TWO-PERSON COOPERATIVE GAMES

Part II: The Axiomatic Approach

J. Nash – 1953

The Approach

"Rather than solve the two-person cooperative game by analyzing the bargaining process, one can attack the problem axiomatically by stating general properties that 'any reasonable solution' should possess. By specifying enough such properties one excludes all but one solution."

Notations

- Two players: N={1,2}, often denoted by *i* or *j*.
- D is the event “disagreement”
- $d = (u_1(D), u_2(D))$
- The value to the two players are ψ_1(B,d) and ψ_2(B,d)

Axiom I: (weak) Pareto efficiency

- The solution is not weakly dominated by any point in B except itself.

If $(u_1,u_2)\in B$ and $u_1 \geq \psi_1$ and $u_2 \geq \psi_2$ then $(u_1,u_2)=(\psi_1,\psi_2)$

- The players never agree on a payoff (ψ_1,ψ_2) when there is a preferable payoff <u>for both</u>.

Axiom I: (weak) Pareto efficiency

- Roth (1977) "The requirement that a solution be Pareto optimal may be thought of as a requirement of collective rationality, since it states that an outcome may not be chosen if there is another outcome which both player agree is preferable."

Axiom II: Invariance to equivalent utility representations.

- Suppose that the bargaining problem (B',d') is obtained from (B,d) by the transformations
  - $u_i \mapsto a_i u_i + b_i, \text{for } i = 1,2 \text{ and } a_i > 0$, which means
  - $d'_i = a_i d_i + b_i$
  - $B' = \{(a_1 u_1 + b_1, a_2 u_2 + b_2) : (u_1, u_2) \in B\}$
- Then $\psi_i(B',d') = a_i\ \psi_i(B,d) + b_i$
- Just the numerical values will be changed.

Axiom III: Symmetry

- The solution does not depend on which player is called player one.

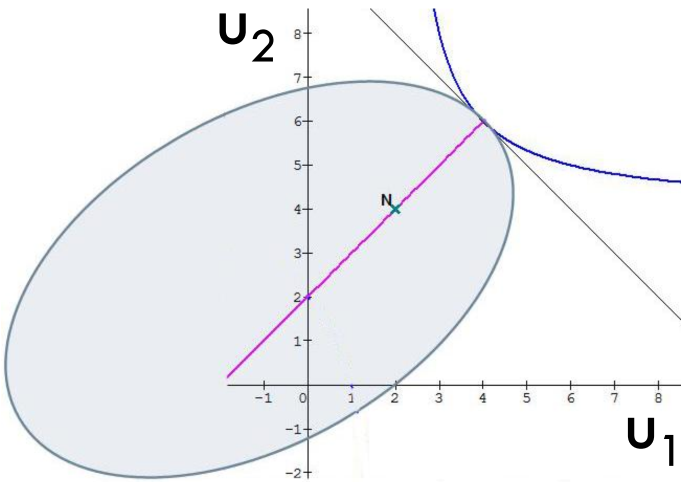
- Consider the two bargaining situations (B,d) and (B',d') such that
  - $d'_1=d_2$ and $d_1=d'_2$
  - $B'=\{(u_1,u_2):(u_2,u_1)\in B\}$

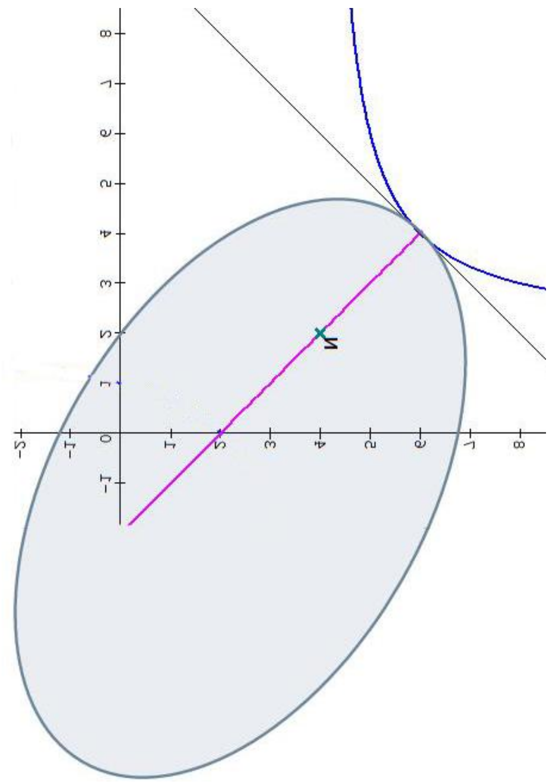
Then the solution point is

$\psi_1(B',d')=\psi_2(B,d)$ and $\psi_2(B',d')=\psi_1(B,d)$

- The difference between the players are the set of strategies and the utility functions.

