Overview part F: Non-transferable utility

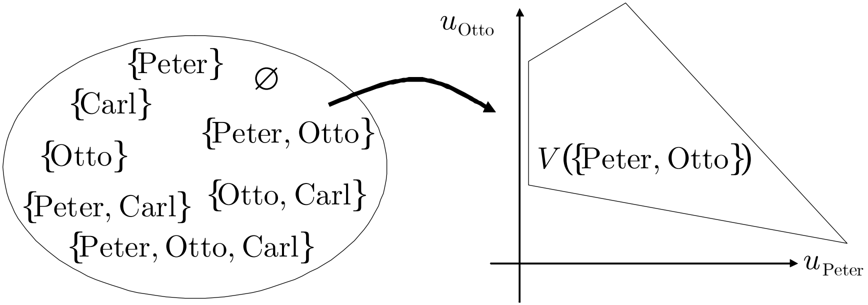
- Exchange economies
- The Nash solution

Overview "The Solow growth model"

- Introduction
- Budget
- Household optimum
- NTU coalition functions and the core
- Edgeworth boxes and coalition functions
- GET: decentralization through prices
- The marriage market

Introduction I

- transferable utility —> To every coalition $K \subseteq N$, a real number $v(K)$ is attributed.
- non-transferable utility —> To every coalition $K \subseteq N$, a set of payoff vectors is attributed.



Set of coalitions

Set of payoff vectors

Introduction II

GET = General Equilibrium Theory

- Agents observe prices and choose their good bundles accordingly.
- All agents (households and firms) are price takers.

The aim is to find prices such that

- all actors behave in a utility, or profit, maximizing way and
- the demand and supply schedules can be fulfilled simultaneously.

—> Walras equilibrium

- existence
- efficiency and core

Special case: marriage market

Definition

The expenditure for a bundle of goods $x = (x_1, x_2, ..., x_\ell)$ at a vector of prices $p = (p_1, p_2, ..., p_\ell)$ is the dot product (or the scalar product):

$$p \cdot x := \sum_{g=1}^{\ell} p_g x_g .$$

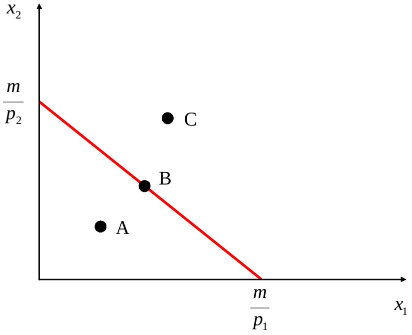
Definition

For $p \in \mathbb{R}^\ell$ and $m \in \mathbb{R}_+$:

$$B(p, m) := \left\{ x \in \mathbb{R}_+^\ell : p \cdot x \leq m \right\}$$

– the money budget.

$$\left\{ x \in \mathbb{R}_+^\ell : p \cdot x = m \right\}$$



Problem

Assume that the household consumes bundle A. Identify the "left-over" in terms of good 1, in terms of good 2 and in money terms.

Problem

What happens to the budget line if

- *price p_1 doubles;*
- *if both prices double?*

Lemma

For any number $\alpha > 0$*:*

$$B(\alpha p, \alpha m) = B(p, m)$$

Problem

Fill in: For any number $\alpha > 0$*:*
$B(\alpha p, m) = B(p, ?)$.

Lemma

The money budget is nonempty, closed and convex. If $p >> 0$ holds, the budget is bounded.

Proof.

- $(0, ..., 0) \in \mathbb{R}_+^\ell$ and $0 \cdot p = 0 \leq m \Rightarrow$ budget is nonempty;
- $x_g \geq 0$, $g = 1, ..., \ell$, $x \cdot p \leq m \Rightarrow$ budget is closed;
- consider x and x' and $k \in [0, 1] \Rightarrow x \cdot p \leq m$ and $x' \cdot p \leq m$ imply: $(kx + (1-k)\,x') \cdot p = kx \cdot p + (1-k)\,x' \cdot p \leq km + (1-k)\,m = m$ $\Rightarrow$ budget is convex;
- If $p >> 0$, $0 \leq x \leq \left(\frac{m}{p_1}, ..., \frac{m}{p_\ell}\right) \Rightarrow$ budget is bounded.

□

Problem

Verify that the budget line's slope is given by $-\frac{p_1}{p_2}$ *(in case of* $p_2 \neq 0$*).*

Definition

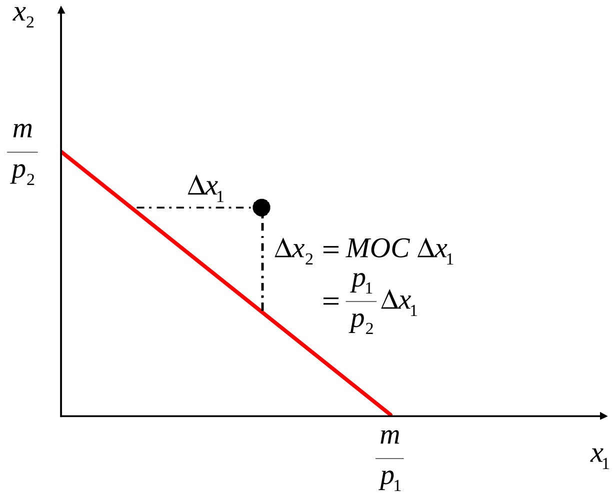
If $p_1 \geqslant 0$ and $p_2 > 0$,

$$MOC\left(x_1\right) = \left|\frac{dx_2}{dx_1}\right| = \frac{p_1}{p_2}$$

– the marginal opportunity cost of consuming one unit of good 1 in terms of good 2.

Budget

Marginal opportunity cost



Endowment budget

Definition

Definition

For $p \in \mathbb{R}^\ell$ and an endowment $\omega \in \mathbb{R}_+^\ell$:

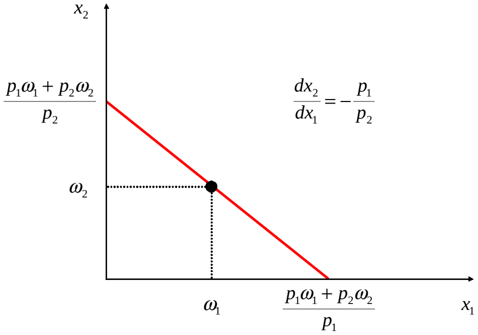
$$B(p, \omega) := \left\{ x \in \mathbb{R}_+^\ell : p \cdot x \leq p \cdot \omega \right\}$$

– the endowment budget.

