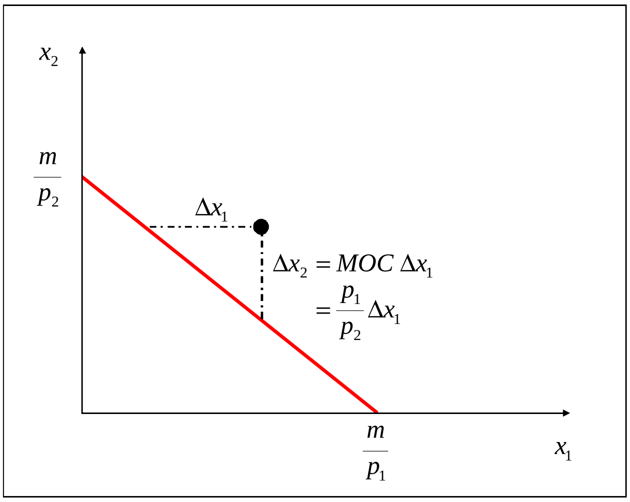

Figure 3. The opportunity cost of one additional unit of good 1

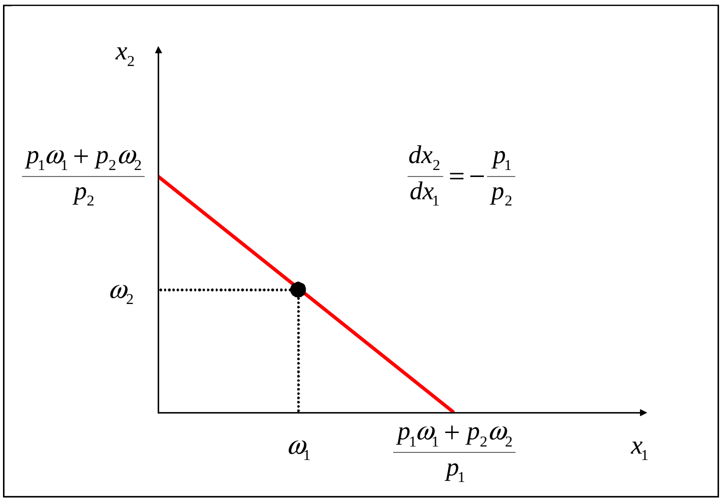

Figure 4. The endowment budget

In case of two goods, the budget line is written as

$$p_1 x_1 + p_2 x_2 = p_1 \omega_1 + p_2 \omega_2$$

and depicted in fig. 4.

Application: consumption today versus consumption tomorrow . We now present three very important examples of endowment budgets. Our first example deals with intertemporal consumption. Consider a household whose monetary income in periods 1 and 2 is ω_1 and ω_2, respectively. His consumption is denoted by x_1 and x_2. We assume that he can borrow ($x_1 > \omega_1$) or lend ($x_1 < \omega_1$). Of course, he can also decide to just consume what he earns ($x_1 = \omega_1$). In either case, he has to break even at the end of the second

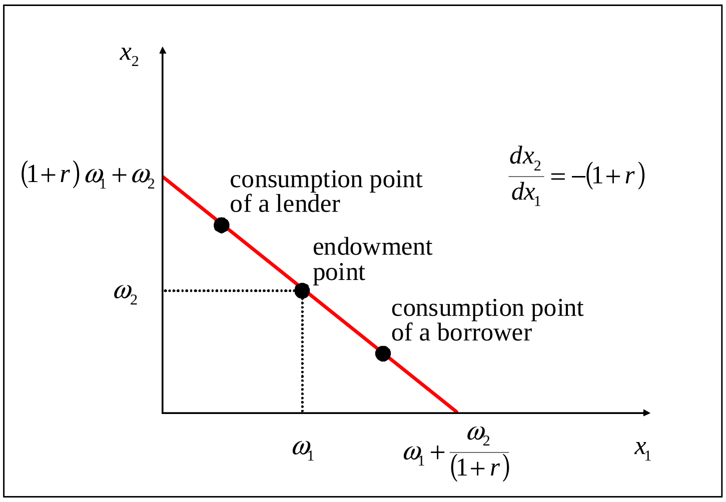

FIGURE 5. Save or borrow?

period. At a given rate of interest r, his second-period consumption is

$$\begin{aligned} x_2 &= \underbrace{\omega_2}_{\substack{\text{second-period}\\ \text{income}}} + \underbrace{(\omega_1 - x_1)}_{\substack{\text{amount borrowed } (<0)\\ \text{or lended } (>0)}} + \underbrace{r\,(\omega_1 - x_1)}_{\substack{\text{interest payed } (<0)\\ \text{or earned } (>0)}} \\ &= \omega_2 + (1+r)\,(\omega_1 - x_1) \end{aligned}$$

We can rewrite the break-even condition (the budget equation) in two different fashions.

- Equalizing the future values of consumption and income yields
$$(1+r)\,x_1 + x_2 = (1+r)\,\omega_1 + \omega_2,$$
while
- the equality of the present values of consumption and income is behind the budget equation
$$x_1 + \frac{x_2}{1+r} = \omega_1 + \frac{\omega_2}{1+r}.$$

Consider also fig. 5 where the present value of the income stream (ω_1, ω_2) is found at the x_1-axis and the future value at the x_2-axis. The marginal opportunity cost of one additional unit of consumption in period 1 is

$$MOC = \left|\frac{dx_2}{dx_1}\right| = 1 + r$$

units of consumption in period 2.

Application: leisure versus consumption. A second application concerns the demand for leisure or, differently put, the supply of labor. We depict the budget line in fig. 6. Recreational hours are denoted by x_R. By definition, the household works $24 - x_R$ hours. For obvious reasons, we have $0 \leq x_R \leq 24 = \omega_R$. Recreational time is the good 1, the second good is real

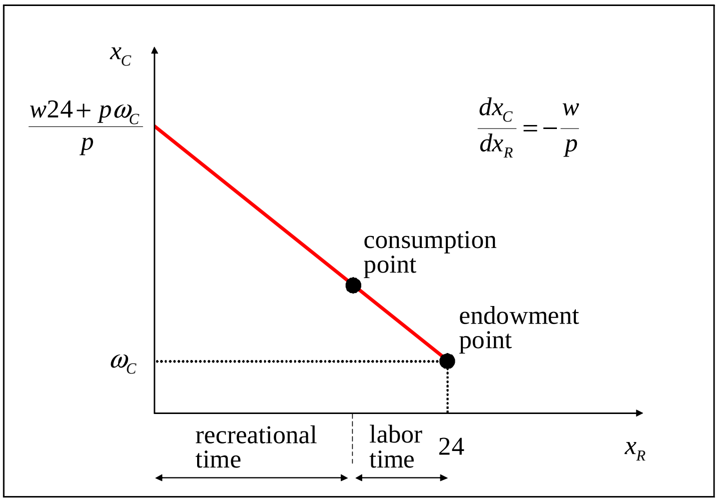

FIGURE 6. Recreational versus labor time

consumption x_C. x_C may stand for the only consumption good (bread) bought and sold at price p. Alternatively, you can think of a bundle of goods x_C and an aggregate price (index) p.

At a wage rate w, the household earns $w(24 - x_R)$. He may also obtain some non-labor income $p\omega_C$ where p is the price index and ω_C the real non-labor income. Thus, the household's consumption in nominal terms is

$$px_C = p\omega_C + w(24 - x_R)$$

which can also be rewritten in endowment-budget form

$$wx_R + px_C = w24 + p\omega_C$$

where $(\omega_C, 24)$ is the endowment point. Thus, the price of leisure is the wage rate. Indeed, if a household chooses to increase its recreational time by one unit, it foregoes w (in monetary consumption terms) or $\frac{w}{p}$ (in real consumption terms). The marginal opportunity cost of one unit of recreational time is

$$MOC = \left|\frac{dx_C}{dx_R}\right| = \frac{w}{p}$$

units of real consumption.

2.3. The household optimum.

2.3.1. *The household's decision situation and problem.* The household's problem can be desribed by the following definition:

DEFINITION XIX.4 (a household's decision situation). *A household's decision situation is a tuple*

$$\begin{aligned} \Delta &= (B, \precsim) \text{ with} \\ B &= B(p,m) \subseteq \mathbb{R}_+^\ell \text{ or } B = B(p,\omega) \subseteq \mathbb{R}_+^\ell \end{aligned}$$

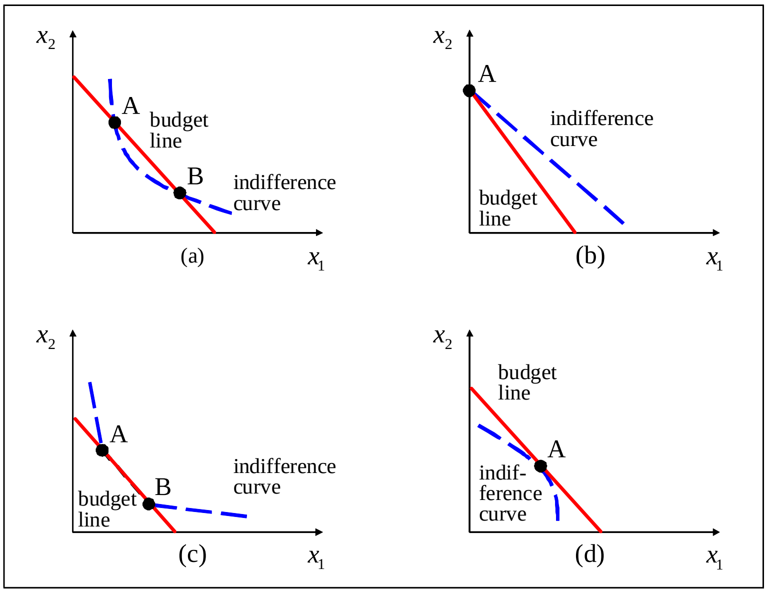

FIGURE 7. Household optima?

where $p \in \mathbb{R}^\ell$ *is a vector of prices and* $\precsim$ *a preference relation on* $\mathbb{R}^\ell_+$. *The household's problem is to find the best-response function given by*

$$x^R(\Delta) := \{x \in B\text{: there is not } x' \in B \text{ with } x' \succ x\}$$

If $\precsim$ *is representable by a utility function* U *on* $\mathbb{R}^\ell_+$, *we have the decision situation* $\Delta = (B, U)$ *and the best-response function*

$$x^R(\Delta) := \arg\max_{x \in B} U(x).$$

Any x^* *from* $x^R(\Delta)$ *is called a household optimum.*

Thus, the household aims to find a highest indifference curve attainable with his budget. As a very obvious corollary from lemma XIX.1, we have

LEMMA XIX.3. *For any number* $\alpha > 0$, *we have* $x^R(\alpha p, \alpha m) = x^R(p, m)$.

EXERCISE XIX.4. *Look at the household situations depicted in fig. 7. Assume monotonicity of preferences. Are the highlighted points A or B optima?*

EXERCISE XIX.5. *Assume a household's decision problem with endowment* $\Delta = (B(p, \omega), \precsim)$. $x^R(\Delta)$ *consists of the bundles* x *that fulfill the two conditions:*

(1) *The household can afford* x*:*

$$p \cdot x \leq p \cdot \omega$$

Marginal willingness to pay:	$MRS = \left\lvert \frac{dx_2}{dx_1} \right\rvert$
If the household consumes one additional unit of good 1, how many units of good 2 can he forgo so as to remain indifferent.	movement on the indifference curve
Marginal opportunity cost:	$MOC = \left\lvert \frac{dx_2}{dx_1} \right\rvert$
If the household consumes one additional unit of good 1, how many units of good 2 does he have to forgo so as to remain within his budget.	movement on the budget line

FIGURE 8. Willingness to pay and opportunity cost

(2) *There is no other bundle y that the household can afford and that he prefers to x:*

$$y \succ x \Rightarrow ??$$

Substitute the question marks by an inequality.

2.3.2. *MRS versus MOC.* A good part of household theory can be couched in terms of the marginal rate of substitution and the marginal opportunity cost. Consider fig. 8. We can ask two questions:

- What is the household's willingness to pay for one additional unit of good 1 in terms of units of good 2? The answer is MRS units of good 2.
- What is the household's cost for one additional unit of good 1 in terms of units of good 2? The answer: MOC units of good 2.

Now, the interplay of the marginal rate of substitution MRS and marginal opportunity cost MOC helps to find the household optimum. Consider

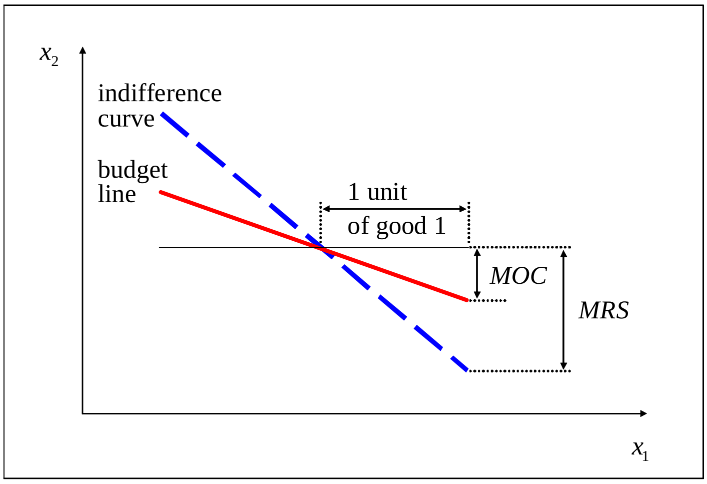

FIGURE 9. Not optimal

the inequality

$$MRS = \underbrace{\left|\frac{dx_2}{dx_1}\right|}_{\substack{\text{absolute value}\\ \text{of the slope of}\\ \text{the indifference}\\ \text{curve}}} > \underbrace{\left|\frac{dx_2}{dx_1}\right|}_{\substack{\text{absolute value}\\ \text{of the slope of}\\ \text{the budget line}}} = MOC.$$

If, now, the household increases his consumption of good 1 by one unit, he can decrease his consumption of good 2 by MRS units and still stay on the same indifference curve. Compare fig. 9. However, the increase of good 1 necessitates a decrease of only $MOC < MRS$ units of good 2. Therefore, the household needs to give up less than he would be prepared to. In case of strict monotonicity, increasing the consumption of good 1 leads to a higher indifference curve.

Thus, we cannot have $MRS > MOC$ at the optimal bundle unless it is impossible to further increase the consumption of good 1. This is the situation depicted in fig. 10.

Thus, if the household consumes both goods in positive quantities, we can derive the optimality condition

$$MRS \stackrel{!}{=} MOC$$

(if both terms are defined).

Alternatively, we can derive this first-order condition with the help of a utility function (if we have one). The household tries to maximize

$$U\left(x_1, \frac{m}{p_2} - \frac{p_1}{p_2}x_1\right).$$

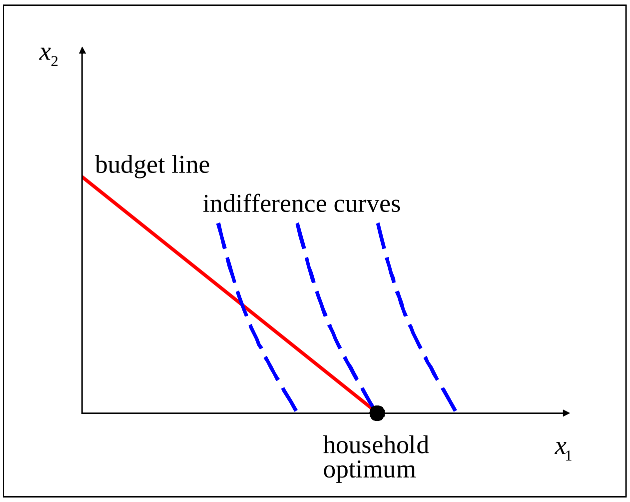

FIGURE 10. The willingness to pay can be higher than the cost.

If the household increases the consumption of good 1 by one unit, we have two effects. First, his utility increases by $\frac{\partial U}{\partial x_1}$. Second, an increase in x_1 leads to a reduction in x_2 by $MOC = \left|\frac{dx_2}{dx_1}\right| = \frac{p_1}{p_2}$ and this reduced consumption of good 2 decreases utility. Therefore, the household increases x_1 as long as

$$\underbrace{\frac{\partial U}{\partial x_1}}_{\substack{\text{marginal benefit}\\ \text{of increasing } x_1}} > \underbrace{\frac{\partial U}{\partial x_2}\left|\frac{dx_2}{dx_1}\right|}_{\substack{\text{marginal cost}\\ \text{of increasing } x_1}}$$

holds. Dividing by $\frac{\partial U}{\partial x_2}$, an increase in x_1 leads to an increase in utility if

$$\begin{aligned} MRS &= \frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}} \text{ (chapter ?? on p. ??)} \\ &> \left|\frac{dx_2}{dx_1}\right| = MOC \end{aligned}$$

holds.