Axiom III: Symmetry

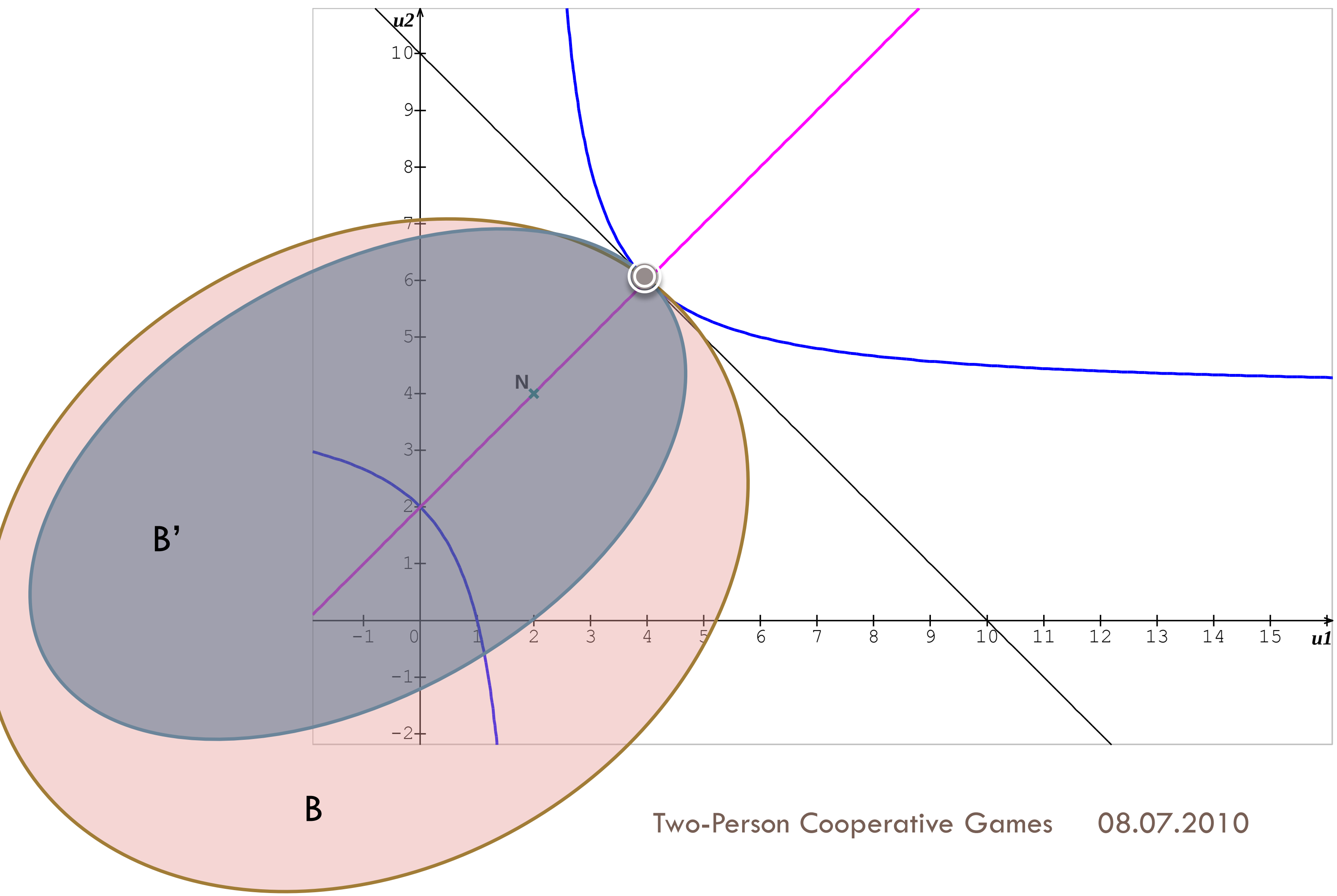




Axiom IV: Independence of irrelevant alternatives

- If (B,d) and (B',d) are two bargaining situations with $B' \subset B$ and $\psi(B,d) \in B'$
- Then $\psi(B,d) = \psi(B',d)$

Axiom IV: Independence of irrelevant alternatives



Discus.

Solution

- Axiom I: Pareto efficiency
- Axiom II: Invariance to equivalent utility representations
- Axiom III: Symmetry
- Axiom IV: Independence of irrelevant alternatives

"For each game (B,d) there is a unique solution (ψ_1, ψ_2) which is a point in B."

Solution

The Nash solution is

$$(d_1, d_2) := \arg \max_{\substack{(u_1,u_2)\in B \\ u_{iN}=\text{const.} \\ u_i \geq u_{iN}}} (u_1 - u_{1N})(u_2 - u_{2N})$$

The solution is the utility pair that maximizes the product of the players' utilities (to bargain).