Endowment budget

A two goods case

budget line: $p_1x_1 + p_2x_2 = p_1\omega_1 + p_2\omega_2$
marginal opportunity cost: $MOC = \left|\frac{dx_2}{dx_1}\right| = \frac{p_1}{p_2}$



Problem

What happens to the budget line if

- *price p_1 doubles;*
- *if both prices double?*

Notation:

- ω_1 and ω_2 – monetary income in t_1 and t_2;
- x_1 and x_2 – consumption in t_1 and t_2;
- household can borrow ($x_1 > \omega_1$), lend ($x_1 < \omega_1$) or consume what it earns ($x_1 = \omega_1$);
- r – rate of interest.

Consumption in t_2:

$$
\begin{aligned}
x_2 &= \underbrace{\omega_2}_{\substack{\text{second-period}\\ \text{income}}} + \underbrace{(\omega_1 - x_1)}_{\substack{\text{amount borrowed } (<0)\\ \text{or lended } (>0)}} + \underbrace{r\,(\omega_1 - x_1)}_{\substack{\text{interest payed } (<0)\\ \text{or earned } (>0)}} \\
&= \omega_2 + (1+r)\,(\omega_1 - x_1)
\end{aligned}
$$

- *borrow* verwandt mit
  - *borgen* und
  - *bergen* („in Sicherheit bringen") wie in *Herberge* („ein das Heer bergender Ort")
- *lend* verwandt mit
  - *Lehen* („zur Nutzung verliehener Besitz") und
  - *leihen*, verwandt mit
    - lateinischstämmig *Relikt* („Überrest") und *Reliquie* („Überbleibsel oder hochverehrte Gebeine von Heiligen") und mit
    - griechischstämmig *Eklipse* („Ausbleiben der Sonne oder des Mondes" $>$ „Sonnen- bzw. Mondfinsternis") und auch mit
    - griechischstämmig *Ellipse* (in der Geometrie ein Langkreis, bei dem die Höhe geringer ist als die Breite und insofern ein Mangel im Vergleich zum Kreis vorhanden ist – agr. *elleipsis* ($\acute{\varepsilon}\lambda\lambda\varepsilon\iota\psi\iota\varsigma$) bedeutet „Ausbleiben" $>$ „Mangel"

2 ways to rewrite the budget equation:

- in future value terms:

$$(1+r)\,x_1 + x_2 = (1+r)\,\omega_1 + \omega_2,$$

- in present value terms:

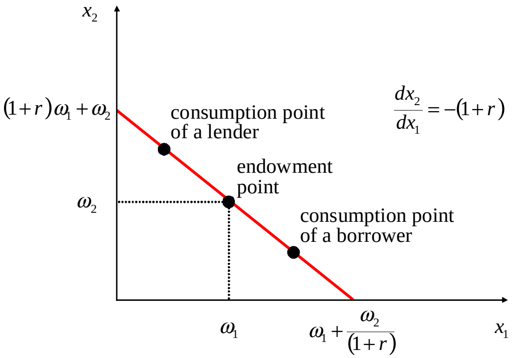
$$x_1 + \frac{x_2}{1+r} = \omega_1 + \frac{\omega_2}{1+r}.$$

Application 1

Intertemporal consumption

budget line: $(1+r)\,x_1 + x_2 = (1+r)\,\omega_1 + \omega_2$

marginal opportunity cost: $MOC = \left|\frac{dx_2}{dx_1}\right| = 1 + r$



Problem

What happens to the budget line if the interest rate decreases?

Notation:

- x_R – recreational hours $(0 \leq x_R \leq 24 = \omega_R) \rightarrow$ good 1;
- household works $24 - x_R$ hours;
- x_C – real consumption $\rightarrow$ good 2;
- w – the wage rate;
- ω_C – the real non-labor income;
- p – the price index.

- Holdshold's consumption in nominal terms:

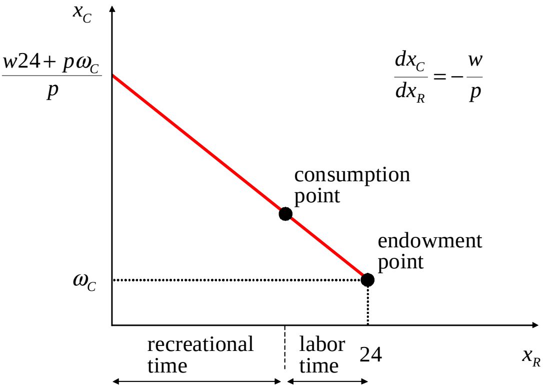
$$px_C = p\omega_C + w\left(24 - x_R\right)$$

- Holdshold's consumption in endowment-budget form:

$$wx_R + px_C = w24 + p\omega_C$$

budget line: $wx_R + px_C = w24 + p\omega_C$

marginal opportunity cost: $MOC = \left|\frac{dx_C}{dx_R}\right| = \frac{w}{p}$



Problem

What happens to the budget line if the wage rate increases?

The household's decision situation

Definition

$$\begin{aligned}\Delta &= (B, \precsim) \text{ with} \\ B &= B(p,m) \subseteq \mathbb{R}_+^\ell \text{ or } B = B(p,\omega) \subseteq \mathbb{R}_+^\ell\end{aligned}$$

– household's decision situation with:

- $p \in \mathbb{R}^\ell$ – a vector of prices;
- $\precsim$ – a preference relation on $\mathbb{R}_+^\ell$.

The household's decision problem

Definition

$$\Delta = (B, U)$$

– the decision situation
with utility function U on $\mathbb{R}_+^\ell$

Definition

$$x^R(\Delta) := \arg\max_{x \in B} U(x)$$

– the best-response function.
– i.e., $x^R(\Delta) = \{x \in B\colon \text{ there is not } x' \in B \text{ with } x' \succ x\}$
Any x^* from $x^R(\Delta)$ – a household optimum.

The household's decision problem

Lemma

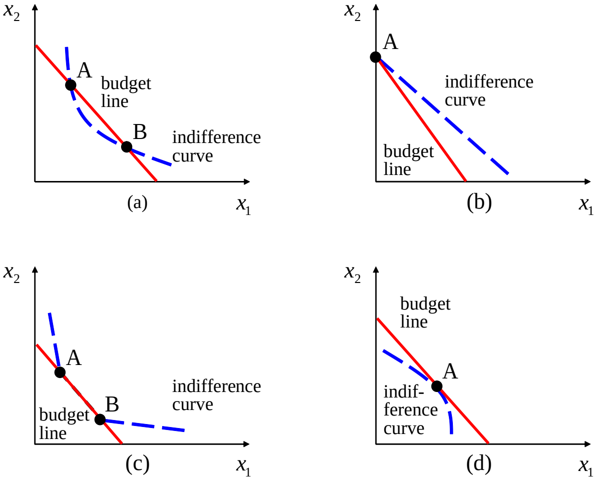
Lemma

For any number $\alpha > 0$*:*

$$x^R(\alpha p, \alpha m) = x^R(p, m)$$

The household's decision problem

Exercise 1



Problem

Assume monotonicity of preferences. Are the highlighted points A or B optima?