The MRS- versus-MOC rule can help to derive the household optimum in some cases:

- Cobb-Douglas utility functions $U(x_1, x_2) = x_1^a x_2^{1-a}$ with $0 < a < 1$ lead to
$$MRS = \frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}} = \frac{a}{1-a}\frac{x_2}{x_1} \stackrel{!}{=} \frac{p_1}{p_2}$$
and, together with the budget line, the household optimum
$$\begin{aligned} x_1(m,p) &= a\frac{m}{p_1}, \\ x_2(m,p) &= (1-a)\frac{m}{p_2}. \end{aligned}$$

- Goods 1 and 2 are perfect substitutes if the utility function is given by $U(x_1, x_2) = ax_1 + bx_2$ with $a > 0$ and $b > 0$. An increase of good 1 enhances utility if
$$\frac{a}{b} = MRS > MOC = \frac{p_1}{p_2}$$
holds so that we find the household optimum
$$x(m,p) = \begin{cases} \left(\frac{m}{p_1}, 0\right), & \frac{a}{b} > \frac{p_1}{p_2} \\ \left\{\left(x_1, \frac{m}{p_2} - \frac{p_1}{p_2}x_1\right) \in \mathbb{R}_+^2 : x_1 \in \left[0, \frac{m}{p_1}\right]\right\} & \frac{a}{b} = \frac{p_1}{p_2} \\ \left(0, \frac{m}{p_2}\right) & \frac{a}{b} < \frac{p_1}{p_2} \end{cases}$$
- Preferences are concave with utility function $U(x_1, x_2) = x_1^2 + x_2^2$. We have the marginal rate of substitution
$$MRS = \frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}} = \frac{2x_1}{2x_2} = \frac{x_1}{x_2}$$
so that
$$\frac{x_1}{x_2} = MRS > MOC = \frac{p_1}{p_2}$$
holds for sufficiently large x_1 which calls for an increase of x_1. Inversely,
$$\frac{x_1}{x_2} = MRS < MOC = \frac{p_1}{p_2}$$
holds for sufficiently large x_2 so that an increase of x_2 seems a good idea. Therefore, we need to compare the extreme bundles $\left(\frac{m}{p_1}, 0\right)$ and $\left(0, \frac{m}{p_2}\right)$ and obtain
$$\begin{aligned} \left(\frac{m}{p_1}\right)^2 + 0^2 &\geq 0^2 + \left(\frac{m}{p_2}\right)^2 \text{ and} \\ p_1 &\leq p_2 \end{aligned}$$
and finally
$$x(m,p) = \begin{cases} \left(\frac{m}{p_1}, 0\right), & p_1 \leq p_2 \\ \left\{\left(\frac{m}{p_1}, 0\right), \left(0, \frac{m}{p_2}\right)\right\} & p_1 = p_2 \\ \left(0, \frac{m}{p_2}\right) & p_1 \geq p_2 \end{cases}$$

2.3.3. *Household optimum and monotonicity.* We now turn to specific implications that can be drawn from the fact that some x^* is a household optimum and that some sort of monotonicity holds.

Lemma XIX.4. *Let x^* be a household optimum of the decision situation $\Delta = (B(p,m), \precsim)$. Then, we have the following implications:*

- *Walras' law: Local nonsatiation implies $p \cdot x^* = m$.*
- *Stict monotonicity implies $p >> 0$.*
- *Local nonsatiation and weak monotonicity imply $p \geq 0$.*

PROOF. We use proofs by contradiction for each statement:

- Because of $x^* \in B$, we can exclude $p \cdot x^* > m$. Assume $p \cdot x^* < m$. Then, the household can afford bundles sufficiently close to x^*. By local nonsatiation, some of these bundles are better than x^*. This is a contradiction to x^* being a household optimum.
- Turning to the second implication, assume a household optimum and a price p_g which is zero or negative. Then, the household can afford more of good g. By strict monotonicity, the household is better off implying the desired contradiction (existence of household optimum).
- Assume a negative price for some good g. By weak monotonicity the household can "buy" additional units of that good without being worse off. Since the price is negative, the household has additional funding for preferred bundles which exist by nonsatiation. Again, a contradiction to the existence of a household optimum follows.

□

3. NTU coalition functions and the core

3.1. Definition of NTU coalition functions. We denote the coalition function without transferable utility by V in order to make the distinction from the coalition function v in the transferable-utility case. V attributes to every coalition $K \neq \emptyset$ a set of utility vectors

$$u_K := (u_i)_{i \in K} \in \mathbb{R}^{|K|}$$

for K's members. The interpretation is similar to transferable utility – the coalition K can achieve any vector from $V(K)$ all by itself.

EXERCISE XIX.6. *Depict*

$$V(\{Peter,\ Otto\}) = \{(u_{Peter}, u_{Otto}) : u_{Peter} \geq 2, u_{Otto} \geq 1, u_{Peter} + u_{Otto} \leq 4\}.$$

DEFINITION XIX.5 (coalition function). *A coalition function V on N for non-transferable utlity associates to every subset K of N a subset of $\mathbb{R}^{|K|}$ such that*

- $V(\emptyset) = \emptyset$ *and*
- $V(K) \neq \emptyset$ *for* $K \neq \emptyset$

hold. $V(K)$ is called coalition K's worth. In order to economize on the use of symbols, we denote the set of all games on N by $\mathbb{V}_N$ and the set of all games (for any player set N) by $\mathbb{V}$.

EXERCISE XIX.7. *Which of the following expressions are formally correct?*

- $V(\{1,2\}) = 1$

- $V(\{1,2\}) = \{1\}$
- $V(\{1,2\}) = (1,2)$
- $V(\{1,2\}) = \emptyset$
- $V(\{1,2\}) = \{(1,2)\}$
- $V(\{1,2\}) = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \le 3, x_2 \le 4, x_1 + x_2 \le 5\right\}$

Normally, V has to obey some other requirements, too, such as convexity or closedness (consult McLean 2002, pp. 2079). A coalition function v with transferable utility can be written as a coalition function V without transferable utility by letting

$$V(K) := \left\{ x_K \in \mathbb{R}^{|K|} : \sum_{i \in K} x_i \le v(K) \right\}$$

for each $K \subseteq N$. If K contains two players, $V(K)$ equals the set of utility tuples lying on or below a line of slope -1.

Exercise XIX.8. *Determine the axis intercepts of the line just mentioned.*

Finally, we want to discuss what superadditivity means in the present context.

Definition XIX.6 (superadditivity). *The coalition function V without transferable utility is called superadditive if, for all coalitions $S, T \subset N$*

$$\begin{aligned} S \cap T &= \emptyset \text{ (}S\text{ and }T\text{ are disjunct),} \\ u_S &\in V(S) \text{ and} \\ u_T &\in V(T) \end{aligned}$$

imply

$$(u_S, u_T) \in V(S \cup T).$$

Here (u_S, u_T) is the vector that contains utilities for the players from S and for the players from T. For two agents superadditivity means that the utility levels that the two agents can achieve individually are still in reach after forming a coalition.

Exercise XIX.9. *Are V_1 and/or V_2 defined on $N = \{1,2,3\}$ and given by*

- $V_1(K) = \begin{cases} \{i\}, & K = \{i\} \\ \{(x_1, x_2) : x_1 \le 1, x_2 \le 4\}, & K = \{1,2\} \\ \{(x_1, x_3) : x_1 \le 2, x_3 \le 3\}, & K = \{1,3\} \\ \{(x_2, x_3) : x_2 \le 4, x_3 \le 5\}, & K = \{2,3\} \\ \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 \le 10\} & K = \{1,2,3\} \end{cases}$

- $V_2(K) = \begin{cases} \{i\}, & K = \{i\} \\ \{(x_1, x_2) : x_1 \leq 1, x_2 \leq 4\}, & K = \{1,2\} \\ \{(x_1, x_3) : x_1 \leq 2, x_3 \leq 2\}, & K = \{1,3\} \\ \{(x_2, x_3) : x_2 \leq 4, x_3 \leq 5\}, & K = \{2,3\} \\ \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 \leq 9\} & K = \{1,2,3\} \end{cases}$

superadditive?

3.2. The core.

Definition XIX.7 (core). *The core of a NTU game V is the set all utility vectors $u = (u_i)_{i \in N} \in \mathbb{R}^n$ that obey feasibility and non-blockability:*

- $u \in V(N)$.
- *There is no coaliton K and no utility vector $u' = (u'_i)_{i \in N}$ such that $u'_K \in V(K)$ holds and $u_i \leq u'_i$ for all $i \in K$ with strict inequality for at least on $i \in K$.*

Non-blockability means: If a coalition K kann achieve u' for its members, u' must not be a coalition-specific Pareto improvement over u. Differently put: If there are a utility vectors u and u' and a colition K such that

$$u'_K \in V(K)$$

and

$$u_K < u'_K$$

hold, u does not lie in the core (since K blocks u).

The core of a NTU coalition function can be empty; Predtetchinski & Herings (2004) specify necessary and sufficient condition for a non-empty core.

4. Edgeworth boxes and coalition functions

4.1. Preview. In order to understand General Equilibrium Theory, the reader needs to know basic household theory (budget, utility functions, indifference curves, household optimum). Chapter II (pp. 16) may also be helpful.

Before delving into the General Equilibrium Theory, we will give you a short preview of where we are heading to. The General Equilibrium Theory has two grand themes. The first is Pareto improvements through exchanges. The second big topic is decentralization through prices.

It is quite possible to add price information into Edgeworth boxes. If household A buys a bundle (x_1^A, x_2^A) with the same worth as his endowment, we have

$$p_1 x_1^A + p_2 x_2^A = p_1 \omega_1^A + p_2 \omega_2^A.$$

Starting from an endowment point, positive prices p_1 and p_2 lead to negatively sloped budget lines for both individuals. In fig. 11, two price lines

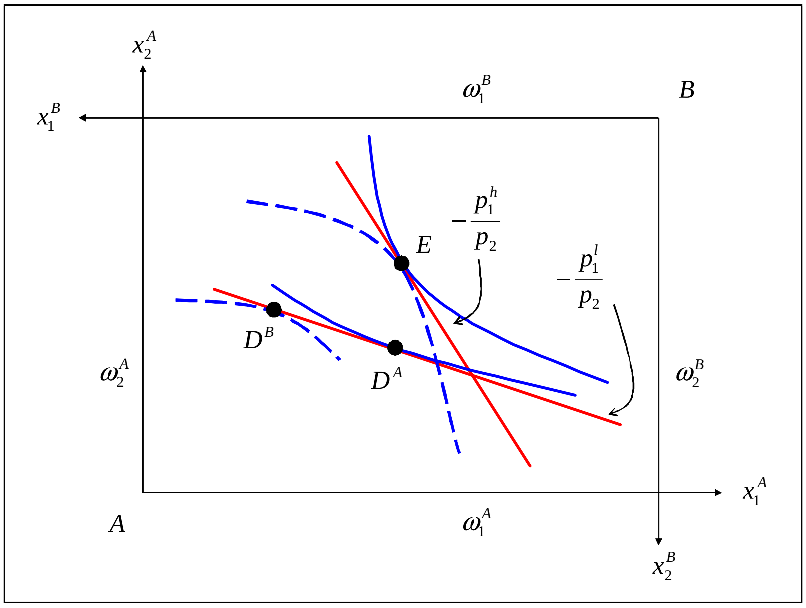

FIGURE 11. Walras equilibrium

with prices $p_1^l < p_1^h$ are depicted. The indifference curves indicate which bundles the households prefer.

EXERCISE XIX.10. *Why do the two price lines in fig. 11 cross at the endowment point* ω*?*

Of course, we would like to know whether these prices are compatible in the sense of allowing both agents to demand the preferred bundle. If that is the case, the prices and the bundles at these prices constitute a Walras equilibrium.

EXERCISE XIX.11. *The low price* p_1^l *is not possible in a Walras equilibrium, because there is excess demand for good* 1 *at this price:*

$$x_1^A + x_1^B > \omega_1^A + \omega_1^B.$$

Do you see that? How about good 2*?*

4.2. The NTU coalition function of an exchange economy. We now proceed to the formal definition of an exchange economy.

DEFINITION XIX.8 (exchange economy). *An exchange economy is a tuple*

$$\mathcal{E} = \left(N, G, \left(\omega^i\right)_{i\in N}, \left(U_i\right)_{i\in N}\right)$$

consisting of

- *the set of agents* $N = \{1, 2, ..., n\}$,
- *the finite set of goods* $G = \{1, ..., \ell\}$,

and for every agent $i \in N$

- *an endowment* $\omega^i = \left(\omega_1^i, ..., \omega_\ell^i\right) \in \mathbb{R}_+^\ell$, *and*
- *a utility function* $U_i : \mathbb{R}_+^\ell \to \mathbb{R}$.

Thus, every agent has property rights on endowments. In the two-agents two-good case, the exchange economy can be depicted by the exchange Edgeworth box:

- The size of the box is given by $\omega = \omega^1 + \omega^2$.
- The utility functions can be depicted by indifference curves (together with numbers indicating strict preference).
- The endowment $\left(\omega^1, \omega^2\right)$ is a specific point within the box.

Definition XIX.9. *Consider an exchange economy* $\mathcal{E}$.

- *A bundle* $(y^i)_{i\in N} \in \mathbb{R}_+^{\ell\cdot n}$ *is an allocation.*
- *An allocation* $(y^i)_{i\in N}$ *is called* K*-feasible if* $\sum_{i\in K} y^i \leq \sum_{i\in K} \omega^i$ *holds.*
- *An allocation* $(y^i)_{i\in N}$ *is called feasible if it is* N*-feasible.*

We can now specify the NTU coalition function associated with a given exchange economy. For $K \neq \emptyset$, we let

$$V(K) := \left\{u_K \in \mathbb{R}^{|K|} : \text{There is a } K\text{-feasible allocation } x \text{ with } u_i \leq U_i(x_i), i \in K\right\}.$$

Thus, for every non-empty coalition K we determine the set of bundles that this coalition possesses where the entries for the non-K players are of no relevance. Every such K-feasible allocation defines the maximal utility levels that the players from K can achieve.

5. GET: decentralization through prices

5.1. Excess Demand and Market Clearance. In this section we deal with the question whether the demand for one good is greater than the supply for this good.

Definition XIX.10. *Assume an exchange economy* $\mathcal{E}$, *a good* $g \in G$ *and a price vector* $p \in \mathbb{R}^\ell$. *If every household* $i \in N$ *has a unique household optimum* $x^i\left(p, \omega^i\right)$, *good* g*'s excess demand is denoted by* $z_g(p)$ *and defined by*

$$z_g(p) := \sum_{i=1}^n x_g^i\left(p, \omega^i\right) - \sum_{i=1}^n \omega_g^i.$$

The corresponding excess demand for all goods $g = 1, ..., \ell$ *is the vector*

$$z(p) := (z_g(p))_{g=1,...,\ell}.$$

The excess demand is a quantity of goods and a vector of quantities of goods respectively. In contrast, the value of the excess demand, which is given by

$$p \cdot z(p),$$

is a scalar amount of money. We remind the reader of Walras' law (p. 356) which immediately implies the following version:

LEMMA XIX.5 (Walras' law). *Every consumer demands a bundle of goods obeying $p \cdot x^i \leq p \cdot \omega^i$ where local nonsatiation implies equality. For all consumers together, we have*

$$p \cdot z(p) = \sum_{i=1}^{n} p \cdot \left(x^i - \omega^i\right) \leq 0$$

and, assuming local-nonsatiation, $p \cdot z(p) = 0$.

Walras' law is of great importance for General Equilibrium Theory. We will later look at the conditions under which excess demand is zero. Then, the problem is to get from $z(p) \cdot p = 0 \in \mathbb{R}$ to $z(p) = 0 \in \mathbb{R}^\ell$.

DEFINITION XIX.11. *A market g is called cleared if excess demand $z_g(p)$ on that market is equal to zero.*

The following two exercises are adapted from Leach (2004, pp. 54) .