Discussion: Axiom I (Pareto efficiency)

- What if this axiom lacks?
- Consider the solution $\psi(B,d)=d$
  - Invariance to equivalent utility representations?
  - Symmetry?
  - Independence of irrelevant alternatives?
- This shows how necessary is this axiom

Discussion: Axiom IV (Independence of irrelevant alternatives)

- All points in B (but the solution point ψ) are *irrelevant*.
- Exclude situations in which the fact that some available agreements influence the outcome.
- "Note that the axiom is satisfied, in particular, by any solution that is defined to be a member of B that maximizes the value of some function." (Rubinstein)

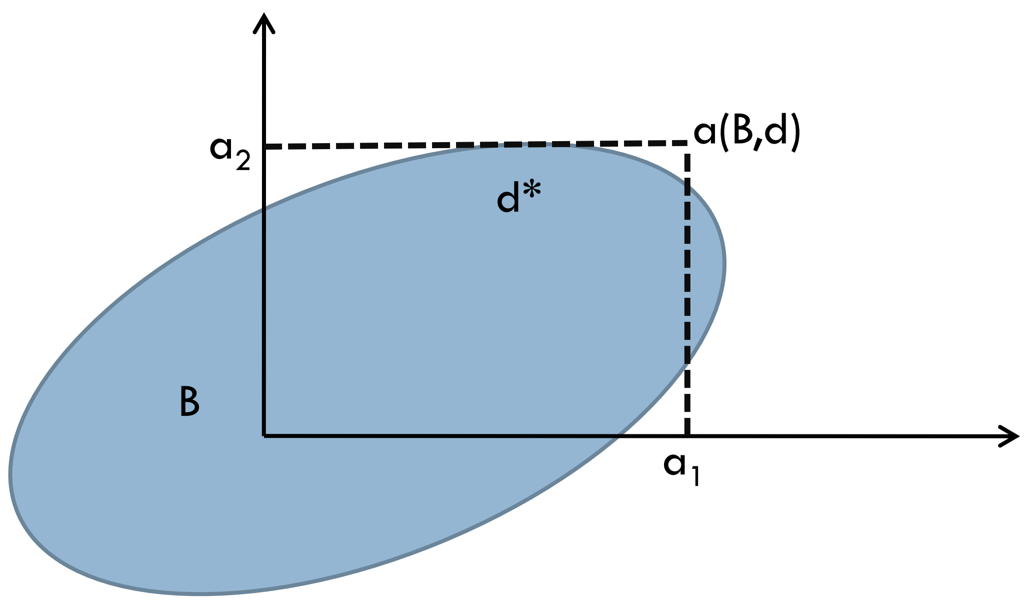
Graph

Discusssion: Axiom IV (suite)

- Nash argues geometrically: “This axiom is equivalent to **an axiom of ‘localization’** of the dependence of the solution point on the shape of the set B. The location of the solution point on the upper-right boundary of B is determined only by the shape of any small segment of the boundary that extends to both sides of it. **It does not depend on the rest of the boundary curve.**”

Idealpunkt a_i

$$a_i(B,d) = \max\{u_i \in R : \text{Es gibt einen Nutzenvektor } u \text{ in } B \text{ mit } u \geq d\}$$



An alternative (among others) Axiom: Individual Monotonicity

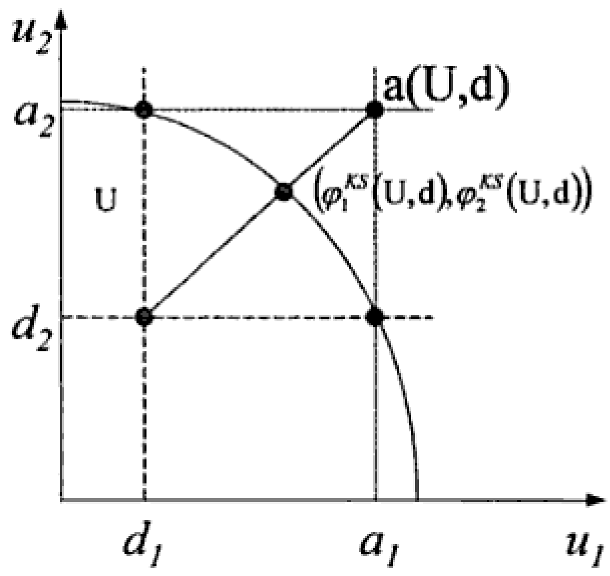
- For any (B',d) and (B,d) with $B' \subset B$ and $a_i = a'_i$ we have $\psi_i(B',d) \leq \psi_i(B,d)$ for $i=1,2$
- For any (B',d) and (B,d) with $B' \subset B$ and $a_i = a'_i$ we have $\psi_j(B',d) \leq \psi_j(B,d)$ for $i \neq j$

An alternative

Axiom I	• Pareto efficiency
Axiom II	• Invariance to equivalent utility representations
Axiom III	• Symmetry
Axiom IV	• ~~Independence of irrelevant alternatives~~
Axiom	• Individual Monotonicity

Kalai-Smorodinsky's solution

Kalai-Smorodinsky's solution



H. Wiese – Kooperative Spieltheorie, S. 292