The household's decision problem

Exercise 2

Problem

Assume a household's decision problem with $\Delta = (B(p, \omega), \precsim)$*.* $x^R(\Delta)$ *consists of the bundles* x *that fulfill the two conditions:*

1. *The household can afford* x*:*

$$p \cdot x \leq p \cdot \omega$$

2. *There is no other bundle* y *that the household can afford and that he prefers to* x*:*

$$y \succ x \Rightarrow ??$$

Substitute the question marks by an inequality.

MRS versus MOC

Marginal willingness to pay: $MRS = \left| \frac{dx_2}{dx_1} \right|$

If the household consumes
one additional unit of good 1,
how many units of good 2
can he forgo so as to remain indifferent.

movement on the indifference curve

Marginal opportunity cost: $MOC = \left| \frac{dx_2}{dx_1} \right|$

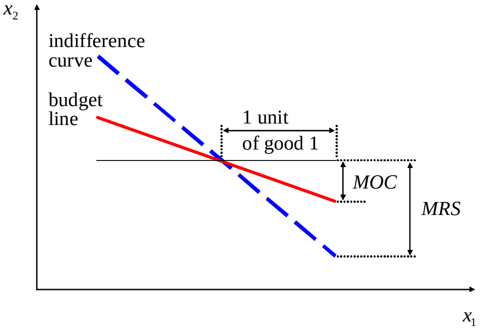
If the household consumes
one additional unit of good 1,
how many units of good 2
does he have to forgo so as to remain
within his budget.

movement on the budget line

MRS versus MOC

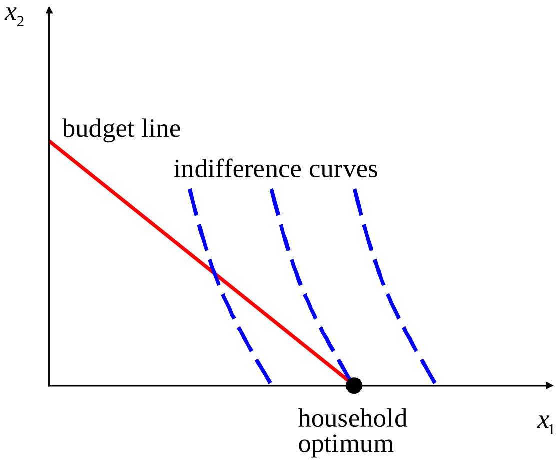
$$MRS = \underbrace{\left|\frac{dx_2}{dx_1}\right|}_{\substack{\text{absolute value}\\ \text{of the slope of}\\ \text{the indifference curve}}} > \underbrace{\left|\frac{dx_2}{dx_1}\right|}_{\substack{\text{absolute value}\\ \text{of the slope of}\\ \text{the budget line}}} = MOC$$

$\Rightarrow$ increase x_1 (if possible)



MRS versus MOC

$MRS > MOC \Rightarrow$ increase x_1 (if possible)



MRS versus MOC

Alternatively: the household tries to maximize $U\left(x_1, \frac{m}{p_2} - \frac{p_1}{p_2}x_1\right)$.

- Consume 1 additional unit of good 1
  - utility increases by $\frac{\partial U}{\partial x_1}$
  - reduction in x_2 by $MOC = \left|\frac{dx_2}{dx_1}\right| = \frac{p_1}{p_2}$ and hence utility decrease by $\frac{\partial U}{\partial x_2}\left|\frac{dx_2}{dx_1}\right|$ (chain rule
- Thus, increase consumption of good 1 as long as

$$\underbrace{\frac{\partial U}{\partial x_1}}_{\substack{\text{marginal benefit}\\ \text{of increasing } x_1}} > \underbrace{\frac{\partial U}{\partial x_2}\left|\frac{dx_2}{dx_1}\right|}_{\substack{\text{marginal cost}\\ \text{of increasing } x_1}}$$

$$\text{or } MRS = \frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}} > \left|\frac{dx_2}{dx_1}\right| = MOC$$

Household optimum

Cobb-Douglas utility function

$U(x_1, x_2) = x_1^a x_2^{1-a}$ with $0 < a < 1$

The two optimality conditions

- $MRS = \frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}} = \frac{a}{1-a}\frac{x_2}{x_1} \stackrel{!}{=} \frac{p_1}{p_2}$ and
- $p_1 x_1 + p_2 x_2 \stackrel{!}{=} m$

yield the household optimum

$$\begin{aligned} x_1^*(m,p) &= a\frac{m}{p_1}, \\ x_2^*(m,p) &= (1-a)\frac{m}{p_2}. \end{aligned}$$

Household optimum

Perfect substitutes

$U(x_1, x_2) = ax_1 + bx_2$ with $a > 0$ and $b > 0$

An increase of good 1 enhances utility if

$$\frac{a}{b} = MRS > MOC = \frac{p_1}{p_2}$$

holds. Therefore

$$x^*(m,p) = \begin{cases} \left(\frac{m}{p_1}, 0\right), & \frac{a}{b} > \frac{p_1}{p_2} \\ \left\{\left(x_1, \frac{m}{p_2} - \frac{p_1}{p_2}x_1\right) \in \mathbb{R}_+^2 : x_1 \in \left[0, \frac{m}{p_1}\right]\right\} & \frac{a}{b} = \frac{p_1}{p_2} \\ \left(0, \frac{m}{p_2}\right) & \frac{a}{b} < \frac{p_1}{p_2} \end{cases}$$

Household optimum

Concave preferences

$U(x_1, x_2) = x_1^2 + x_2^2$

An increase of good 1 enhances utility if

$$\frac{x_1}{x_2} = \frac{2x_1}{2x_2} = \frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}} = MRS > MOC = \frac{p_1}{p_2}$$

holds. Therefore, corner solution unless prices are equal:

$$x^*(m, p) = \begin{cases} \left(\frac{m}{p_1}, 0\right), & p_1 \leq p_2 \\ \left\{\left(\frac{m}{p_1}, 0\right), \left(0, \frac{m}{p_2}\right)\right\} & p_1 = p_2 \\ \left(0, \frac{m}{p_2}\right) & p_1 \geq p_2 \end{cases}$$

Household optimum and monotonicity

Lemma

Let x^* be a household optimum of $\Delta = \left(B\left(p, m\right), \precsim\right) \Rightarrow$

- *local nonsatiation: $p \cdot x^* = m$ (Walras' law);*
- *strict monotonicity: $p >> 0$;*
- *local nonsatiation and weak monotonicity: $p \geq 0$.*

Proof.