EXERCISE XIX.12. *Consider a market where the excess demand of three individuals 1, 2, and 3 is given by*

$$z_1(p) = \frac{8}{p} - 4, z_2(p) = \frac{4}{p} - 2, z_3(p) = \frac{12}{p} - 2.$$

Find the market-clearing price. Is individual 3 a buyer or a seller?

EXERCISE XIX.13. *Abba (A) and Bertha (B) consider buying two goods 1 and 2, and face the price p for good 1 in terms of good 2. Think of good 2 as the numeraire good with price 1. Abba's and Bertha's utility functions, u_A and u_B, respectively, are given by $u_A\left(x_1^A, x_2^A\right) = \sqrt{x_1^A} + x_2^A$ and $u_B\left(x_1^B, x_2^B\right) = \sqrt{x_1^B} + x_2^B$. Endowments are $\omega^A = (18, 0)$ and $\omega^B = (0, 10)$. Find the bundles demanded by these two agents. Then find the price p that fulfills $\omega_1^A + \omega_1^B = x_1^A + x_1^B$ and $\omega_2^A + \omega_2^B = x_2^A + x_2^B$.*

In the above exercise, what, if only market 1 is cleared? The following lemma shows that local nonsatiation excludes this possibility.

LEMMA XIX.6 (Market clearance). *In case of local nonsatiation,*

(1) *if all markets but one are cleared, the last one also clears or its price is zero,*
(2) *if at prices $p \gg 0$ all markets but one are cleared, all markets clear.*

PROOF. If $\ell - 1$ markets are cleared, the excess demand on these markets is 0. Without loss of generality, markets $g = 1, ..., \ell - 1$ are cleared. Applying Walras's law we get

$$\begin{aligned} 0 &= p \cdot z(p) \\ &= p_\ell z_\ell(p), \end{aligned}$$

and hence both claims. □

5.2. Walras equilibrium.

5.2.1. *Definition.* Are there prices for all ℓ goods, for which all individual demands are possible at the same time? Differently put, is there a price vector $\widehat{p}$, such that the demand for all ℓ goods does not exceed the initial endowment:

DEFINITION XIX.12. *A price vector $\widehat{p}$ and the corresponding demand system $\left(\widehat{x}^i\right)_{i=1,\dots,n} = \left(x^i\left(\widehat{p},\omega^i\right)\right)_{i=1,\dots,n}$ is called a Walras equilibrium if*

$$\sum_{i=1}^{n} \widehat{x}^i \leq \sum_{i=1}^{n} \omega^i$$

or

$$z\left(\widehat{p}\right) \leq 0$$

holds.

The equilibrium condition requires that

1. all households choose an optimal bundle, i.e., every household i chooses the bundle of goods $x^i\left(\widehat{p},\omega^i\right)$ (or a bundle from $x^i\left(\widehat{p},\omega^i\right)$) at given prices $\widehat{p}$,
2. the resulting allocation is feasible, or, differently put, for every good, the quantity demanded is not larger than the available quantity.

EXERCISE XIX.14. *Rewrite the equilibrium condition*

$$\sum_{i=1}^{n} x^i\left(\widehat{p},\omega^i\right) \leq \sum_{i=1}^{n} \omega^i$$

so that it is clear that the inequalities must hold for each good.

The equilibrium condition excludes that the demand for one good is greater than the supply for this good. The reader might find this definition of the equilibrium confusing at the first glance. Why do we not define equilibrium through the equality of supply and demand? The definition is weaker and we will show in the next section that under certain conditions a not positive excess demand implies an excess demand of zero.

5.2.2. *Market clearing in the Walras equilibrium.* In this section, we will present the conditions for which a market in equilibrium has an excess demand of zero, i.e. the market is cleared. Consider the following definitions and lemmata:

DEFINITION XIX.13. *A good is called free if its price is equal to zero.*

LEMMA XIX.7 (free goods). *Assume local nonsatiation and weak monotonicity for all households. If $\left[\widehat{p},\left(\widehat{x}^i\right)_{i=1,\dots,n}\right]$ is a Walras equilibrium and the excess demand for a good is negative, this good must be free.*

PROOF. Assume, to the contrary, that $p_g > 0$ holds. We obtain a contradiction to Walras law for local nonsatiation:

$$\begin{aligned} p \cdot z(p) &= \underbrace{p_g z_g(p)}_{<0} + \sum_{\substack{g'=1, \\ g' \neq g}}^{\ell} p_{g'} z_{g'}(p) \quad (z_g(p) < 0) \\ &< \sum_{\substack{g'=1, \\ g' \neq g}}^{\ell} \underbrace{p_{g'}}_{\geq 0} \underbrace{z_{g'}(p)}_{\leq 0} \text{ (lemma XIX.4, p. 356)} \\ &\leq 0. \end{aligned}$$

□

Finally (for now), we need to define the desiredness of goods:

DEFINITION XIX.14. *A good is desired if the excess demand at price zero is positive.*

LEMMA XIX.8 (desiredness). *If all goods are desired and if local nonsatiation and weak monotonicity hold and if $\widehat{p}$ is a Walras equilibrium, then $z(\widehat{p}) = 0$.*

PROOF. Suppose that there is a good g with $z_g(\widehat{p}) < 0$. Then g must be a free good according to lemma XIX.7 and have a positive excess demand by the definition of desiredness, $z_g(\widehat{p}) > 0$. □

5.2.3. *Example: The Cobb-Douglas Exchange Economy with Two Agents.* We remember from chapter **??** that income m and Cobb-Douglas utility function

$$u(x_1, x_2) = x_1^a x_2^{1-a}$$

implies the household optimum

$$\begin{aligned} x_1 &= a\frac{m}{p_1}, \\ x_2 &= (1-a)\frac{m}{p_2}. \end{aligned}$$

Consider, now, individual 1 with Cobb-Douglas utility function u^1 and parameters a_1 (for good 1) and $1 - a_1$ (for good 2). The initial endowment of individual 1 equals $\omega^1 = (1, 0)$. Individual 2 possesses a Cobb-Douglas utility function u^2 with parameters a_2 (for good 1) and $1 - a_2$ (for good 2). His initial endowment is $\omega^2 = (0, 1)$. Parameters a_1 and a_2 obey the following conditions: $0 < a_1 < 1$ and $0 < a_2 < 1$. Both goods are desired and local strict monotonicity holds. According to lemma XIX.8, the market is in equilibrium only if it is cleared. Substituting the value of the endowment for income, we get the demand for good 1 for individual 1 :

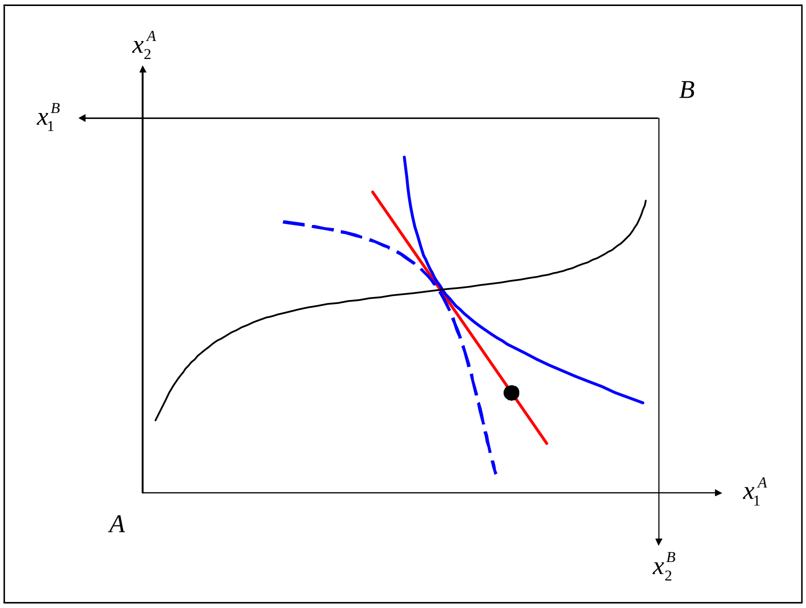

FIGURE 12. The General Equilibrium in the exchange Edgeworth box

$$\begin{aligned}
&x_1^1\left(p_1, p_2, \omega^1 \cdot p\right) \\
&= a_1 \frac{\omega^1 \cdot p}{p_1} \\
&= a_1
\end{aligned}$$

and the demand for good 1 for individual 2

$$\begin{aligned}
&x_1^2\left(p_1, p_2, \omega^2 \cdot p\right) \\
&= a_2 \frac{\omega^2 \cdot p}{p_1} \\
&= a_2 \frac{p_2}{p_1}.
\end{aligned}$$

Assuming positive prices, lemma XIX.6 (p. 362) says that both markets are cleared if one is cleared. Market 1 is cleared if demand equals supply, i.e., if

$$a_1 + a_2 \frac{p_2}{p_1} = 1,$$

which is equivalent to

$$\frac{p_2}{p_1} = \frac{1 - a_1}{a_2}.$$

All prices, which satisfy these equations, are equilibrium prices. Obviously, only relative prices are determined.

Figure 12 sketches the equilibrium in the two-goods case.

5.3. Existence of the Walras equilibrium.

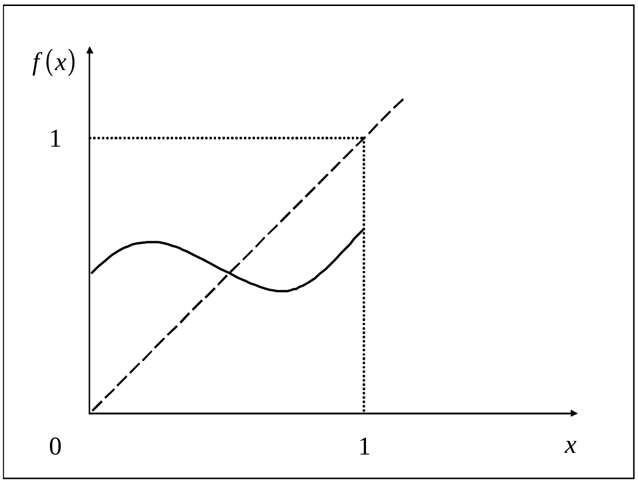

FIGURE 13. Brouwer fixed-point theorem

5.3.1. *Proposition.* So far we have not questioned the existence of the Walras equilibrium. Fortunately, the following theorem holds:

THEOREM XIX.1 (Existence of the Walras Equilibrium). *If aggregate excess demand is a continuous function (in prices), if the value of the excess demand is zero and if the preferences are strictly monotonic, there exists a price vector* $\widehat{p}$ *such that* $z(\widehat{p}) \leq 0$.

The proof of this theorem uses Brouwer's fixed-point theorem. Therefore, we introduce this theorem in the next section and then present the proof of the proposition in section 2.5.3.

5.3.2. *Brouwer fixed-point theorem.*

THEOREM XIX.2. *Suppose* $f : M \to M$ *is a function on the nonempty, compact and convex set* $M \subseteq \mathbb{R}^\ell$. *If* f *is continuous, there exists* $x \in M$ *such that* $f(x) = x$. x *is called a fixed point.*

Note that the range of f is included in M. One can figure out the conclusion from Brouwer's fixed-point theorem for the one-dimensional case by means of a continuous function on the unit interval. If either $f(0) = 0$ or $f(1) = 1$ hold, one fixed point is found. Otherwise, fig. 13 shows a continuous function fulfilling $f(0) > 0$ and $f(1) < 1$. The graph of such a figure cuts the 45°-line. The projection of this intersection point onto the x- or the y-axis is the sought-after fixed point.

The fixed-point theorem can be nicely illustrated. Put a handkerchief on the square $[0,1] \times [0,1]$ from $\mathbb{R}^2$. This subset is nonempty, compact and convex. A continuous function

$$f : [0,1] \times [0,1] \to [0,1] \times [0,1]$$

corresponds to the following process:

- rumple the handkerchief,
- put the rumpled handkerchief again on the square and
- press it flat.

The handkerchief must not be torn because tearing corresponds to a non-continuous f . Brouwer's fixed-point theorem now claims that there is at least one spot on the handkerchief which, before and after rumpling, comes to lie on the same place of the square. Alternatively, one can imagine stirring cake dough with a wooden spoon so that the dough does not lose its coherence. At least one participle of the dough does not change its place despite the stirring movements.

We do not prove the theorem but ask you to try the following exercise.

Exercise XIX.15. *Assume, one of the requirements for the fixed-point theorem does not hold. Show, by a counter example, that there can be a function such that there is no fixed point. Specifically, assume that*
a) M is not compact
b) M is not convex
c) f is not continuous.

May-be, German-speaking people may like to learn Brouwer's fixed-point theorem by memorizing the poem due to Hans-Jürgen Podszuweit (found in Homo Oeconomicus, XIV (1997), p. 537):

Das Nilpferd hört perplex:
Sein Bauch, der sei konvex.
Und steht es vor uns nackt,
sieht man: Er ist kompakt.
Nimmt man 'ne stetige Funktion
von Bauch
in Bauch
– Sie ahnen schon –,
dann nämlich folgt aus dem
Brouwer'schen Theorem:
Ein Fixpunkt muß da sein.
Dasselbe gilt beim Schwein
q.e.d.

5.3.3. *Proof of the existence theorem XIX.1.* In order to apply Brouwer's fixed-point theorem to proposition XIX.1, we first construct a convex and compact set. The prices of the ℓ goods could be normed such that the sum of the nonnegative (!, we have strict monotonicity) prices equals 1. Just divide all prices by the sum of the prices. We can restrict our search for equilibrium prices to the $\ell - 1$- dimensional unit simplex:

$$S^{\ell-1} = \left\{ p \in \mathbb{R}_+^\ell : \sum_{g=1}^{\ell} p_g = 1 \right\}.$$

$S^{\ell-1}$is nonempty, compact (closed and bounded as a subset of $\mathbb{R}^{\ell-1}$) and convex.

Exercise XIX.16. *Draw $S^1 = S^{2-1}$.*

The idea of the proof is as follows: First, we define a continuous function f on this (nonempty, compact and convex) set. Brouwer's theorem says that there is at least one fixed point of this function. Second, we show that such a fixed point fulfills the condition of the Walras equilibrium.

The abovementioned continuous function

$$f = \begin{pmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ f_\ell \end{pmatrix} : S^{\ell-1} \to S^{\ell-1}$$

is defined by

$$f_g(p) = \frac{p_g + \max(0, z_g(p))}{1 + \sum_{g'=1}^{\ell} \max\left(0, z_{g'}(p)\right)}, g = 1, ..., \ell$$

f is continuous because every f_g, $g = 1, ..., \ell$, is continuous. The latter is continuous because z (according to our assumption) and max are continuous functions. Finally, we can confirm that f is well defined, i.e., that $f(p)$ lies in $S^{\ell-1}$ for all p from $S^{\ell-1}$:

$$\begin{aligned}
\sum_{g=1}^{\ell} f_g(p) &= \sum_{g=1}^{\ell} \frac{p_g + \max(0, z_g(p))}{1 + \sum_{g'=1}^{\ell} \max\left(0, z_{g'}(p)\right)} \\
&= \frac{1}{1 + \sum_{g'=1}^{\ell} \max\left(0, z_{g'}(p)\right)} \sum_{g=1}^{\ell} (p_g + \max(0, z_g(p))) \\
&= \frac{1}{1 + \sum_{g'=1}^{\ell} \max\left(0, z_{g'}(p)\right)} \left(1 + \sum_{g=1}^{\ell} \max(0, z_g(p))\right) \\
&= 1.
\end{aligned}$$

The function f increases the price of a good g in case of $f_g(p) > p_g$, only, i.e. if

$$\frac{p_g + \max(0, z_g(p))}{1 + \sum_{g'=1}^{\ell} \max\left(0, z_{g'}(p)\right)} > p_g$$

or

$$\max\left(0,z_g\left(p\right)\right)>p_g\sum_{g'=1}^{\ell}\max\left(0,z_{g'}\left(p\right)\right)$$

or

$$\frac{\max\left(0,z_g\left(p\right)\right)}{\sum_{g'=1}^{\ell}\max\left(0,z_{g'}\left(p\right)\right)}>\frac{p_g}{\sum_{g'=1}^{\ell}p_{g'}}$$

holds.

The last formula has a nice interpretation: when the relative excess demand for a good is greater than the relative price for the same good (as measured by the sum of the excess demand, respectively the sum of the prices), the function f increases the price. Behind f, we imagine the workings of the Walras auctioneer, who changes prices upon observing excess demands. This so-called tâtonnement may (or may not) converge towards the equilibrium price vector.