- Assume: $p \cdot x^* < m \Rightarrow$ household can afford bundles close to x^*. Some of them are better than x^* (local nonsatiation). Contradiction!
- Assume $p_g \leq 0 \Rightarrow$ household can be made better off by consuming more of good g (strict monotonicity). Contradiction!
- Assume $p_g < 0 \Rightarrow$ household can "buy" additional units of g without being worse off (weak monotonicity). Household has additional funding for preferred bundles (nonsatiation). Contradiction!

Definition of NTU coalition functions I

- v – coalition function with transferable utility
- V – coalition function without transferable utility

V attributes to every coalition $K \neq \emptyset$ a set of utility vectors

$$u_K := (u_i)_{i \in K} \in \mathbb{R}^{|K|}$$

for K's members.

Problem

Depict

$$\begin{aligned} & V(\{Peter, Otto\}) \\ = \; & \{(u_{Peter}, u_{Otto}) : u_{Peter} \geq 2, u_{Otto} \geq 1, u_{Peter} + u_{Otto} \leq 4\}. \end{aligned}$$

Definition of NTU coalition functions II

Definition (coalition function)

A coalition function V on N for non-transferable utlity associates to every subset K of N a subset of $\mathbb{R}^{|K|}$ such that

- $V(\emptyset) = \emptyset$ and
- $V(K) \neq \emptyset$ for $K \neq \emptyset$

hold.

Problem

Which of the following expressions are formally correct?

- $V(\{1,2\}) = 1,\ V(\{1,2\}) = \{1\},\ V(\{1,2\}) = (1,2)$
- $V(\{1,2\}) = \emptyset,\ V(\{1,2\}) = \{(1,2)\}$
- $V(\{1,2\}) = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq 3, x_2 \leq 4, x_1 + x_2 \leq 5\right\}$

Definition of NTU coalition functions III

Definition (superadditivity)

The coalition function V without transferable utility is called superadditive if, for all coalitions $S, T \subset N$

$$\begin{aligned} S \cap T &= \emptyset \text{ (}S\text{ and }T\text{ are disjunct)}, \\ u_S &\in V(S) \text{ and} \\ u_T &\in V(T) \end{aligned}$$

imply

$$(u_S, u_T) \in V(S \cup T).$$

Definition of NTU coalition functions IV

Problem

Is V_2 defined on $N = \{1,2,3\}$ and given by

$$V_2(K) = \begin{cases} \{i\}, & K = \{i\} \\ \{(x_1, x_2) : x_1 \leq 1, x_2 \leq 4\}, & K = \{1,2\} \\ \{(x_1, x_3) : x_1 \leq 2, x_3 \leq 2\}, & K = \{1,3\} \\ \{(x_2, x_3) : x_2 \leq 4, x_3 \leq 5\}, & K = \{2,3\} \\ \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 \leq 9\} & K = \{1,2,3\} \end{cases}$$

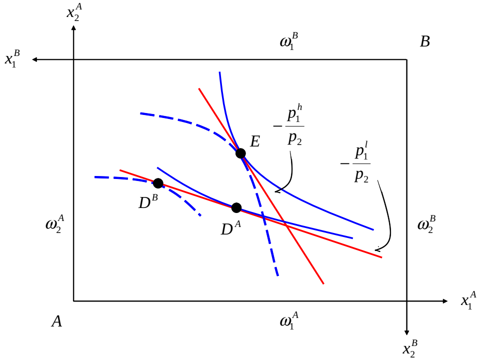
superadditive?

The core

Definition (core)

The core of a NTU game V is the set all utility vectors $u = (u_i)_{i \in N} \in \mathbb{R}^n$ that obey feasibility and non-blockability:

- $u \in V(N)$.
- There is no coaliton K and no utility vector $u' = (u'_i)_{i \in N}$ such that $u'_K \in V(K)$ holds and $u_i \leq u'_i$ for all $i \in K$ with strict inequality for at least on $i \in K$.



The low price p_1^l is not possible in a Walras equilibrium, because there is excess demand for good 1 at this price:

$$x_1^A + x_1^B > \omega_1^A + \omega_1^B.$$

Definition (exchange economy)

An exchange economy is a tuple

$$\mathcal{E} = \left(N, G, \left(\omega^i\right)_{i \in N}, \left(U_i\right)_{i \in N} \right)$$

consisting of

- the set of agents $N = \{1, 2, ..., n\}$,
- the finite set of goods $G = \{1, ..., \ell\}$,

and for every agent $i \in N$

- an endowment $\omega^i = \left(\omega_1^i, ..., \omega_\ell^i\right) \in \mathbb{R}_+^\ell$, and
- a utility function $U_i : \mathbb{R}_+^\ell \to \mathbb{R}$.

Two-agents two-good case —> exchange Edgeworth box

Exchange theory: positive theory

feasible allocations

Definition

Consider an exchange economy $\mathcal{E}$.

- A bundle $(y^i)_{i\in N} \in \mathbb{R}_+^{\ell\cdot n}$ is an allocation.
- An allocation $(y^i)_{i\in N}$ is called K-feasible if $\sum_{i\in K} y^i \leq \sum_{i\in K} \omega^i$ holds.
- An allocation $(y^i)_{i\in N}$ is called feasible if it is N-feasible.

For $K \neq \emptyset$, we let

$$V(K) := \left\{ u_K \in \mathbb{R}^{|K|} : \exists\ K\text{-feasible allocation } x \text{ with } u_i \leq U_i(x_i), i \in K \right\}.$$

non-empty coalition K
—> set of bundles that this coalition possesses
—> every K-feasible allocation defines the maximal utility levels that the players from K can achieve.

Definition

Assume an exchange economy $\mathcal{E}$, a good $g \in G$ and a price vector $p \in \mathbb{R}^\ell$. If every household $i \in N$ has a unique household optimum $x^i\left(p, \omega^i\right)$, good g's excess demand is denoted by $z_g\left(p\right)$ and defined by

$$z_g\left(p\right) := \sum_{i=1}^n x_g^i\left(p, \omega^i\right) - \sum_{i=1}^n \omega_g^i.$$

The corresponding excess demand for all goods $g = 1, ..., \ell$ is the vector

$$z\left(p\right) := \left(z_g\left(p\right)\right)_{g=1,\ldots,\ell}.$$

The value of the excess demand is given by

$$p \cdot z\left(p\right).$$

Lemma (Walras' law)

Every consumer demands a bundle of goods obeying $p \cdot x^i \leq p \cdot \omega^i$ *where local nonsatiation implies equality. For all consumers together, we have*

$$p \cdot z(p) = \sum_{i=1}^{n} p \cdot \left(x^i - \omega^i\right) \leq 0$$

and, assuming local-nonsatiation, $p \cdot z(p) = 0$.

Definition

A market g is called cleared if excess demand $z_g(p)$ on that market is equal to zero.

Problem

Abba (A) and Bertha (B) consider buying two goods 1 *and* 2, *and face the price p for good* 1 *in terms of good* 2. *Think of good* 2 *as the numeraire good with price* 1. *Abba's and Bertha's utility functions,* u_A *and* u_B, *respectively, are given by* $u_A\left(x_1^A, x_2^A\right) = \sqrt{x_1^A} + x_2^A$ *and* $u_B\left(x_1^B, x_2^B\right) = \sqrt{x_1^B} + x_2^B$. *Endowments are* $\omega^A = (18, 0)$ *and* $\omega^B = (0, 10)$. *Find the bundles demanded by these two agents. Then find the price p that fulfills* $\omega_1^A + \omega_1^B = x_1^A + x_1^B$ *and* $\omega_2^A + \omega_2^B = x_2^A + x_2^B$.

Lemma (Market clearance)

In case of local nonsatiation,

1. *if all markets but one are cleared, the last one also clears or its price is zero,*
2. *if at prices $p \gg 0$ all markets but one are cleared, all markets clear.*

Proof.

If $\ell - 1$ markets are cleared, the excess demand on these markets is 0. Without loss of generality, markets $g = 1, ..., \ell - 1$ are cleared. Applying Walras's law we get

$$0 = p \cdot z(p) = p_\ell z_\ell(p).$$

□

Definition

A price vector $\widehat{p}$ and the corresponding demand system $\left(\widehat{x}^i\right)_{i=1,...,n} = \left(x^i\left(\widehat{p}, \omega^i\right)\right)_{i=1,...,n}$ is called a Walras equilibrium if

$$\sum_{i=1}^{n} \widehat{x}^i \leq \sum_{i=1}^{n} \omega^i$$

or

$$z\left(\widehat{p}\right) \leq 0$$

holds.

Definition

A good is called free if its price is equal to zero.

Lemma (free goods)

Assume local nonsatiation and weak monotonicity for all households. If $\left[\widehat{p}, \left(\widehat{x}^i\right)_{i=1,\ldots,n}\right]$ *is a Walras equilibrium and the excess demand for a good is negative, this good must be free.*

Proof.

Assume, to the contrary, that $p_g > 0$ holds. We obtain a contradiction to Walras law for local nonsatiation:

$$p \cdot z(p) = \underbrace{p_g z_g(p)}_{<0} + \sum_{\substack{g'=1,\\ g' \neq g}}^{\ell} p_{g'} z_{g'}(p) \quad (z_g(p) < 0)$$

$$< \sum_{\substack{g'=1,\\ g' \neq g}}^{\ell} \underbrace{p_{g'}}_{\substack{\geq 0 \\ \text{(local nonsatiation and} \\ \text{weak monotonicity)}}} \underbrace{z_{g'}(p)}_{\substack{\leq 0 \\ \text{(definition} \\ \text{Walras equilibrium)}}}$$

$$\leq 0.$$

Definition

A good is desired if the excess demand at price zero is positive.

Lemma (desiredness)

If all goods are desired and if local nonsatiation and weak monotonicity hold and if $\widehat{p}$ *is a Walras equilibrium, then* $z(\widehat{p}) = 0$.

Proof.

Suppose that there is a good g with $z_g(\widehat{p}) < 0$. Then g must be a free good according to the lemma on free goods and have a positive excess demand by the definition of desiredness, $z_g(\widehat{p}) > 0$. □

Parameters a_1 and a_2 and endowments $\omega^1 = (1, 0)$ and $\omega^2 = (0, 1)$

- Agent 1's demand for good 1:

$$x_1^1\left(p_1, p_2, \omega^1 \cdot p\right) = a_1 \frac{\omega^1 \cdot p}{p_1} = a_1.$$

- Agent 2's demand for good 1:

$$x_1^2\left(p_1, p_2, \omega^2 \cdot p\right) = a_2 \frac{\omega^2 \cdot p}{p_1} = a_2 \frac{p_2}{p_1}.$$

Parameters a_1 and a_2 and endowments $\omega^1 = (1, 0)$ and $\omega^2 = (0, 1)$

- Agent 1's demand for good 1:

$$x_1^1 \left(p_1, p_2, \omega^1 \cdot p\right) = a_1 \frac{\omega^1 \cdot p}{p_1} = a_1.$$

- Agent 2's demand for good 1:

$$x_1^2 \left(p_1, p_2, \omega^2 \cdot p\right) = a_2 \frac{\omega^2 \cdot p}{p_1} = a_2 \frac{p_2}{p_1}.$$

- Market 1 is cleared if

$$a_1 + a_2 \frac{p_2}{p_1} = 1 \text{ or } \frac{p_2}{p_1} = \frac{1 - a_1}{a_2}$$

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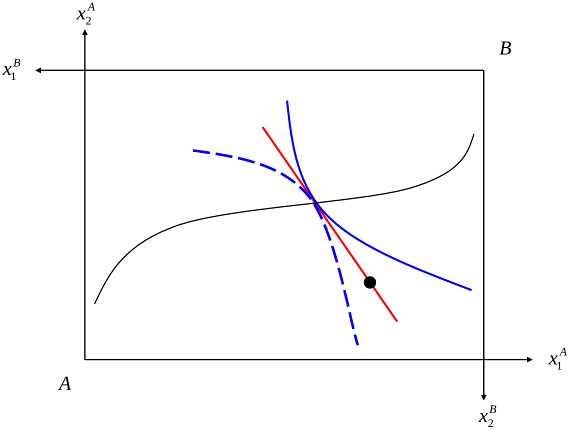
- How about the market for good 2?

Exchange theory: positive theory

Example: The Cobb-Douglas Exchange Economy with Two Agents



Theorem (Existence of the Walras Equilibrium)

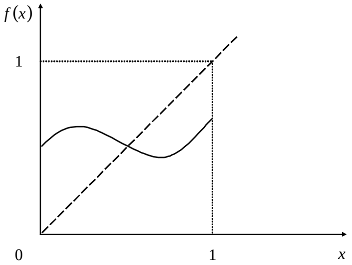
If aggregate excess demand is a continuous function (in prices), if the value of the excess demand is zero and if the preferences are strictly monotonic, there exists a price vector $\widehat{p}$ *such that* $z(\widehat{p}) \leq 0$.

Theorem

Suppose $f : M \to M$ *is a function on the nonempty, compact and convex set* $M \subseteq \mathbb{R}^\ell$. *If* f *is continuous, there exists* $x \in M$ *such that* $f(x) = x$. x *is called a fixed point.*

Continuous function on the unit interval.