We now complete the proof: according to Brouwer's fixed-point theorem there is one $\widehat{p}$ such that

$$\widehat{p}=f\left(\widehat{p}\right),$$

from which we have

$$\widehat{p}_g=\frac{\widehat{p}_g+\max\left(0,z_g\left(\widehat{p}\right)\right)}{1+\sum_{g'=1}^{\ell}\max\left(0,z_{g'}\left(\widehat{p}\right)\right)}$$

and finally

$$\widehat{p}_g\sum_{g'=1}^{\ell}\max\left(0,z_{g'}\left(\widehat{p}\right)\right)=\max\left(0,z_g\left(\widehat{p}\right)\right)$$

for all $g=1,...,\ell$.

Next we multiply both sides for all goods $g=1,...,\ell$ by $z_g(\widehat{p})$:

$$z_g(\widehat{p})\widehat{p}_g\sum_{g'=1}^{\ell}\max\left(0,z_{g'}\left(\widehat{p}\right)\right)=z_g(\widehat{p})\max\left(0,z_g\left(\widehat{p}\right)\right)$$

and summing up over all g yields

$$\sum_{g=1}^{\ell}z_g(\widehat{p})\widehat{p}_g\sum_{g'=1}^{\ell}\max\left(0,z_{g'}\left(\widehat{p}\right)\right)=\sum_{g=1}^{\ell}z_g(\widehat{p})\max\left(0,z_g\left(\widehat{p}\right)\right).$$

By Walras' law, the left-hand expression is equal to zero. The right one consists of a sum of expressions, which are equal either to zero or to $\left(z_g\left(\widehat{p}\right)\right)^2$. Therefore, $z_g\left(\widehat{p}\right)\leq 0$ for all $g=1,...,\ell$. This is what we wanted to show.

6. Pareto optimality and the core

6.1. The first welfare theorem from the point of view of general equilibrium analysis. We now turn to general equilibrium analysis and consider the total system of markets simultaneously. For an exchange economy, we will be able to show more than just Pareto efficiency (compare chapter II, pp. 17). We will show that every Walras allocation lies in the core in case of strict monotonicity. The core presented in this section is related to the core introduced in chapter III. While in that chapter, the core is defined within the framework of coalition functions, we present a definition for allocations in the present section. In both cases, a core is defined by Pareto efficiency and the impossibility to block. As in chapter III, we call a subset $S \subseteq N$ a coalition.

A coalition S can block an allocation if it can present an allocation that improves the lot of its members and that can be afforded by S :

Definition XIX.15 (blockable allocation, core). *Let* $\mathcal{E} = \left(N, G, \left(\omega^i\right)_{i\in N}, (U_i)_{i\in N}\right)$ *be an exchange economy. A coalition* $S \subseteq N$ *is said to block an allocation* $(y^i)_{i\in N}$, *if an allocation* $\left(z^i\right)_{i\in N}$ *exists such that*

- $U_i\left(z^i\right) \geq U_i\left(y^i\right)$ *for all* $i \in S$, $U_i\left(z^i\right) > U_i\left(y^i\right)$ *for some* $i \in S$ *and*
- $\sum_{i\in S} z^i \leq \sum_{i\in S} \omega^i$

hold.

An allocation is not blockable if there is no coalition can block it. The set of all feasible and non-blockable allocations is called the core of an exchange economy.

Within the Edgeworth box, the core can be depicted graphically. We see the endowment point and the associated exchange lense in fig. 14. Every household (considered a one-man coalition) blocks any allocation that lies below the indifference curve cutting his endowment point. Therefore, the core is contained inside the exchange lense. Both households together block any allocation that is not Pareto efficient. Thus, the core is the intersection of the exchange lense and the contract curve, roughly speaking.

We now turn to a remarkable claim:

Theorem XIX.3. *Assume an exchange economy* $\mathcal{E}$ *with local non-satiation and weak monotonicity. Every Walras allocation lies in the core.*

Proof. Consider a Walras allocation $\left(\widehat{x}^i\right)_{i\in N}$. Lemma XIX.4 (p. 356) implies

$$\widehat{p} \overset{(1)}{>>} 0$$

where $\widehat{p}$ is the equilibrium price vector.

Assume, now, that $\left(\widehat{x}^i\right)_{i\in N}$ does not lie in the core. Since a Walras allocation is feasible, there exists a coalition $S \subseteq N$ that can block $\left(\widehat{x}^i\right)_{i\in N}$. I.e., there is an allocation $\left(z^i\right)_{i\in N}$ such that

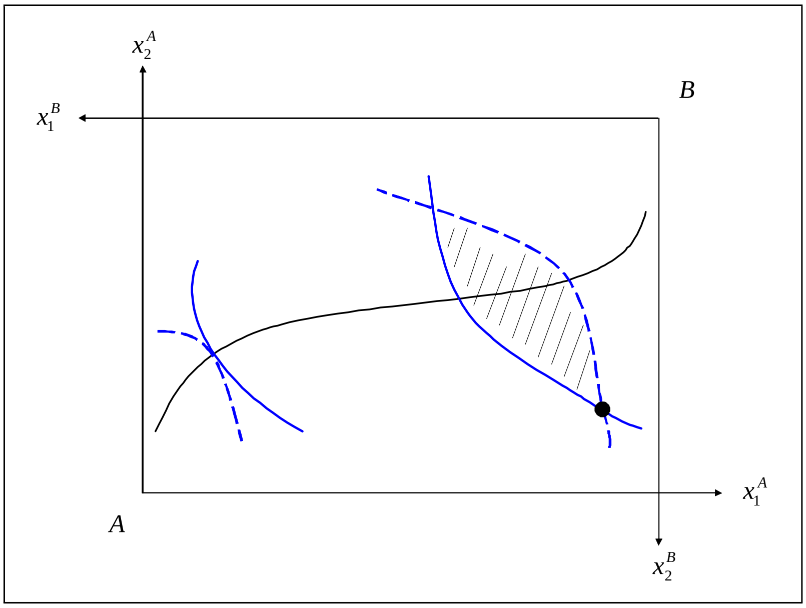

FIGURE 14. Exchange lense and core

- $U_i(z^i) \geq U_i(\widehat{x}^i)$ for all $i \in S$, $U_i(z^i) > U_i(\widehat{x}^i)$ for some $i \in S$ and
- $\sum_{i \in S} z^i \leq \sum_{i \in S} \omega^i$.

The second point, together with (1), leads to the implication

$$\widehat{p} \cdot \left(\sum_{i \in S} z^i - \sum_{i \in S} \omega^i \right) \leq 0. \tag{XIX.1}$$

The first point implies

$$\begin{aligned} \widehat{p} \cdot z^i \overset{(2)}{\geq} \widehat{p} \cdot \widehat{x}^i &= \widehat{p} \cdot \omega^i \text{ for all } i \in S \text{ (by local nonsatiation) and} \\ \widehat{p} \cdot z^j \overset{(3)}{>} \widehat{p} \cdot \widehat{x}^j &= \widehat{p} \cdot \omega^j \text{ for some } j \in S \text{ (otherwise, } \widehat{x}^j \text{ is not an optimum).} \end{aligned}$$

Summing over all these households from S yields

$$\begin{aligned} \widehat{p} \cdot \sum_{i \in S} z^i &= \sum_{i \in S} \widehat{p} \cdot z^i \text{ (distributivity)} \\ &> \sum_{i \in S} \widehat{p} \cdot \omega^i \text{ (above inequalities (2) and (3))} \\ &= \widehat{p} \cdot \sum_{i \in S} \omega^i \text{ (distributivity).} \end{aligned}$$

This inequality can be rewritten as

$$\widehat{p} \cdot \left(\sum_{i \in S} z^i - \sum_{i \in S} \omega^i \right) > 0$$

contradicting eq. XIX.1. $\square$

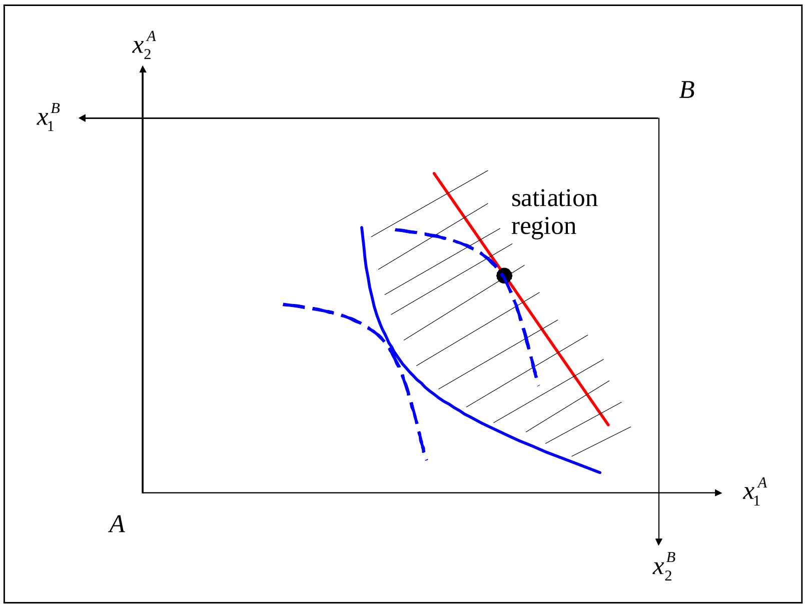

FIGURE 15. A non-efficient equilibrium

We now consider a case where a Walras allocation does not lie in the core. Consider fig. 15. The lower-left agent's preferences violate non-satiation. He is indifferent between all the bundles in the shaded area that comprises the highlighted endowment point and the price line. The equilibrium point is the point of tangency between that price line and the upper-right agent's indifference curve. This point is not Pareto efficient. The lower-left agent could forego some units of both goods without harming himself.

7. The marriage market

7.1. Consistent and feasible allocations and the NTU game. Roth & Sotomayor (1992) present a marriage-market model that can also be applied to other matching problems (for example, employers and employees). Assume a set of men

$$M = \{m_1, ..., m_k\}$$

and a set of women

$$W = \{w_1, ..., w_n\}.$$

Each man $m \in M$ possesses a utility function $U_m : W \cup \{m\} \to \mathbb{R}$ where w_1 means marrying woman 1 while m stands for celibacy (singleness). Similarly, every womand $w \in W$ has a utility function U_w on $M \cup \{w\}$.

EXERCISE XIX.17. *What does* $U_{w_1}(m_1) > U_{w_1}(w_1) > U_{w_1}(m_2)$ *mean?*

In order to simplify the analysis, we follow the authors in assuming that all the preferences are strict. The tuple of all $k + n$ utility functions is denoted by $\mathbf{U}$.

DEFINITION XIX.16 (marriage market). *A marriage market* $(M, W, \mathbf{U})$ *consists of disjunct sets of individuals* M *and* W *and utility functions* $\mathbf{U} = (U_i)_{i \in M \cup W}$ *with domain* $W \cup \{m\}$ *for every* $m \in M$ *and domain* $M \cup \{w\}$ *for every* $w \in W$.

In comparison with an exchange economy, the players themselves are the object of preferences. This explains the emotionality reigning in this market as many readers may know from their own experience.

An allocation in a goods market attaches good bundles to players. In contrast, players show up in the domain and in the range of a marriage-market allocation.

DEFINITION XIX.17 (allocation). *For a marriage market* $(M, W, \mathbf{U})$, *the function*

$$\mu : M \cup W \to M \cup W$$

is called an allocation if the two requirements

- $\mu(m) \in \{m\} \cup W$ *for all* $m \in M$ *and*
- $\mu(w) \in \{w\} \cup M$ *for all* $w \in W$

are fulfilled.

Thus, men can be singles or attached to a woman – Adam and Eve, not Adam and Steve.

EXERCISE XIX.18. *Which players are characterized by* $\mu(\mu(i)) = i$*?*

Allocations as defined above do not reflect traditional marriages. After all, we may have $\mu(m) = w$ but not $\mu(w) = m$ for some pair $m \in M$, $w \in W$.

DEFINITION XIX.18 (consistent allocation). *For a marriage market* $(M, W, \mathbf{U})$, *an allocation* μ *is called consistent if* $\mu(\mu(i)) = i$ *holds for all* $i \in M \cup W$.

Single individuals i are defined by $\mu(i) = i$ and fulfull the consistency condition by

$$\mu(\mu(i)) = \mu(i) = i.$$

Assume a feasible allocation μ and a man $m \in M$ who is not single. By $\mu(m) \in \{m\} \cup W$, he is attached to a women $w \in W$ $(\mu(m) = w)$. Consistency then implies

$$m = \mu(\mu(m)) = \mu(w)$$

so that the woman w is attached to the very same man – a marriage relation.

DEFINITION XIX.19. *Consider a consistent allocation* μ *in a marriage market* $(M, W, \mathbf{U})$. μ *is called* K*-feasible if* $\mu(K) \subseteq K$ *holds.*

K-feasibility means that every individual from K is single or has a marriage partner in K. Similarly, a blocking coalition in an exchange economy can only redistribute goods this coalition possesses (K-feasibility, see p. **??**). Every consistent allocation μ is $M \cup W$-feasible.

Very similar to the exchange economy, we can define the associated NTU coalition function V by

$$V(K) := \left\{ u_K \in \mathbb{R}^{|K|} : \text{There is a feasible allocation } \mu \text{ with } u_i \leq U_i(\mu(i)), i \in K \right\}.$$

7.2. Core. We now turn to the question of whether a feasible allocation is stable. One important point is that no (wo)man has a partner she (or he) would rather like to do without.

Definition XIX.20 (acceptability). *An agent i finds another individual j acceptable if $U_i(j) > U_i(i)$ holds.*

The idea of the above definition is that nobody can be married against his (or her) will. However, the fact a man m does not have his favorite woman w as spouse does not speak against the stability of the underlying allocation – w might find another man m' more attrative to whom she is actually married.

Definition XIX.21 (from individual rationality to the core). *Let μ be a consistent (or $M \cup W$-feasible) allocation.*

- *μ is called individually rational if $U_i(\mu(i)) \geq U_i(i)$ holds for all $i \in N$ (non-blockability by one-man coalitions).*
- *μ is called pairwise rational if there is no pair of players $(m, w) \in M \times W$ such that*

$$\begin{aligned} U_m(w) &> U_m(\mu(m)) \text{ and} \\ U_w(m) &> U_w(\mu(w)) \end{aligned}$$

  hold (non-blockability by heterosexual pairs).
- *μ is called Pareto optimal if there is no consistent allocation μ' that fulfills*

$$\begin{aligned} U_i(\mu'(i)) &\geq U_i(\mu(i)) \text{ for all } i \in M \cup W \text{ and} \\ U_j(\mu'(j)) &> U_j(\mu(j)) \text{ for at least one } j \in M \cup W \end{aligned}$$

  (non-blockability by the grand coalition).
- *μ lies in the core if there is not coalition $K \subseteq M \cup W$ and no K-feasible allocation μ' such that*

$$\begin{aligned} U_i(\mu'(i)) &\geq U_i(\mu(i)) \text{ for all } i \in K \text{ and} \\ U_j(\mu'(j)) &> U_j(\mu(j)) \text{ for at least one } j \in K \end{aligned}$$

  holds (non-blockability by any coalition).

Pairwise rationality means that there is no man and no woman such that both of them could improve their lot by marrying. The improvement can mean breaking off existing marriages or giving up celibacy. Pareto optimality and the core are defined with reference to feasibility and non-blockability by the grand or by any coalition, respectively.

Remember that K-feasibility includes $\mu'(K) \subseteq K$ – players from a blocking coalition K have to restrict their search of partners to players within K. The inclusion $\mu'(K) \subseteq K$ is also fulfilled by individual rationality. After all, every $\{i\}$-feasible allocation μ' obeys $\mu'(i) = i$. Also, pairwise rationality underlies this inclusion – the blocking coalition $\{m, w\}$ forms a pair.

Exercise XIX.19. *What is the connection between individual rationality and acceptability?*

Theorem XIX.4. *Let $(M, W, \mathbf{U})$ be a marriage market. The set of consistent allocations that are individually rational and pairwise rational is the core.*

Proof. We first show that a consistent allocation is both individually and pairwise rational belongs to the core. In order to do so, we assume an allocation μ outside the core. We can assume that μ is consistent. Thus, there exists a coalition K that can block μ by suggesting a K-feasible allocation μ' that fulfills

$$\begin{aligned} U_i\left(\mu'(i)\right) &\geq U_i(\mu(i)) \text{ for all } i \in K \text{ and} \\ U_j\left(\mu'(j)\right) &> U_j(\mu(j)) \text{ for at least one } j \in K. \end{aligned}$$

Let us focus on individual j that is strictly better off under μ' than under μ. We can distinguish two cases:

- j is single or married under μ and (re)marries under μ'.