- $f(0) = 0$ or $f(1) = 1$
  —> fixed point is found
- $f(0) > 0$ and $f(1) < 1$
  —> the graph cuts the 45°-line
  —> fixed point is found



Real-life examples:

- rumpling a handkerchief
- stirring cake dough

Problem

Assume, one of the requirements for the fixed-point theorem does not hold. Show, by a counter example, that there can be a function such that there is no fixed point. Specifically, assume that

a) M is not compact

b) M is not convex

c) f is not continuous.

Hans-Jürgen Podszuweit (found in Homo Oeconomicus, XIV (1997), p. 537):

Das Nilpferd hört perplex:
Sein Bauch, der sei konvex.
Und steht es vor uns nackt,
sieht man: Er ist kompakt.
Nimmt man 'ne stetige Funktion
von Bauch
in Bauch
– Sie ahnen schon –,
dann nämlich folgt aus dem
Brouwer'schen Theorem:
Ein Fixpunkt muß da sein.
Dasselbe gilt beim Schwein
q.e.d.

- Constructing a convex and compact set:
  Norm prices of the ℓ goods such that the sum of the nonnegative (!, we have strict monotonicity) prices equals 1. We can restrict our search for equilibrium prices to the $\ell - 1$- dimensional unit simplex:

$$S^{\ell-1} = \left\{ p \in \mathbb{R}_+^\ell : \sum_{g=1}^{\ell} p_g = 1 \right\}.$$

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- $S^{\ell-1}$is nonempty, compact (closed and bounded as a subset of $\mathbb{R}^{\ell-1}$) and convex.
- problem: Draw $S^1 = S^{2-1}$.

- The idea of the proof: First, we define a continuous function f on this (nonempty, compact and convex) set. Brouwer's theorem says that there is at least one fixed point of this function. Second, we show that such a fixed point fulfills the condition of the Walras equilibrium.

- The idea of the proof: First, we define a continuous function f on this (nonempty, compact and convex) set. Brouwer's theorem says that there is at least one fixed point of this function. Second, we show that such a fixed point fulfills the condition of the Walras equilibrium.
- The abovementioned continuous function

$$f = \begin{pmatrix} f_1 \\ f_2 \\ . \\ . \\ . \\ f_\ell \end{pmatrix} : S^{\ell-1} \to S^{\ell-1}$$

is defined by

$$f_g(p) = \frac{p_g + \max(0, z_g(p))}{1 + \sum_{g'=1}^{\ell} \max(0, z_{g'}(p))}, g = 1, ..., \ell$$

- f is continuous because every f_g, $g = 1, ..., \ell$, is continuous. The latter is continuous because z (according to our assumption) und max are continuous functions. Finally, we can confirm that f is well defined, i.e., that $f(p)$ lies in $S^{\ell-1}$ for all p from $S^{\ell-1}$:

$$\begin{aligned}
\sum_{g=1}^{\ell} f_g(p) &= \sum_{g=1}^{\ell} \frac{p_g + \max(0, z_g(p))}{1 + \sum_{g'=1}^{\ell} \max(0, z_{g'}(p))} \\
&= \frac{1}{1 + \sum_{g'=1}^{\ell} \max(0, z_{g'}(p))} \sum_{g=1}^{\ell} (p_g + \max(0, z_g(p))) \\
&= \frac{1}{1 + \sum_{g'=1}^{\ell} \max(0, z_{g'}(p))} \left(1 + \sum_{g=1}^{\ell} \max(0, z_g(p))\right) \\
&= 1.
\end{aligned}$$

- The function f increases the price of a good g in case of $f_g(p) > p_g$, only, i.e. if
$$\frac{p_g + \max(0, z_g(p))}{1 + \sum_{g'=1}^{\ell} \max(0, z_{g'}(p))} > p_g$$
or
$$\frac{\max(0, z_g(p))}{\sum_{g'=1}^{\ell} \max(0, z_{g'}(p))} > \frac{p_g}{\sum_{g'=1}^{\ell} p_{g'}}$$
holds.

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holds.

- Interpretation: Increase price if its relative excess demand is greater than its relative price.

—> f = Walras auctioneer

—> tâtonnement

- We now complete the proof: according to Brouwer's fixed-point theorem there is one $\widehat{p}$ such that

$$\widehat{p} = f\left(\widehat{p}\right),$$

from which we have

$$\widehat{p}_g = \frac{\widehat{p}_g + \max\left(0, z_g\left(\widehat{p}\right)\right)}{1 + \sum_{g'=1}^{\ell} \max\left(0, z_{g'}\left(\widehat{p}\right)\right)}$$

and finally

$$\widehat{p}_g \sum_{g'=1}^{\ell} \max\left(0, z_{g'}\left(\widehat{p}\right)\right) = \max\left(0, z_g\left(\widehat{p}\right)\right)$$

for all $g = 1, ..., \ell$.

- Next we multiply both sides for all goods $g = 1, ..., \ell$ by $z_g(\widehat{p})$:

$$z_g(\widehat{p})\widehat{p}_g \sum_{g'=1}^{\ell} \max(0, z_{g'}(\widehat{p})) = z_g(\widehat{p}) \max(0, z_g(\widehat{p}))$$

and summing up over all g yields

$$\sum_{g=1}^{\ell} z_g(\widehat{p})\widehat{p}_g \sum_{g'=1}^{\ell} \max(0, z_{g'}(\widehat{p})) = \sum_{g=1}^{\ell} z_g(\widehat{p}) \max(0, z_g(\widehat{p})) .$$

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- By Walras' law, the left-hand expression is equal to zero. The right one consists of a sum of expressions, which are equal either to zero or to $\left(z_g(\widehat{p})\right)^2$. Therefore, $z_g(\widehat{p}) \leq 0$ for all $g = 1, ..., \ell$. This is what we wanted to show.

Definition (blockable allocation, core)

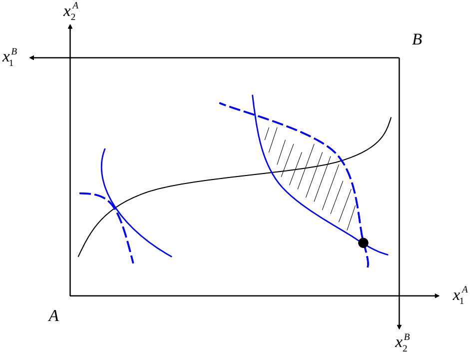
Let $\mathcal{E} = \left(N, G, \left(\omega^i\right)_{i \in N}, \left(U_i\right)_{i \in N}\right)$ be an exchange economy. A coalition $S \subseteq N$ is said to block an allocation $\left(y^i\right)_{i \in N}$, if an allocation $\left(z^i\right)_{i \in N}$ exists such that

- $U_i\left(z^i\right) \geq U_i\left(y^i\right)$ for all $i \in S$, $U_i\left(z^i\right) > U_i\left(y^i\right)$ for some $i \in S$ and
- $\sum_{i \in S} z^i \leq \sum_{i \in S} \omega^i$

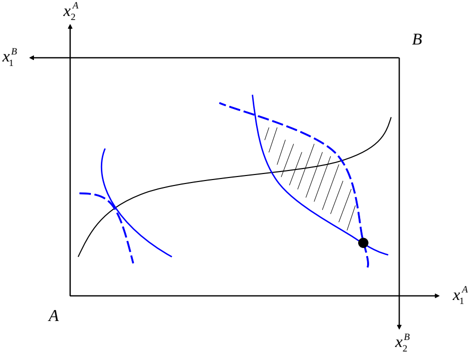
hold.