  In this case both j and his (or her) spouse $\mu'(j) \in K$ (!) are strictly better off because we work with strict preferences. Then μ is not pairwise rational.
- j is married under μ and single under μ'.

  This second case implies that j is better off as a single contadicting individual rationality.

Exercise XIX.20. *Complete the proof by showing that every allocation from the core is both individually and pairwise rational.*

8. Topics and literature

The main topics in this chapter are

- money budget
- endowment budget
- marginal opportunity cost
- feasibility
- objective function
- indirect utility function
- marginal utility of income
- labor supply
- intertemporal consumption
- demand function
- exchange economy
- excess demand
- market clearing
- Walras equilibrium
- first welfare theorem
- second welfare theorem
- Walras' law
- free goods
- Brouwer's fixed-point theorem
- positive theory
- non-transferable utility
- endowment
- Walras equilibrium
- Walras allocation
- marriage market
- core

We point to the careful but difficult survey of cooperative game theory for non-transferable utility by McLean (2002). A careful introduction into General Equilibrium Theory is presented by Hildenbrandt & Kirman (1988).

9. Solutions

Exercise XIX.1

For any number $\alpha > 0$, we have $B(\alpha p, m) = B\left(p, \frac{m}{\alpha}\right)$.

Exercise XIX.2

Fig. 16 shows the left-over in terms of good 1 and good 2. In order to obtain the left-over in money terms, we need to multiply the left-over of good 1 with p_1 (or the left-over of good 2 with p_2).

Exercise XIX.3

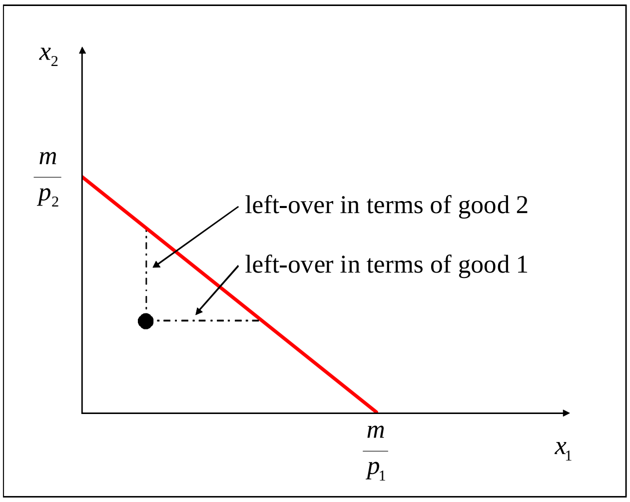

FIGURE 16. The left-over

Solving $p_1x_1 + p_2x_2 = m$ for x_2 yields $x_2 = \frac{m}{p_2} - \frac{p_1}{p_2}x_1$ so that the derivative of x_2 (as a function of x_1) is $-\frac{p_1}{p_2}$.

Exercise XIX.4

In subfigure (a), points A and B do not correspond to an optimum. The preferences are strictly convex and every point between A and B is better than A or B. Subfigure (b) depicts perfect substitutes. Point A is the household optimum. In subfigure (c), points A and B are optima but so are all the points in between. Turning to subfigure (c), the point of tangency A is the worst bundle of all the bundles on the budget line. There are two candidates for household optima in this case of concave preferences, the two extreme bundles $\left(\frac{m}{p_1}, 0\right)$ and $\left(0, \frac{m}{p_2}\right)$.

Exercise XIX.5

If y is better than x, the household optimum, the household cannot afford y :

$$y \succ x \Rightarrow p \cdot y > p \cdot \omega$$

Exercise XIX.6

The utility levels achievable by Peter and Otto are depicted in fig. **??** dargestellt.

Exercise XIX.7

$V(\{1,2\})$ is a non-empty subset of $\mathbb{R}^2$ which is true for the two last expressions, only.

Exercise XIX.8

All axis intercepts are $v(K)$.

Exercise XIX.9

V_1 is superadditive while V_2 is not. We do not show superadditivity of V_1 but just consider a few examples. Take $S = \{1\}$ and $T = \{2\}$. You find $1 \in V_1(\{1\})$ and $2 \in V_1(\{2\})$ and, in line with superadditivity, $(1,2) \in$

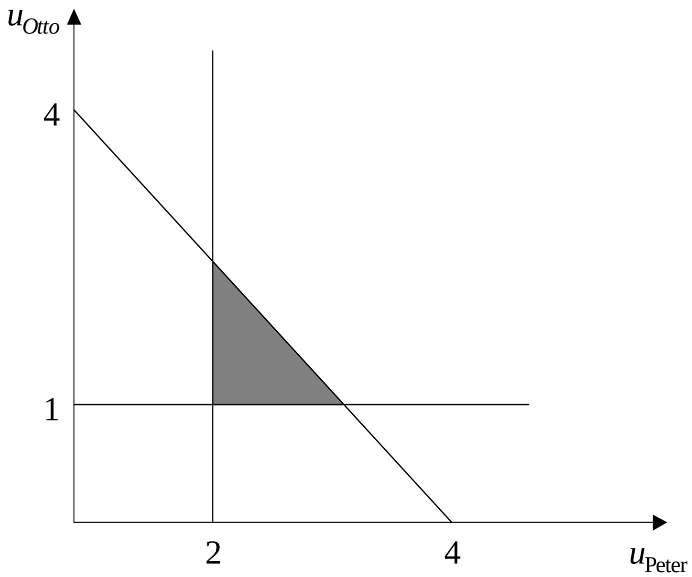

FIGURE 17. Payoff vectors

$V_1(\{1,2\})$. Similarly, for $S = \{1\}$ and $T = \{2,3\}$ we have $1 \in V_1(\{1\})$, $(4,5) \in V_1(\{2,3\})$ and $(1,4,5) \in V_1(\{1,2,3\})$.

V_2 is not superadditive because of $1 \in V_2(\{1\})$ and $3 \in V_2(\{3\})$, but $(1,3) \notin V_2(\{1,3\})$. Superadditivity can also be disproved by looking at $S = \{1\}$ and $T = \{2,3\}$ aufzeigen.

Exercise XIX.10

The individual is always free to consume ω. If he wants to consume another bundle, the prices are relevant.

Exercise XIX.11

Markets clear for p_1^h, but not for p_1^l. At price p_1^l, individuals A and B want to consume more of good 1 than they possess together. Just note that D^A is to the right of D^B. At price p_1^l, there is excess supply of good 2 :

$$x_2^A + x_2^B < \omega_2^A + \omega_2^B.$$

Exercise XIX.12

From

$$\begin{aligned} z_1(p) + z_2(p) + z_3(p) &= \frac{8}{p} - 4 + \frac{4}{p} - 2 + \frac{12}{p} - 2 \\ &= \frac{24}{p} - 8 \end{aligned}$$

we obtain the market clearing price

$$p^* = 3.$$

For individual 3, we have $z_3(3) = \frac{12}{3} - 2 = 2$. He is a buyer.

Exercise XIX.13

Abba will find the household optimum by letting

$$px_1^A + x_2^A = 18p \text{ and } |MRS| = \frac{MU_1}{MU_2} = \frac{\frac{1}{2\sqrt{x_1^A}}}{1} = \frac{p}{1}.$$

Solving the second equation for x_1^A, we obtain

$$x_1^A = \frac{1}{4p^2}.$$

Substituting into the first yields

$$p\frac{1}{4p^2} + x_2^A = 18p \text{ and hence}$$
$$x_2^A = 18p - \frac{1}{4p}.$$

Bertha's optimal bundle is

$$x_1^B = \frac{1}{4p^2}, x_2^B = 10 - \frac{1}{4p}.$$

Equation $\omega_1^A + \omega_1^B = x_1^A + x_1^B$ can be written as

$$18 = x_1^A + x_1^B = \frac{2}{4p^2}$$

and we find

$$p = \frac{1}{6}.$$

The same price obtains from

$$10 = x_2^A + x_2^B = 18p - \frac{1}{4p} + 10 - \frac{1}{4p}.$$

Exercise XIX.14

We can rewrite

$$\sum_{i=1}^{n} x^i\left(\widehat{p}, \omega^i\right) \leq \sum_{i=1}^{n} \omega^i$$

as

$$\sum_{i=1}^{n}\left(x_1^i\left(\widehat{p}, \omega^i\right), ..., x_\ell^i\left(\widehat{p}, \omega^i\right)\right) \leq \sum_{i=1}^{n}\left(\omega_1^i, ..., \omega_\ell^i\right)$$

which just means

$$\sum_{i=1}^{n} x_g^i\left(\widehat{p}, \omega^i\right) \leq \sum_{i=1}^{n} \omega_g^i \text{ for all } g = 1, ..., \ell.$$

Exercise XIX.15

a) There are two cases for which M is not compact: If M is not bounded, e.g. $M = \mathbb{R}$, the function $f(x) = x + 1$ maps M onto M but is does not have a fixed point. For M being open, e.g. the function $f(x) = \frac{1+x}{2}$ with $M = (0, 1)$ has no fixed point.

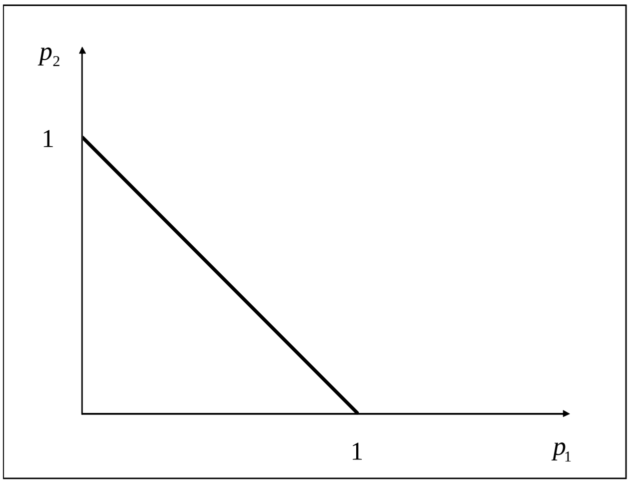

FIGURE 18. The 1- dimensional unit simplex

b) With $M = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$, the function $f(x) = \frac{1}{2}$ has no fixed point.
c) Let $M = [0,1]$. Then $f = \begin{cases} 1, \text{ if } 0 \leq x \leq \frac{2}{3} \\ 0, \text{ if } \frac{2}{3} < x \leq 1 \end{cases}$ has no fixed point.

Exercise XIX.16

S^1 is the one-dimensional segment as shown in fig. 18.

Exercise XIX.17

Woman 1 would rather marry man 1 than stay single, but would prefer to stay alone over marrying man 2.

Exercise XIX.18

For single individuals or for individuals with a spouse (read on!).

Exercise XIX.19

Individual rationality means that every married individual has an acceptable partner.

Exercise XIX.20

Let μ be an allocation from the core of the marriage game $(M, W, \mathbf{U})$. Then μ is consistent and there is no allocation $K \subseteq M \cup W$ that can block μ. In particular, this is true for one-man (one-woman) coalitions and for heterosexual pairs. Thus, we obtain individual rationality (no individual can be married although he prefers to be single) and pairwise rationality (no pair can be prevented from marrying if both of them so prefer).

10. Further exercises without solutions

PROBLEM XIX.1.

Sketch budget lines or the displacements of budget lines for the following examples:

- Time $T = 18$ and money $m = 50$ for football F (good 1) or basket ball B (good 2) with prices
  - $p_F = 5$, $p_B = 10$ in monetary terms,
  - $t_F = 3$, $t_B = 2$ and temporary terms
- Two goods, bread (good 1) and other goods (good 2). Transfer in kind with and without prohibition to sell:
  - $m = 20$, $p_B = 2$, $p_{\text{other}} = 1$
  - Transfer in kind: $B = 5$

CHAPTER XX

Die Nash-Lösung

1. Introduction

Eines der berühmtesten Konzepte der Koalitionsfunktion ohne transferierbaren Nutzen ist die Nash-Lösung (oder Nash-Verhandlungslösung). Sie hat mit dem Begriff des Nash-Gleichgewichtes, das in die nichtkooperative Spieltheorie gehört, außer dem Bezug zu John Nash, nichts zu tun. Die hier folgenden Ausführungen sind durch Thomson (1994) und Peters (1992) inspiriert. Nash (1953) selbst hat seine Theorie für zwei Personen expliziert. Die Verallgemeinerung auf n Personen ist jedoch nicht schwer.

2. Threat points

$V(K)$ determines the payoffs that a player from K can obtain only in relation to the payoffs of the other players. For future reference, we are interested in the payoffs a player might possibly get:

Definition XX.1 (maximal payoff). *Let K be a coalition and i a player from K. Player i's maximal payoff for K and V is denoted by $m_i^K(V)$ and given by*

$$m_i^K(V) := \max\left\{x_i \in \mathbb{R} : \text{There is a payoff vector } u \in V(K) \subseteq \mathbb{R}^{|K|} \text{ with } u_i = x_i\right\}.$$

We sometimes write m_i^K rather than $m_i^K(V)$.

We will assume that $m_i^K(V)$ always exists (which is true if $V(K)$ is closed).

Exercise XX.1. *Determine $m_1^{\{1,2\}}$ for*

$$V(\{1,2\}) = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \le 3, x_2 \le 4, x_1 + x_2 \le 5\right\}.$$

Die Besonderheit der Nash-Lösung besteht darin, dass die Auszahlungen $V(K)$, die Koalitionen K mit $1 < |K| < n$ erzielen können, keine Rolle spielen. Es kommt nur darauf an, was die große Koalition und die Einerkoalitionen erzielen können. Für die Einerkoalitionen beschränkt man sich zudem auf ihre maximale Auszahlungen:

$$d_i := m_i^{\{i\}}, i \in N.$$

$d = (d_i)_{i \in N}$ nennt man auch den Drohpunkt oder Status-quo-Punkt. Er gibt diejenigen Nutzen an, die die Agenten bei Abbruch der Verhandlungen erreichen können.

Wir definieren nun ein Verhandlungsspiel (N, V) als ein spezielles Koalitionsspiel bei nichttransferierbarem Nutzen (siehe McLean (2002, S. 2080)):

DEFINITION XX.2. *Ein Verhandlungsspiel* (N, V) *ist ein Koalitionsspiel bei nichttransferierbarem Nutzen, das die folgenden Bedingungen erfüllt:*

- $(d_i)_{i\in N}$ *ist zulässig:* $(d_i)_{i\in N} \in V(N)$.
- *In* $V(N)$ *existiert mindestens ein Vektor* x *mit* $x >> d$.

Häufig setzt man $U := V(N)$ *(die so genannte Verhandlungsmenge) und schreibt dann auch* (U, d) *anstelle von* V. *Die Menge der* n*-Personen-Verhandlungsspiele bezeichnet man mit* $\mathbb{V}_n$.

In Abb. 1, die sich auf den Zwei-Personen-Fall bezieht, gibt es offenbar viele Nutzenvektoren, die beide Spieler im Vergleich zu d besser stellen. Von besonderer Wichtigkeit ist die Nutzengrenze (die so genannte Nutzenmöglichkeitenkurve), die die Menge der Pareto-optimalen Nutzenkombinationen wiedergibt. Streng genommen bräuchten wir noch einige weitere „technische" Bedingungen (u.a. Konvexität); wir wollen sie jedoch nicht näher erläutern. Eine ausführlichere Einführung in die Nash-Verhandlungs-Lösung bietet Binmore (1992, Kap. 5).

Bevor Sie sich an die nächste Aufgabe machen, wollen wir den so genannten Idealpunkt definieren:

DEFINITION XX.3. *Idealpunkt heißt der durch*

$$a(U, d) = (a_i(U, d))_{i\in N}$$

und

$$a_i(U, d) := \max\{u_i \in \mathbb{R} : \textit{Es gibt einen Nutzenvektor } u \textit{ in } U \textit{ mit } u \geq d\}$$

definierte Punkt aus $\mathbb{R}^n$.

$a_i(U, d)$ ist die maximale Auszahlung, die Spieler i unter all denjenigen Punkten aus U erreichen kann, die alle Spieler gegenüber dem Drohpunkt vorziehen („rechts oberhalb" des Drohpunktes). Im Allgemeinen liegt $a(U, d)$ nicht in U.

EXERCISE XX.2. *Skizzieren Sie den Kern der zu Abbildung 1 gehörigen charakteristischen Funktion ohne transferierbaren Nutzen. Zeichnen Sie auch* $a_1(U, d)$, $a_2(U, d)$ *und* $a(U, d)$ *ein.*

3. Axiome für Verhandlungslösungen

Das Ziel dieses und des nächsten Abschnittes besteht darin, Verhandlungsspielen Auszahlungsvektoren zuordnen zu können. Dazu definieren wir zunächst allgemein eine solche Lösung:

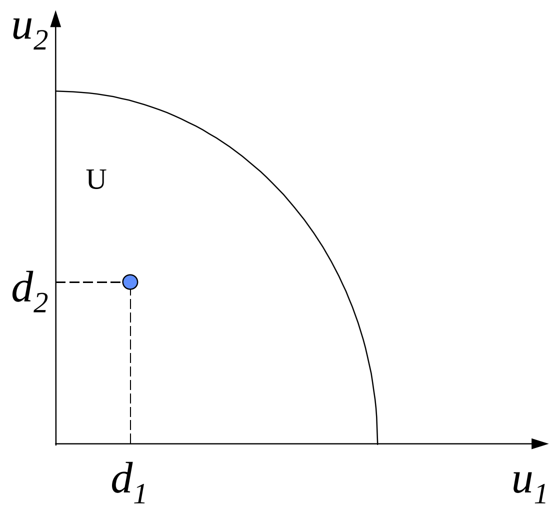

FIGURE 1. Ausgangssituation für die Nash-Verhandlungslösung

DEFINITION XX.4. *Die Abbildung*

$$\psi : \mathbb{V}_n \to \mathbb{R}^n$$

heißt punktwertige Verhandlungslösung auf N.

EXERCISE XX.3. *Wie sollte Ihrer Meinung nach eine Lösung auf $\{1,2\}$ von d_1 abhängen?*

Wir präsentieren zunächst Axiome, die Verhandlungslösungen erfüllen könnten oder sollten. Dabei beschränken wir uns auf den Fall $N = \{1,2\}$; die Verallgemeinerung auf n Spieler ist jedoch leicht.

Pareto-Axiom: Die Auszahlung $(\psi_1(U,d), \psi_2(U,d))$ ist zulässig und durch $\{1,2\}$ nicht blockierbar:

- $(\psi_1(U,d), \psi_2(U,d)) \in U$ und
- $(u_1,u_2) > (\psi_1(U,d), \psi_2(U,d))$ impliziert $(u_1,u_2) \notin U$.

EXERCISE XX.4. *Welche Auszahlungen in Abb. 1 auf S. 385 sind Pareto-effizient?*

Symmetrie-Axiom: Für zwei Verhandlungsprobleme

$$(U,d) \text{ und } (U',d')$$

gelte

$$d_1' = d_2,\ d_2' = d_1$$

und

$$U' = \{(u_1,u_2) : (u_2,u_1) \in U\}.$$

Dann folgen

$$\begin{aligned}\psi_1\left(U',d'\right) &= \psi_2\left(U,d\right) \text{ und}\\ \psi_2\left(U',d'\right) &= \psi_1\left(U,d\right).\end{aligned}$$

Die Lösung darf also nicht von der Benennung der Spieler abhängen. Das Symmetrieaxiom ist insofern angreifbar, als es in Verhandlungen sicherlich Aspekte gibt, die nicht in U und d „eingefangen" sind: die Persönlichkeit der Verhandelnden, ein möglicher „Heimvorteil" eines Spielers oder Ähnliches.

Axiom über die Invarianz bei affinen Transformationen: Für $a_1, a_2 > 0$, b_1, b_2 beliebig und zwei Verhandlungsprobleme

$$(U,d) \text{ und } \left(U',d'\right)$$

gelte

$$\begin{aligned}d_1' &= a_1 \cdot d_1 + b_1,\\ d_2' &= a_2 \cdot d_2 + b_2\end{aligned}$$

und

$$U' = \{(a_1 \cdot u_1 + b_1, a_2 \cdot u_2 + b_2) : (u_1,u_2) \in U\}.$$

Dann folgen

$$\begin{aligned}\psi_1\left(U',d'\right) &= a_1\psi_1\left(U,d\right) + b_1 \text{ und}\\ \psi_2\left(U',d'\right) &= a_2\psi_2\left(U,d\right) + b_2.\end{aligned}$$

Wir erinnern daran (siehe Abschnitt **??** auf S. **??** ff.), dass von Neumann-Morgenstern-Nutzenfunktionen u nur bis auf positive affine Transformationen eindeutig bestimmt sind. Diese Invarianz, so fordert das Axiom, soll sich auch auf die Verhandlungslösung übertragen. Allerdings ist die vorangehende Begründung nur für von Neumann-Morgenstern-Nutzenfunktionen stichhaltig. Beispielsweise kann man nicht die Nutzenwerte einer Cobb-Douglas-Nutzenfunktion im Falle einer Tauschökonomie nehmen.

Axiom über die Unabhängigkeit irrelevanter Alternativen: Für zwei Verhandlungsprobleme

$$(U,d) \text{ und } \left(U',d'\right)$$

gelte

$$d_1' = d_1,\, d_2' = d_2$$

und

$$U' \subseteq U.$$

Dann folgt aus

$$\left(\psi_1\left(U,d\right), \psi_2\left(U,d\right)\right) \in U'$$

bereits

$$\left(\psi_1\left(U',d'\right), \psi_2\left(U',d'\right)\right) = \left(\psi_1\left(U,d\right), \psi_2\left(U,d\right)\right).$$

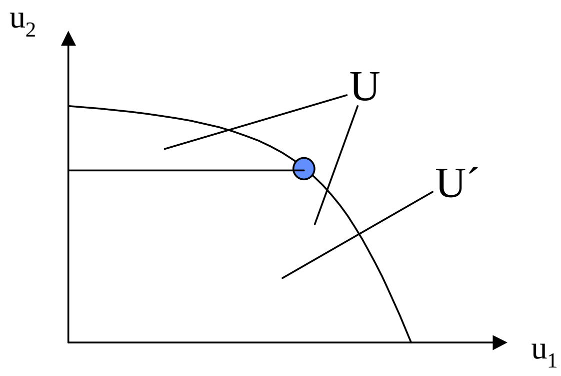

FIGURE 2. Irrelevante Alternativen?

Falls sich bei einem Verhandlungsproblem eine bestimmte Verhandlungslösung ergibt, so sollte diese Verhandlungslösung also nach Möglichkeit auch dann weiter bestehen bleiben, wenn die Menge der erreichbaren Nutzenkombinationen kleiner wird.

Dieses Axiom ist vielleicht am problematischsten: Betrachten wir dazu Abb. 2. Die Menge U enthält die Menge U'. Wenn der hervorgehobene Punkt, der sowohl in U als auch in U' liegt, die Lösung bei U ist, gilt dies auch für U'. Intuitiv sollte man allerdings meinen, dass sich die Verhandlungsposition für Spieler 2 bei U' schlechter darstellt als bei U.

Schließlich präsentieren wir zwei recht plausible Monotonie-Axiome. Das erste bezieht sich auf die Verhandlungsmenge, das zweite auf den Drohpunkt. Intuitiv besagt das Verhandlungsmengen-Monotonie-Axiom Folgendes: Wenn man für jeden Nutzenwert u_2 die u_1-Nutzenwerte erhöht, erhält Spieler 1 eine höhere Auszahlung. Formal schreiben wir dies so:

Axiom über die Verhandlungsmengen-Monotonie: Für $N = \{1, 2\}$ und $i, j \in N$, $i \neq j$, erhält bei

$$\begin{aligned} U' &\supseteq U \text{ und} \\ a_j\left(U', d\right) &= a_j\left(U, d\right) \end{aligned}$$

Spieler i bei U' mindestens so viel wie bei U :

$$\psi_i\left(U', d\right) \geq \psi_i\left(U, d\right).$$

(Die $a_j\left(U, d\right)$ haben wir auf S. 384 definiert.)

Dieses Axiom ist in Abb. 3 angedeutet, wobei wir $i = 1$ und $j = 2$ gesetzt haben.

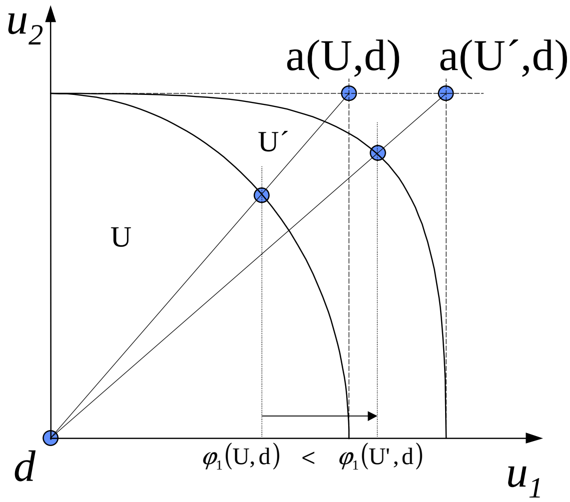

FIGURE 3. Monotonie-Axiom

Axiom über die Drohpunkt-Monotonie: Für $N = \{1, 2\}$ und $i, j \in N$, $i \neq j$, erhält bei

$$d'_i \geq d_i \text{ und } d_j = d'_j$$

Spieler i bei d' mindestens so viel wie bei d :

$$\psi_i \left(U, d'\right) \geq \psi_i \left(U, d\right).$$

4. Die Nash-Verhandlungslösung

Ähnlich wie bei der Shapley-Lösung kann man auch für Verhandlungs-Lösungen zeigen, dass bestimmte Axiome äquivalent zu einer bestimmten Berechnungsart sind. Es stellt sich heraus, dass einige Axiome gleichbedeutend damit sind, dass man nach demjenigen Nutzenbündel (u_1, u_2) sucht, das

$$(u_1 - d_1)(u_2 - d_2)$$

maximiert. Hierbei sind allerdings nur Nutzenbündel „rechts oberhalb" von d zulässig; ansonsten wäre die individuelle Rationalität verletzt. Wir setzen zudem voraus, dass es genau ein Nutzentupel gibt, das diese Maximierungsaufgabe löst.

THEOREM XX.1. *Genau dann, wenn eine Lösung auf* $\mathbb{V}_2$ *die Axiome*

- *Pareto-Axiom,*
- *Symmetrie-Axiom,*
- *Axiom über die Invarianz bei affinen Transformationen und*
- *Axiom über die Unabhängigkeit irrelevanter Alternativen*

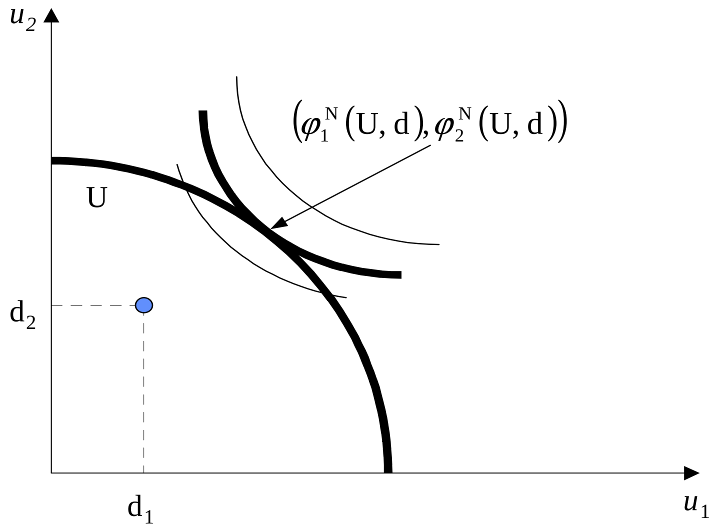

FIGURE 4. Die Nash-Verhandlungslösung

erfüllt, lautet sie

$$\underset{\substack{(u_1,u_2)\in U,\\ u_1\geq d_1,u_2\geq d_2}}{\operatorname{argmax}} \ (u_1-d_1)(u_2-d_2).$$

Diese Lösung nennen wir die (symmetrische) Nash-Verhandlungslösung und bezeichnen sie mit φ^N.

Dieses Theorem lässt sich ohne weiteres auf n Spieler übertragen. Man hat dann für das Produkt aus n Faktoren das Maximum zu suchen. Einen Beweis liefern wir nicht und verweisen stattdessen wiederum auf das Lehrbuch von Binmore (1992).

Es gibt eine einfache graphische Veranschaulichung der Nash-Verhandlungslösung. Wir betrachten die Iso-Produkt-Kurven, d.h. die Menge derjenigen Tupel (u_1,u_2), für die

$$(u_1-d_1)(u_2-d_2)=k$$

für verschiedene Konstanten k gilt. Dies sind Hyperbeln mit nach (d_1,d_2) verschobenem Ursprung. Die Nash-Verhandlungslösung ist diejenige Nutzenkombination, bei der eine Hyperbel die Verhandlungsmenge U genau einmal berührt (siehe Abbildung 4).

Betrachten wir ein sehr einfaches Beispiel: Ein Kuchen der Größe 1 sei zwischen zwei Spielern aufzuteilen. Damit haben wir

$$U=\{(u_1,u_2):u_1+u_2\leq 1,u_1,u_2\geq 0\}.$$

Der Status-quo-Punkt d sei ein Punkt aus U mit

$$d_1 + d_2 < 1.$$

Sind die vier Axiome erfüllt, erhalten wir

$$\begin{aligned}\left(\varphi_1^N(U,d), \varphi_2^N(U,d)\right) &= \underset{\substack{(u_1,u_2)\in U,\\ u_1\geq d_1, u_2\geq d_2}}{\operatorname{argmax}} (u_1 - d_1)(u_2 - d_2) \\ &= \left(\frac{1}{2} + \frac{1}{2}(d_1 - d_2), \frac{1}{2} + \frac{1}{2}(d_2 - d_1)\right).\end{aligned}$$

Exercise XX.5. *Stimmt das?*

Die Interpretation dieses Ergebnisses ist offensichtlich: Jeder Spieler profitiert von einem eigenen hohen Drohpunkt (er hat „nicht viel zu verlieren") und von einem niedrigen Drohpunkt des Verhandlungspartners (dieser ist auf den Handel angewiesen). Die Drohpunkt-Monotonie ist für die Nash-Verhandlungslösung also erfüllt.

Man könnte nun das obige Kuchen-Aufteilungsbeispiel so variieren, dass einer der Spieler risikoavers ist, während der andere risikoneutral bleibt. Wir werden sehen, dass der risikoneutrale Agent einen größeren Teil des Kuchens erhält als der risikoaverse. Ein Beispiel für eine Risikoaversion ausdrückende von Neumann-Morgenstern-Nutzenfunktion ist die Wurzelfunktion.

Exercise XX.6. *Bestimmen Sie die Nash-Verhandlungslösung im Falle von*

$$U = \{(u_1, u_2) : x_1 + x_2 \leq 1, u_1 = \sqrt{x_1}, u_2 = x_2, x_1 \geq 0, x_2 \geq 0\}$$

und

$$d_1 = 0, d_2 = 0.$$

Hinweis: Bestimmen Sie zunächst die Nutzenmöglichkeitenkurve, also u_2 als Funktion von u_1.

Eine Frage bezüglich der Axiomatisierung haben wir offen gelassen. Wie steht es mit der Verhandlungsmengen-Monotonie auf S. 387? Lässt sie sich vielleicht aus den anderen Axiomen folgern? Nein! Sie ist nicht erfüllt. Diese Behauptung kann man aus Abb. 5 ablesen. Offenbar ist hier U' eine echte Obermenge von U und auch $a_2(U', d) = a_2(U, d)$ ist erfüllt. Trotzdem erhält Spieler 1 bei U' weniger als bei U.

Dass die Nash-Lösung Monotonie verletzt, ist sicherlich ein großer Nachteil dieses Lösungskonzeptes. Wir werden in Abschnitt 6 monotone Alternativen zur Nash-Lösung präsentieren.