An allocation is not blockable if there is no coalition can block it. The set of all feasible and non-blockable allocations is called the core of an exchange economy.

- Core in the Edgeworth box: Every household (considered a one-man coalition) blocks any allocation that lies below the indifference curve cutting his endowment point.



- Core in the Edgeworth box: Every household (considered a one-man coalition) blocks any allocation that lies below the indifference curve cutting his endowment point.
- Therefore, the core is contained inside the exchange lense.



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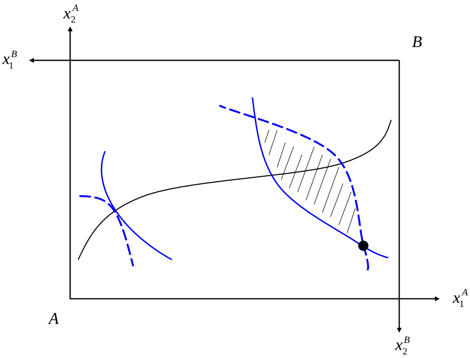
- Both households together block any allocation that is not Pareto efficient.



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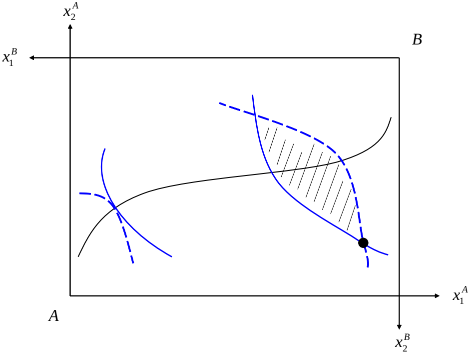
- Therefore, the core is contained inside the exchange lense.
- Both households together block any allocation that is not Pareto efficient.
- Thus, the core is the intersection of the exchange lense and the contract curve.



Theorem

Assume an exchange economy $\mathcal{E}$ with local non-satiation and weak monotonicity. Every Walras allocation lies in the core.

- Consider a Walras allocation $\left(\widehat{x}^i\right)_{i\in N}$. A lemma from above implies

$$\widehat{p} \overset{(1)}{>>} 0$$

where $\widehat{p}$ is the equilibrium price vector.

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- Assume, now, that $\left(\widehat{x}^i\right)_{i\in N}$ does not lie in the core. Since a Walras allocation is feasible, there exists a coalition $S \subseteq N$ that can block $\left(\widehat{x}^i\right)_{i\in N}$. I.e., there is an allocation $\left(z^i\right)_{i\in N}$ such that

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  - $\sum_{i\in S} z^i \leq \sum_{i\in S} \omega^i$.

- The second point, together with (1), leads to the implication

$$\widehat{p} \cdot \left(\sum_{i \in S} z^i - \sum_{i \in S} \omega^i \right) \leq 0.$$

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- The first point implies

$$\widehat{p}\cdot z^i \overset{(2)}{\geq} \widehat{p}\cdot\widehat{x}^i = \widehat{p}\cdot\omega^i \text{ for all } i\in S \text{ (by local nonsatiation) and}$$
$$\widehat{p}\cdot z^j \overset{(3)}{>} \widehat{p}\cdot\widehat{x}^j = \widehat{p}\cdot\omega^j \text{ for some } j\in S \text{ (otherwise, } \widehat{x}^j \text{ is not an optim}$$

- Summing over all these households from S yields

$$\begin{aligned}\widehat{p}\cdot\sum_{i\in S} z^i &= \sum_{i\in S}\widehat{p}\cdot z^i \text{ (distributivity)}\\ &> \sum_{i\in S}\widehat{p}\cdot\omega^i \text{ (above inequalities (2) and (3))}\\ &= \widehat{p}\cdot\sum_{i\in S}\omega^i \text{ (distributivity).}\end{aligned}$$

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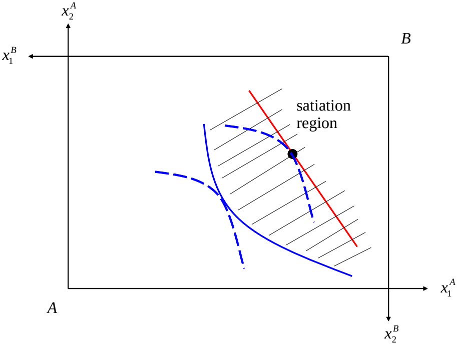
- This inequality can be rewritten as

$$\widehat{p}\cdot\left(\sum_{i\in S}z^{i}-\sum_{i\in S}\omega^{i}\right)>0,$$

contradicting the inequality noted above.

- Example where a Walras allocation does not lie in the core: The lower-left agent's preferences violate non-satiation.



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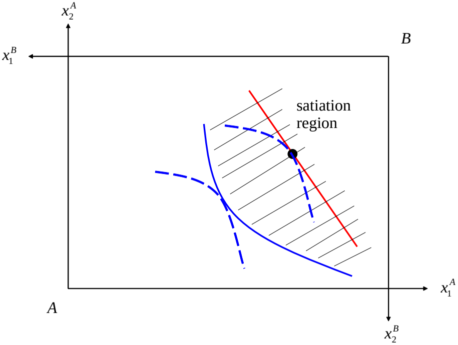
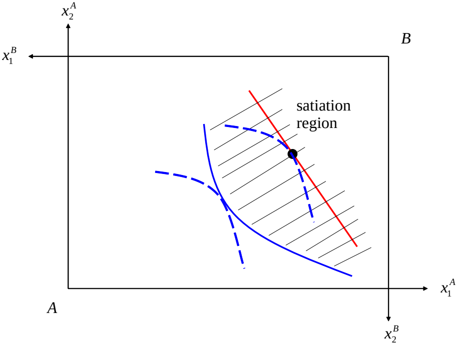
- The equilibrium point is the point of tangency between that price line and the upper-right agent's indifference curve.



- Example where a Walras allocation does not lie in the core: The lower-left agent's preferences violate non-satiation.
- The equilibrium point is the point of tangency between that price line and the upper-right agent's indifference curve.
- This point is not Pareto-efficient. The lower-left agent could forego some units of both goods without harming himself.