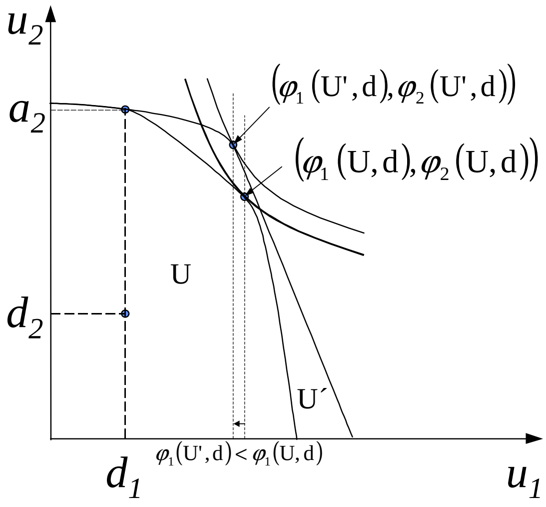

FIGURE 5. Verletzung der Verhandlungsmengen-Monotonie

5. Asymmetrische Nash-Lösungen

5.1. Definition. In der Literatur findet sich neben der symmetrischen Nash-Lösung (die wir in den vorangehenden Abschnitten erläutert haben) auch eine asymmetrische Lösung, die auf Harsanyi & Selten (1972) zurückgeht. In diese Lösung gehen neben der Verhandlungsmenge $U \subseteq \mathbb{R}^n$ und dem Drohpunkt $d \in \mathbb{R}^n$ Gewichte $\omega = (\omega_1, ..., \omega_n) > 0$ ein. (Man hat $\omega_i \geq 0$ für alle $i \in N$ und $\omega_i > 0$ für mindestens ein $i \in N$.)

DEFINITION XX.5. *Die asymmetrische Nash-Lösung ist durch*

$$\varphi_i^N(U, d, \omega) := \begin{cases} \arg_i \max_{\substack{u \in U, \\ u \geq d}} (u_1 - d_1)^{\omega_1} (u_2 - d_2)^{\omega_2}, & \omega_i > 0 \\ d_i, & \omega_i = 0 \end{cases}$$

definiert.

Falls ein Spieler $i \in N$ das Gewicht Null aufweist, erhält er also als Auszahlung nur seinen Drohwert d_i. Falls sein Gewicht positiv ist, erhält er aus dem Auszahlungsvektor, der das obige Produkt maximiert, seine Komponente (daher das i neben arg). Die Gewichte müssen sich nicht zu 1 ergänzen, man kann dies jedoch ohne Beschränkung der Allgemeinheit fordern. Denn $(u_1 - d_1)^{\omega_1} (u_2 - d_2)^{\omega_2}$ wird für genau dieselben (u_1, u_2) maximiert wie

$$\left[(u_1 - d_1)^{\omega_1} (u_2 - d_2)^{\omega_2}\right]^{\frac{1}{\omega_1+\omega_2}} = (u_1 - d_1)^{\frac{\omega_1}{\omega_1+\omega_2}} (u_2 - d_2)^{\frac{\omega_2}{\omega_1+\omega_2}}.$$

Axiomatisch lässt sich die asymmetrische Nash-Lösung so wie die symmetrische Nash-Lösung charakterisieren. Lediglich das Symmetrie-Axiom muss nicht mehr erfüllt sein.

Als Beispiel betrachten wir

$$U = \{(u_1, u_2) : x_1 + x_2 \leq 1, u_1 = \sqrt{x_1}, u_2 = x_2, x_1 \geq 0, x_2 \geq 0\}$$

mit der dazugehörigen Nutzenmöglichkeitenkurve

$$u_2 = x_2 = 1 - x_1 = 1 - (u_1)^2,$$

dem Drohpunkt $d = (0, 0)$ und dem Gewichtsvektor $\omega = (1, 2)$.

Damit erhält man das zu maximierende Produkt

$$\begin{aligned} & (u_1 - d_1)^1 (u_2 - d_2)^2 \\ = \; & u_1 \left(1 - u_1^2\right)^2 \\ = \; & u_1 + u_1^5 - 2u_1^3. \end{aligned}$$

Differenzieren und Nullsetzen ergeben

$$1 + 5u_1^4 - 6u_1^2 \stackrel{!}{=} 0$$

bzw.

$$u_1^4 - \frac{6}{5}u_1^2 + \frac{1}{5} \stackrel{!}{=} 0,$$

woraus sich zunächst (Anwendung der quadratischen Lösung auf u_1^2)

$$u_1^2 = 1 \text{ oder } u_1^2 = \frac{1}{5}$$

und dann die Lösungen

$$1, -1, \frac{1}{\sqrt{5}}, -\frac{1}{\sqrt{5}}$$

ergeben. Die negativen Nutzenwerte fallen wegen $d = 0$ weg. $u_1 = 1$ können wir ausschließen, weil dadurch $u_2 = 0$ impliziert wird und sich damit ein Nash-Produkt von Null ergibt, während das Nash-Produkt bei $\frac{1}{\sqrt{5}}$

$$u_1 \left(1 - u_1^2\right)^2 = \frac{1}{\sqrt{5}} \left(1 - \frac{1}{5}\right)^2 = \frac{16}{5^3}\sqrt{5} > 0$$

ist. Wir erhalten also (mit $\frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$)

$$\varphi^N (U, d, \omega) = \left(\frac{\sqrt{5}}{5}, \frac{4}{5}\right).$$

Gegenüber der symmetrischen Lösung (siehe Aufg. XX.6), hat sich Spieler 2 von $\frac{2}{3}$ auf $\frac{4}{5}$ verbessert, dies ist eine Auswirkung seines höheren Gewichtes.

Exercise XX.7. *Berechnen Sie die asymmetrische Nash-Lösung für $\omega_1 = 2$ und $\omega_2 = 1$. Ist die Auszahlung für Spieler 1 dieses Mal höher als $\frac{\sqrt{3}}{3}$? Hinweis: Die Aufgabe ist leichter als die obige Beispielrechnung!*

5.2. Interpretation der Gewichte. Wie kann es nun zu unterschiedlichen Gewichten kommen? In der Literatur wird in der Regel darauf verwiesen, dass die Spieler über unterschiedliches Verhandlungsgeschick oder unterschiedliche Verhandlungsmacht verfügten. Das ist natürlich noch recht vage. Hinter den unterschiedlichen Gewichten kann sich durchaus Konkretes verbergen; wir zitieren hierzu drei unterschiedliche Ansätze:

Kalai (1977) generiert die Asymmetrie zwischen zwei Spielern dadurch, dass einer der Spieler eine größere Familie repräsentiert als der andere. Der Verhandlungspartner mit der großen Familie hat dann ein größeres Gewicht. Vielleicht verhandelt er härter, weil sein Anteil durch viele Köpfe geteilt werden muss, oder er ist mächtiger, weil er eine größere Familie zur Unterstützung im Rücken hat. Die Verhandlungsmacht eines Spielers ist nun mit seiner Familiengröße zu gewichten.

Rubinstein (1982) ist es gelungen, ein nichtkooperatives Kuchen-Verhandlungsspiel für zwei Spieler zu definieren, dessen Auszahlungen im teilspielperfekten Gleichgewicht gleich der asymmetrischen Nash-Lösung sind. Die beiden Verhandelnden machen Angebot und Gegenangebot; sie werden jedoch dadurch, dass der „Kuchen" in jeder Verhandlungsrunde schrumpft, unter Druck gesetzt, sich schnell zu einigen. Und diese Schrumpfungsfaktoren, so kann Rubinstein zeigen, sind gerade die Gewichte der Nash-Verhandlungslösung! Der ungeduldige Spieler (mit kleinem Schrumpffaktor) wird einen geringeren Teil des Kuchens bekommen als der geduldige Spieler. Der Rubinstein'sche Aufsatz bietet zudem ein schönes Beispiel für das Nash-Programm; die mithilfe der kooperativen Spieltheorie gewonnenen Auszahlungen werden dabei durch die Auszahlungen im Gleichgewicht eines geeigneten nichtkooperativen Spiels repliziert. Eine leicht verständliche Darstellung des Rubinstein-Modells bietet Wiese (2002, S. 323 ff.).

Der dritte Ansatz führt die Nash-Lösung mit der Shapley-Lösung zusammen. Diesen Ansatz wollen wir im nächsten Abschnitt etwas ausführlicher darstellen. Bisher gibt es lediglich ein Arbeitspapier dazu, die Autoren sind Laruelle & Valenciano (2003). Federico Valenciano hat den Ansatz während der „14th Conference on Game Theory at Stony Brook" im Juli 2003 präsentiert, auf der übrigens sowohl John Nash als auch Lloyd Shapley anwesend waren. Robert Aumann (siehe S. **??**) fasste das Lösungskonzept mit den Worten „Nash to the power of Shapley" zusammen.

5.3. Shapley-Indizes als Gewichte.

5.3.1. *Vorgehen.* Laruelle & Valenciano (2003) verknüpfen in wunderbarer Weise die Nash-Lösung mit der Shapley-Lösung: Die Verhandlungen über den zu wählenden Auszahlungspunkt in U finden, und das ist die Grundidee des Beitrags der Autoren, „unter dem Schatten einer Wahlregel" statt. Man wird dadurch auf die asymmetrische Nash-Lösung geführt, wobei die

Gewichte gerade die Shapley-Auszahlungen der die Wahlregel darstellenden einfachen Koalitionsfunktion sind.

Wir werden nun in fünf Schritten vorgehen:

- Zunächst haben wir einige wenige formale Festlegungen zu treffen.
- Dann präsentieren wir die Formalisierung der Wahlregel.
- Anschließend definieren wir eine Koalitionsfunktion ohne transferierbaren Nutzen, die sowohl auf der Verhandlungsmenge U und dem Drohpunkt d als auch auf der Wahlregel beruht.
- Dann deuten wir kurz an, welche Axiome wir für die einzuführende Lösung benötigen.
- Schließlich präsentieren wir die Laruelle-Valenciano-Lösung.

5.3.2. *Eine kurze formale Vorbemerkung.* Laruelle & Valenciano (2003) gehen davon aus, dass die Spieler einem Nash'schen Tupel

$$(U, d)$$

gegenüberstehen, wobei U eine (hinreichend schöne) Teilmenge des $\mathbb{R}^n$, die Verhandlungsmenge, und $d \in \mathbb{R}^n$ den Drohpunkt darstellen. Wir benötigen nun die Festlegungen

$$\begin{aligned} U^K &:= \left\{ x^K \in \mathbb{R}^{|K|} : x \in U \right\} \text{ und} \\ H\left(d^K\right) &:= \left\{ x^K \in \mathbb{R}^{|K|} : x^K \leq d^K \right\}. \end{aligned}$$

U^K ist also eine Teilmenge von $\mathbb{R}^{|K|}$, die dadurch gebildet wird, dass man die Auszahlungen der Spieler aus $N \backslash K$ streicht. $H\left(d^K\right)$ ist die so genannte umfassende Hülle von $d^K = (d_i)_{i \in K}$, die neben d^K selbst alle Auszahlungen „links unterhalb" von d^K enthält.

5.3.3. *Die Wahlregel.* Die Wahlregel wird durch ein einfaches Spiel v repräsentiert. Koalitionen K mit $v(K) = 1$ heißen Gewinnkoalitionen und können „sich durchsetzen". Die Autoren treffen folgende zwei Annahmen:

- Die große Koalition kann durchsetzen, was sie möchte, $v(N) = 1$.
- v ist nicht widersprüchlich; das Komplement jeder Gewinnkoalition ist also unterlegen.

Die Menge der einfachen Spiele, die diese Eigenschaften haben, nennen wir G_n^W.

5.3.4. *Die Koalitionsfunktion ohne transferierbaren Nutzen.* Die durch v repräsentierte Wahlregel dient nun dazu, eine Koalitionsfunktion ohne transferierbaren Nutzen zu definieren. Dazu setzt man für $K \subseteq N$

$$V_{(U,d,v)}(K) := \begin{cases} U^K, & v(K) = 1 \\ H\left(d^K\right), & v(K) \neq 1. \end{cases}$$

Die Spieler aus K können also eine Auszahlungsmenge U^K realisieren, falls sie eine Gewinnkoalition darstellen. Ansonsten können sie lediglich ihre

Drohauszahlungen realisieren. Diese Definition ist das zentrale Element im Ansatz der beiden Autoren. Sie zeigen, dass diese Koalitionsfunktion eine Verallgemeinerung sowohl der Nash'schen Verhandlungsspiele als auch der einfachen Spiele darstellt:

- Nimmt man nun als Spezialfall die Wahlregel v, bei der nur die große Koalition eine Gewinnkoalition ist, ergibt sich
$$V_{(U,d,v)}(K) := \begin{cases} U, & K = N \\ H\left(d^K\right), & \text{sonst.} \end{cases}$$
Dann hat man ein Nash'sches Verhandlungsspiel.
- Laruelle & Valenciano (2003) begründen, dass man für
$$U = \left\{x \in \mathbb{R}^n : \sum_{i \in N} x_i \leq 1\right\} \text{ und}$$
$$d = (0, ..., 0)$$
eine Koalitionsfunktion ohne transferierbaren Nutzen erhält, die der Koalitionsfunktion v (mit transferierbarem Nutzen) entspricht.

5.3.5. *Lösungen und Axiome.* Laruelle und Valenciano suchen nach einer Lösung in Abhängigkeit von $U \subseteq \mathbb{R}^n, d \in \mathbb{R}^n$ und $v \in G_n^W$:

Definition XX.6. *Die Abbildung*
$$\psi : \mathbb{V}_n \times G_n^W \to \mathbb{R}^n$$
heißt punktwertige Verhandlungslösung auf N.

Den Autoren gelingt es, ihre Lösung durch sechs Axiome zu charakterisieren. Diese Axiome sind im Wesentlichen diejenigen, die für die Nash-Lösung bzw. die Shapley-Lösung benötigt werden. Wir deuten sie hier an, ohne sie formal präzise wiederzugeben:

- ψ soll das Pareto-Axiom der Nash-Lösung (S. 385) erfüllen.
- ψ soll unabhängig davon sein, wie die Spieler benannt sind.
- ψ soll von irrelevanten Alternativen unabhängig sein (S. 386).
- ψ soll die Invarianz bei affinen Transformationen (S. 386) erfüllen.
- Ein Nullspieler $i \in N$ in $v \in G_n^W$ soll die Auszahlung d_i erhalten.
- Schließlich soll das Transferaxiom (bzw. eine Variante dieses Axioms) bezüglich der (einfachen!) Spiele aus G_n^W erfüllt sein (siehe Kap. **??**, S. **??**).

5.3.6. *Die Laruelle-Valenciano-Lösung.* Nun endlich präsentieren wir die Laruelle-Valenciano-Lösung samt ihrer Axiomatisierung:

Definition XX.7. *Sei eine Verhandlungsmenge $U \subseteq \mathbb{R}^n$, ein Drohpunkt $d \in \mathbb{R}^n$ und eine Wahlregel $v \in G_n^W$ mit $\varphi_i(v) > 0$ für alle $i \in N$ gegeben.*

Die Laruelle-Valenciano-Lösung φ^{LV} ist gleich der asymmetrischen Nash-Lösung für (U,d), wobei die Gewichte gleich den Shapley-Auszahlungen für v sind:

$$\varphi^{LV}(U,d,v) = \varphi^{N}(U,d,\varphi(v)).$$

THEOREM XX.2. *Eine Lösung*

$$\psi : \mathbb{V}_n \times G_n^W \to \mathbb{R}^n$$

erfüllt genau dann die sechs Axiome aus Abschnitt 5.3.5, wenn $\psi = \varphi^{LV}$ gilt.

Zudem ist φ^{LV} eine Verallgemeinerung der Nash- und der Shapley-Lösung:

- Ist v eine symmetrische Koalitionsfunktion, ist die Laruelle-Valenciano-Lösung gleich der symmetrischen Nash-Lösung,
$$\varphi^{LV}(U,d,v) = \varphi^{N}(U,d).$$
- Für
$$\begin{aligned} U &= \left\{ x \in \mathbb{R}^n : \sum_{i \in N} x_i \leq 1 \right\} \text{ und} \\ d &= (0,...,0) \end{aligned}$$
erhält man
$$\varphi^{LV}(U,d,v) = \varphi(v).$$

Die Laruelle-Valenciano-Lösung verbindet also auf äußerst elegante Art und Weise die Nash-Lösung mit der Shapley-Lösung. Zu Recht bezeichnete Robert Aumann während der auf S. 393 erwähnten Konferenz dieses Resultat als „mind blowing". Man könnte es auch als Hinweis dafür nehmen, dass die Shapley-Lösung im Bereich des transferierbaren Nutzens und die Nash-Lösung im Bereich des nichttransferierbaren Nutzens ihre Vorrangstellungen verdient haben. Beispielsweise bezeichnet auch Myerson (1994, S. 69) diese beiden Lösungen als „the most conceptually elegant and appealing solution theories in cooperative game theory".