Matching of

- employers and employees
- students and internships or
- men and women:

$$M = \{m_1, ..., m_k\}, W = \{w_1, ..., w_n\}$$

with utility functions

$$U_m : W \cup \{m\} \to \mathbb{R}$$

Problem

What does $U_{w_1}(m_1) > U_{w_1}(w_1) > U_{w_1}(m_2)$ mean?

Assumption: all the preferences are strict

Definition (marriage market)

A marriage market $(M, W, \mathbf{U})$ consists of disjunct sets of individuals M and W and utility functions $\mathbf{U} = (U_i)_{i \in M \cup W}$ with domain $W \cup \{m\}$ for every $m \in M$ and domain $M \cup \{w\}$ for every $w \in W$.

- the players themselves are the object of preferences, hence
- emotionality reigning in this market

The marriage market

allocations

Definition (allocation)

For a marriage market $(M, W, \mathbf{U})$, the function

$$\mu : M \cup W \rightarrow M \cup W$$

is called an allocation if the two requirements

- $\mu(m) \in \{m\} \cup W$ for all $m \in M$ and
- $\mu(w) \in \{w\} \cup M$ for all $w \in W$

are fulfilled.

Thus, men can be singles or attached to a woman
—> Adam and Eve, not Adam and Steve.

Problem

Which players are characterized by $\mu(\mu(i)) = i$?

Definition (consistent allocation)

For a marriage market $(M, W, \mathbf{U})$, an allocation μ is called consistent if $\mu(\mu(i)) = i$ holds for all $i \in M \cup W$.

Single individuals i are defined by $\mu(i) = i$ and fulfull the consistency condition by

$$\mu(\mu(i)) = \mu(i) = i.$$

Assume a feasible allocation μ and a man $m \in M$ who is not single. By $\mu(m) \in \{m\} \cup W$, he is attached to a women $w \in W$ ($\mu(m) = w$). Consistency then implies

$$m = \mu(\mu(m)) = \mu(w)$$

so that the woman w is attached to the very same man – a marriage relation.

The marriage market

feasible allocations and the NTU game

Definition

Consider a consistent allocation μ in a marriage market $(M, W, \mathbf{U})$. μ is called K-feasible if $\mu(K) \subseteq K$ holds.

K-feasibility means that every individual from K is single or has a marriage partner in K. Similarly, a blocking coalition in an exchange economy can only redistribute goods this coalition possesses. Every consistent allocation μ is $M \cup W$-feasible.

Very similar to the exchange economy, we can define the associated NTU coalition function V by

$$V(K) := \left\{ u_K \in \mathbb{R}^{|K|} : \exists \text{ feasible allocation } \mu \text{ with } u_i \leq U_i(\mu(i)), i \in K \right\}.$$

Definition (acceptability)

An agent i finds another individual j acceptable if $U_i(j) > U_i(i)$ holds.

—> nobody can be married against his (or her) will

However, if

- I fancy Sandra Bullock but
- she prefers another man,

the underlying allocation may well be stable.

The core

from individual rationality to the core I

Definition (from individual rationality to the core)

Let μ be a consistent (or $M \cup W$-feasible) allocation.

- μ is called individually rational if $U_i(\mu(i)) \geq U_i(i)$ holds for all $i \in N$ (non-blockability by one-man coalitions).
- μ is called pairwise rational if there is no pair of players $(m, w) \in M \times W$ such that

$$\begin{aligned} U_m(w) &> U_m(\mu(m)) \text{ and} \\ U_w(m) &> U_w(\mu(w)) \end{aligned}$$

hold (non-blockability by heterosexual pairs).

Definition

- μ is called Pareto optimal if there is no consistent allocation μ' that fulfills

$$\begin{array}{rcll} U_i\left(\mu'\left(i\right)\right) & \geq & U_i\left(\mu\left(i\right)\right) & \text{for all } i \in M \cup W \text{ and} \\ U_j\left(\mu'\left(j\right)\right) & > & U_j\left(\mu\left(j\right)\right) & \text{for at least one } j \in M \cup W \end{array}$$

  (non-blockability by the grand coalition).

- μ lies in the core if there is not coalition $K \subseteq M \cup W$ and no K-feasible allocation μ' such that

$$\begin{array}{rcll} U_i\left(\mu'\left(i\right)\right) & \geq & U_i\left(\mu\left(i\right)\right) & \text{for all } i \in K \text{ and} \\ U_j\left(\mu'\left(j\right)\right) & > & U_j\left(\mu\left(j\right)\right) & \text{for at least one } j \in K \end{array}$$

  holds (non-blockability by any coalition).

The core

comments

Pairwise rationality: no man and no woman exist such that both can improve their lot by marrying

- breaking off existing marriages or
- giving up celibacy

Pareto optimality is defined with reference to feasibility and non-blockability by the grand coalition.
K-feasibility —> $\mu'(K) \subseteq K$

- individual rationality: every $\{i\}$-feasible allocation μ' obeys $\mu'(i) = i$
- pairwise rationality: the blocking coalition $\{m, w\}$ forms a pair.

Problem

What is the connection between individual rationality and acceptability?

Theorem

Let $(M, W, \mathbf{U})$ be a marriage market. The set of consistent allocations that are individually rational and pairwise rational is the core.

First part of the proof: A consistent allocation that is both individually and pairwise rational belongs to the core.

Assume a consistent allocation μ outside the core. Thus, there exists a coalition K that can block μ by suggesting a K-feasible allocation μ' that fulfills

$$\begin{aligned} U_i\left(\mu'(i)\right) &\geq U_i\left(\mu(i)\right) \text{ for all } i \in K \text{ and} \\ U_j\left(\mu'(j)\right) &> U_j\left(\mu(j)\right) \text{ for at least one } j \in K. \end{aligned}$$

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Let us focus on individual j that is strictly better off under μ' than under μ. We can distinguish two cases:

- j is single or married under μ and (re)marries under μ'.
  In this case both j and his (or her) spouse $\mu'(j) \in K$ (!) are strictly better off because we work with strict preferences. Then μ is not pairwise rational.
- j is married under μ and single under μ'.
  This second case implies that j is better off as a single contadicting individual rationality.

Further exercises

Sketch budget lines or the displacements of budget lines for the following examples:

- Time $T = 18$ and money $m = 50$ for football F (good 1) or basket ball B (good 2) with prices
  - $p_F = 5$, $p_B = 10$ in monetary terms,
  - $t_F = 3$, $t_B = 2$ and temporary terms
- Two goods, bread (good 1) and other goods (good 2). Transfer in kind with and without probhibition to sell:
  - $m = 300$, $p_B = 2$, $p_{\text{other}} = 1$
  - Transfer in kind: $B = 50$