6. Alternativen zur Nash-Lösung

Es gibt eine Vielzahl von Alternativen zur Nash-Lösung. Aber lediglich die Kalai-Smorodinsky-Lösung (Kalai & Smorodinsky (1975)) werden wir etwas ausführlicher darstellen. Bei diesem Lösungskonzept spielt der durch

$$a_i(U,d) := \max\{u_i \in \mathbb{R} : \text{Es gibt einen Nutzenvektor } u \text{ in } U \text{ mit } u \geq d\}$$

definierte Punkt $a(U,d) = (a_i(U,d))_{i\in N}$ eine zentrale Rolle (siehe S. 384). Dieser so genannte Idealpunkt und die Kalai-Smorodinsky-Lösung sind in Abb. 6 für zwei Spieler eingezeichnet. Die Kalai-Smorodinsky-Lösung erhält

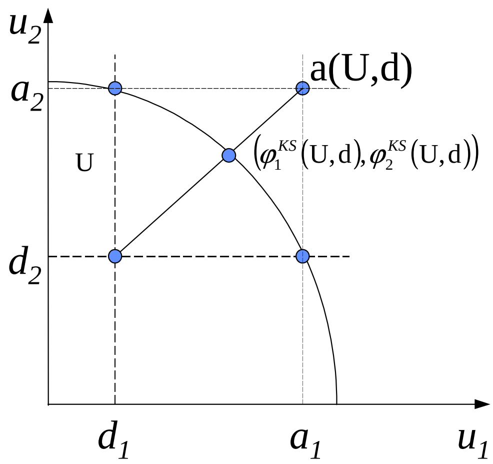

FIGURE 6. Die Kalai-Smorodinsky-Lösung

man als Schnittpunkt der Nutzengrenze von U mit der Strecke, die $a(U,d)$ mit d verbindet. Wir bezeichnen sie mit φ^{KS}.

THEOREM XX.3. *Genau dann, wenn eine Lösung ψ auf $\mathbb{V}_2$ die Axiome*

- *Pareto-Axiom,*
- *Symmetrie-Axiom,*
- *Axiom über die Invarianz bei affinen Transformationen und*
- *Axiom über die individuelle Monotonie für zwei Spieler*

erfüllt, gilt

$$\psi = \varphi^{KS}.$$

Während sich die Axiomatisierungen im Falle der Nash-Lösung und auch bei der asymmetrischen Nash-Lösung ohne weiteres auf n Spieler erweitern lassen, gilt die Axiomatisierung der Kalai-Smorodinsky-Lösung nur für den Zwei-Personen-Fall. Die Berechnung von φ^{KS} lässt sich jedoch ohne größere Schwierigkeiten auch für beliebige Spielermengen bewerkstelligen.

Man beachte, dass die Axiome für die Kalai-Smorodinsky-Lösung sich von denjenigen für die Nash-Lösung nur im vierten Axiom unterscheiden. Anstelle des Axioms über die Unabhängigkeit irrelevanter Alternativen bei Nash tritt bei Kalai und Smorodinsky das Axiom über die individuelle Monotonie für zwei Spieler. Beide Lösungskonzepte erfüllen die Drohpunkt-Monotonie (S. 388).

Nun nehmen wir wieder unser Kuchen-Beispiel. Die Verhandlungsmenge sei also

$$U = \{(u_1, u_2) : u_1 + u_2 \leq 1, u_1 \geq 0, u_2 \geq 0\}$$

und der Drohpunkt d erfülle

$$d_1 + d_2 < 1.$$

Der Idealpunkt ist durch

$$\begin{aligned} a_1(U,d) &= \max\{u_1 \in \mathbb{R} : \text{Es gibt einen Nutzenvektor } u \text{ in } U \text{ mit } u \geq d\} \\ &= \max\{u_1 \in \mathbb{R} : u_1 + u_2 \leq 1 \text{ und } u_2 \geq d_2\} \\ &= 1 - d_2 > d_1 \end{aligned}$$

und analog durch

$$a_2(U,d) = 1 - d_1$$

gegeben.

Aus Gründen der Schreibökonomie verwenden wir nun a_i für $a_i(U,d)$. Wir haben den Schnittpunkt der Nutzengrenze mit der d und a verbindenden Strecke zu finden. Die Nutzengrenze erfüllt $u_2 = 1-u_1$; die Verbindungsstrecke zwischen d und a kann man mithilfe der Punkt-Steigungs-Formel beschreiben, wobei wir vom Punkt d ausgehen und von dort $(a_1 - d_1) > 0$ Einheiten in Richtung u_1 und $(a_2 - d_2) > 0$ Einheiten in Richtung u_2 „marschieren":

$$\underbrace{\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} + \beta \begin{pmatrix} a_1 - d_1 \\ a_2 - d_2 \end{pmatrix}}_{\substack{\text{durch } \beta \text{ parametrisierter Punkt} \\ \text{auf der Linie von } d \text{ zu } a}} = \underbrace{\begin{pmatrix} u_1 \\ 1 - u_1 \end{pmatrix}}_{\substack{\text{Punkt auf der} \\ \text{Nutzengrenze}}}.$$

Wir haben somit zwei Gleichungen mit den zwei Unbekannten u_1 und β. Nach Substitution von $a_1 = 1 - d_2$ und $a_2 = 1 - d_1$ kann man diese folgendermaßen schreiben:

$$\begin{aligned} d_1 + (1 - d_2 - d_1)\beta &= u_1 \text{ und} \\ d_2 + (1 - d_1 - d_2)\beta &= 1 - u_1. \end{aligned}$$

Setzt man die obere in die untere ein, erhält man zunächst

$$d_2 + (1 - d_1 - d_2)\beta = 1 - (d_1 + (1 - d_2 - d_1)\beta)$$

und sodann

$$\beta = \frac{1}{2}.$$

Hieraus ergibt sich die Kalai-Smorodinsky-Lösung für unser Kuchen-Beispiel als

$$\begin{aligned} \varphi^{KS}(U,d) &= (d_1 + \beta(a_1 - d_1), d_2 + \beta(a_2 - d_2)) \\ &= \left(d_1 + \frac{1}{2}(1 - d_2 - d_1), d_2 + \frac{1}{2}\beta(1 - d_1 - d_2)\right) \\ &= \left(\frac{1}{2} + \frac{1}{2}(d_1 - d_2), \frac{1}{2} + \frac{1}{2}(d_2 - d_1)\right). \end{aligned}$$

Für dieses Beispiel stimmen φ^{KS} und φ^{N} überein. Die Übereinstimmung ist jedoch im Falle von

$$U' = \{(u_1, u_2) : x_1 + x_2 \leq 1, u_1 = \sqrt{x_1}, u_2 = x_2, x_1 \geq 0, x_2 \geq 0\}$$

und

$$d_1 = 0, d_2 = 0$$

nicht mehr gegeben, wie wir nun zeigen wollen. Bevor die tatsächliche Rechnung erfolgt (siehe die nächste Übung), können wir jedoch das Monotonie-Axiom anwenden. Die Nutzengrenze für U' ist durch $u_2 = x_2 = 1 - x_1 = 1 - (u_1)^2$ gegeben. Wegen $(u_1)^2 \leq u_1$, $0 \leq x_1 \leq 1$, und $-(u_1)^2 \geq -u_1$ ist U' eine Obermenge von U. Gleichheit ergibt sich bei 0 und 1, sodass alle Voraussetzungen für das Monotonie-Axiom (S. 387) erfüllt sind. Wir wissen also, dass die Kalai-Smorodinsky-Lösung für beide Spieler eine Auszahlung von mindestens $\frac{1}{2}$ vorsehen muss.

Exercise XX.8. *Berechnen Sie die Kalai-Smorodinsky-Lösung für das variierte Kuchen-Beispiel. Hinweis: Die quadratische Gleichung* $(u_1)^2 + u_1 - 1 = 0$ *hat die zwei Lösungen* $-\frac{1}{2} + \frac{1}{2}\sqrt{5} \approx 0,62$ *und* $-\frac{1}{2} - \frac{1}{2}\sqrt{5}$.

Neben der Nash-Lösung und der Kalai-Smorodinsky-Lösung gibt es

- die egalitäre Lösung (ausgehend von d gewinnt jeder Spieler gleich viel hinzu),
- die diktatorische Lösung (ein Spieler $i \in N$, der Diktator, erhält $a_i(U, d)$, während sich die übrigen Spieler mit d_j, $j \neq i$, zu begnügen haben),
- die utilitaristische Lösung (die Summe der Auszahlungen wird maximiert) und
- etliche andere Lösungen (siehe Thomson (1994) und die dort angegebene Literatur).

7. Neue Begriffe

- Koalitionsfunktion ohne transferierbaren Nutzen
- Maximal erreichbarer Nutzen
- Superadditivität
- Anfangsausstattung
- Allokationen
- Walras-Gleichgewicht
- Walras-Allokation
- Heiratsmarkt
- Kern (Zulässigkeit, Nicht-Blockade)
- 1. Hauptsatz der Wohlfahrtstheorie
- Nash-Verhandlungs-Lösung
- Drohpunkt

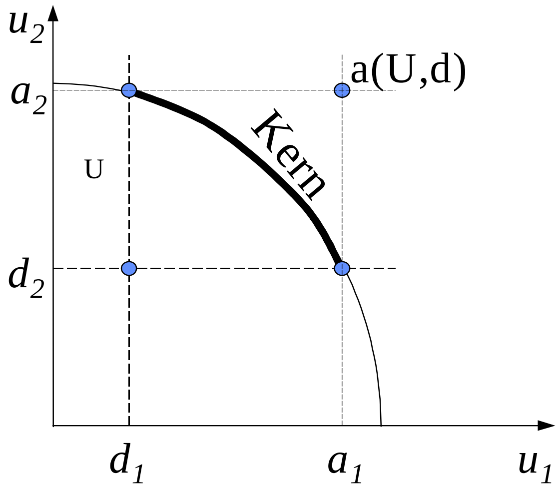

FIGURE 7. Der Kern des Verhandlungsproblems

- Verhandlungsmenge
- Verhandlungsspiel
- Nutzenmöglichkeitenkurve
- Idealpunkt
- Punktwertige Verhandlungslösung
- Axiome
  - Invarianz bei affinen Transformationen
  - Unabhängigkeit irrelevanter Alternativen
  - Individuelle Monotonie für zwei Spieler
  - Verhandlungsmengen-Monotonie
  - Drohpunkt-Monotonie
- Asymmetrische Nash-Lösung
- Laruelle-Valenciano-Lösung
- Kalai-Smordinsky-Lösung

8. Lösungen zu den Übungen

Exercise XX.1

Have you found $m_1^{\{1,2\}} = 3$?

8.0.7. *XX.2.* Der Kern umfasst die Nutzenkombinationen, gegen die keine Koalition Einwand erhebt. Die große Koalition, also beide Spieler zusammen, besteht auf einen Punkt auf der Nutzenmöglichkeitskurve, während Spieler 1 eine Auszahlung unterhalb von d_1 und Spieler 2 eine Auszahlung unterhalb von d_2 nicht akzeptieren werden. Der Kern, der in Abbildung 7 eingezeichnet ist, ist also gleich der Nutzengrenze zwischen $a_1(U,d)$ und $a_2(U,d)$.

8.0.8. *XX.3.* Je weniger ein Spieler auf Verhandlungen angewiesen ist, desto stärker ist seine Verhandlungsposition. Deshalb sollte man erwarten, dass der in Verhandlungen erreichbare Nutzen mit d_1 ansteigt.

8.0.9. *XX.4.* Die gesamte Nutzenmöglichkeitenkurve ist Pareto-effizient, nicht nur der Kern.

8.0.10. *XX.5.* Ja, das ist richtig. Nach Einsetzen von $u_2 = 1 - u_1$ haben wir

$$\begin{aligned} & (u_1 - d_1)(1 - u_1 - d_2) \\ = \; & -u_1^2 + u_1(1 + d_1 - d_2) - d_1 + d_1 d_2 \end{aligned}$$

zu maximieren. Durch Differenzieren und Nullsetzen erhält man

$$\varphi_1^N(U, d) = \frac{1}{2} + \frac{1}{2}(d_1 - d_2)$$

und dann unter Beachtung von $u_2 = 1 - u_1$

$$\varphi_2^N(U, d) = \frac{1}{2} + \frac{1}{2}(d_2 - d_1).$$

8.0.11. *XX.6.* Die Nutzenmöglichkeitenkurve ergibt sich so:

$$u_2 = x_2 = 1 - x_1 = 1 - (u_1)^2.$$

Damit erhält man das zu maximierende Produkt

$$\begin{aligned} & (u_1 - d_1)(u_2 - d_2) \\ = \; & (u_1)\left(1 - u_1^2\right) \\ = \; & u_1 - u_1^3. \end{aligned}$$

Differenzieren und Nullsetzen ergeben

$$u_1^2 = \frac{1}{3}$$

und

$$u_2 = x_2 = 1 - u_1^2 = \frac{2}{3}.$$

Man erhält also wegen $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ die Nash-Lösung

$$\left(\frac{\sqrt{3}}{3}, \frac{2}{3}\right).$$

8.0.12. *XX.7.* Wir erhalten das zu maximierende Produkt

$$(u_1 - d_1)^2 (u_2 - d_2)^1 = u_1^2\left(1 - u_1^2\right) = u_1^2 - u_1^4$$

Differenzieren und Nullsetzen ergeben

$$u_1\left(2 - 4u_1^2\right) \stackrel{!}{=} 0,$$

woraus

$$u_1 \stackrel{!}{=} 0,\ u_1 \stackrel{!}{=} \frac{\sqrt{2}}{2} \text{ oder } u_1 \stackrel{!}{=} -\frac{\sqrt{2}}{2}$$

folgen. Wie im Haupttext ausgeführt, kommt nur $\frac{\sqrt{2}}{2}$ in Frage:

$$\varphi^N(U,d,\omega) = \left(\frac{\sqrt{2}}{2}, \frac{1}{2}\right).$$

Gegenüber der symmetrischen Lösung hat sich Spieler 1 von $\frac{\sqrt{3}}{3} \approx 0,58$ auf $\frac{\sqrt{2}}{2} \approx 0,71$ verbessert, während sich Spieler 2 von $\frac{2}{3}$ auf $\frac{1}{2}$ verschlechtert hat.

8.0.13. *XX.8.* Der Idealpunkt ist durch

$$\begin{aligned} a_1\left(U',d\right) &= \max\{u_1 \in \mathbb{R} : \text{Es gibt einen Nutzenvektor } u \text{ in } U \text{ mit } u \geq (0,0)\} \\ &= \max\left\{u_1 \in \mathbb{R} : (u_1)^2 + u_2 \leq 1 \text{ und } u_2 \geq 0\right\} \\ &= 1 \end{aligned}$$

und durch

$$\begin{aligned} a_2\left(U',d\right) &= \max\{u_2 \in \mathbb{R} : \text{Es gibt einen Nutzenvektor } u \text{ in } U \text{ mit } u \geq (0,0)\} \\ &= \max\left\{u_2 \in \mathbb{R} : (u_1)^2 + u_2 \leq 1 \text{ und } u_1 \geq 0\right\} \\ &= 1 \end{aligned}$$

definiert.

Nun geht es ganz ähnlich weiter wie im Haupttext. Zunächst ergibt sich, mit anders definiertem Idealpunkt, wiederum

$$\underbrace{\begin{pmatrix}0\\0\end{pmatrix} + \beta\begin{pmatrix}1-0\\1-0\end{pmatrix}}_{\substack{\text{durch } \beta \text{ parametrisierter Punkt} \\ \text{auf der Linie von } d=(0,0) \text{ zu } a=(1,1)}} = \underbrace{\begin{pmatrix}u_1\\1-(u_1)^2\end{pmatrix}}_{\substack{\text{Punkt auf der} \\ \text{Nutzengrenze}}}.$$

Damit ergeben sich die beiden Gleichungen

$$\begin{aligned} \beta &= u_1 \text{ und} \\ \beta &= 1-(u_1)^2, \end{aligned}$$

woraus

$$u_1 = -\frac{1}{2} + \frac{1}{2}\sqrt{5}$$

wegen $-\frac{1}{2} - \frac{1}{2}\sqrt{5} < 0$ folgt. Somit erhält man die Kalai-Smorodinsky-Lösung für U' und $d = 0$ als

$$\begin{aligned} \varphi^{KS}(U,d) &= \left(u_1, 1-(u_1)^2\right) \\ &= \left(-\frac{1}{2} + \frac{1}{2}\sqrt{5}, 1 - \left(-\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)^2\right) \\ &= \left(-\frac{1}{2} + \frac{1}{2}\sqrt{5}, -\frac{1}{2} + \frac{1}{2}\sqrt{5}\right) \\ &\approx (0,62, 0,62). \end{aligned}$$

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