Towards an evolutionary cooperative game theory

Andre Casajus and Harald Wiese

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Two game theories

- Noncooperative game theory
  = strategy-oriented game theory
- Strategies, payoff functions
- Nash equilibrium

- Cooperative game theory
  = payoff-oriented game theory
- Coalition functions
- Core, Shapley value

Two game theories

but only one evolutionary game theory

By evolutionary game theory, we normally understand **evolutionary noncooperative game theory**:

- Players are drawn at random from a large population.
- They are programmed to play a certain (mixed) strategy and
- the strategy that does better than other strategies grows faster

- i.e., more players use the successful strategies.

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- How about an **evolutionary cooperative game theory**?
- John Nash received a grant from the NSF to develop a new 'evolutionary' solution concept for cooperative games.

Evolutionary cooperative game theory

overview I

Idea: Agents are programmed to assume a certain player role.

- Agents' payoff —> fitness —> proliferation
- $(s_1(0), ..., s_n(0))$ —> payoffs —> $(s_1(1), ..., s_n(1))$ —>

Problem: $s_i(t)$ will not be natural numbers
Solution:

- extended coalition function defined for coalitions $(s_1(t), ..., s_n(t))$
- we use the Lovasz extension v^ℓ $(u_T^\ell(s) = \min_{i \in T} s_i)$

Problem: extensions cannot be an input for the (standard) Shapley value.
Solution:

- continuous Shapley value introduced by Aumann and Shapley (1974)
- which uses derivatives

Evolutionary cooperative game theory

overview II

Problem: Lovasz extensions v^ℓ are not differentiable
Solution:

- approximation with differentiable functions
- leading to continuous Shapley payoffs and
- their limits that feed into
- a replicator dynamic = differential equation which is

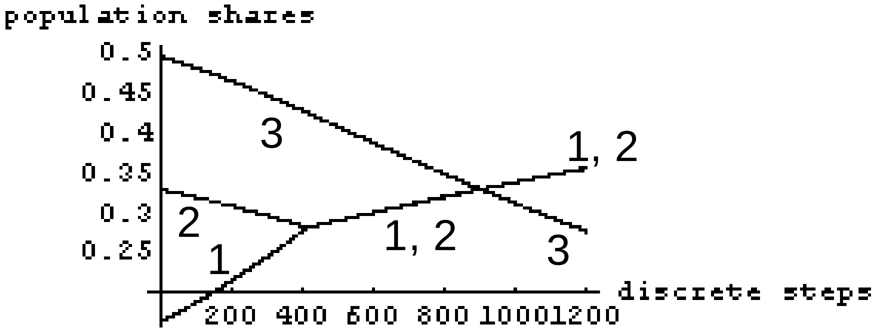
Problem: not solvable by standard means
Solution:

- consider discrete version of replicator dynamic,
- increase number of steps and decrease step length and
- let go towards infinity and zero, respectively.

Results

unanimity game for three players: the two productive players 1 and 2 win

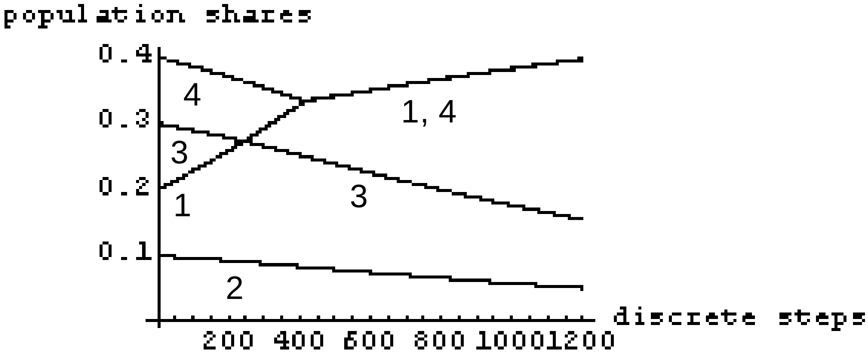
2 time periods with step length $\frac{1}{600}$
$(x_1(0), x_2(0), x_3(0)) = \left(\frac{1}{6}, \frac{2}{6}, \frac{1}{2}\right) \rightarrow \left(\frac{1}{2}, \frac{1}{2}, 0\right)$



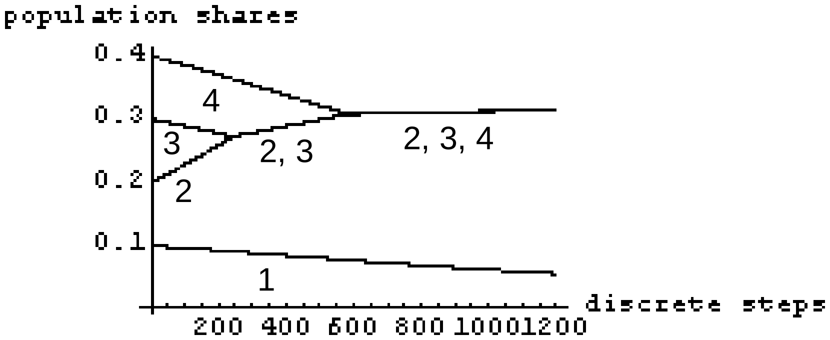
Results

apex game: the apex player teams up with player 4

2 time periods with step length $\frac{1}{600}$
$(x_1(0), x_2(0), x_3(0), x_4(0)) = \left(\frac{2}{10}, \frac{1}{10}, \frac{3}{10}, \frac{4}{10}\right) \to \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right)$



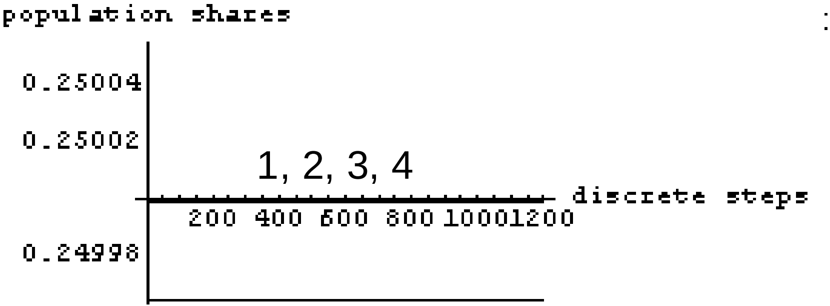
2 time periods with step length $\frac{1}{600}$
$(x_1(0), x_2(0), x_3(0), x_4(0)) = \left(\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}\right) \longrightarrow \left(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$



Results

apex game with identical starting shares

2 time periods with step length $\frac{1}{600}$
$(x_1(0), x_2(0), x_3(0), x_4(0)) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) \rightarrow \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$



And now the mathematical details

- Coalition functions as vectors
- Measuring agents
- The Lovasz extension and its approximation
- The continuous Shapley value
- The replicator dynamics

Payoff vectors

A payoff vector x for N is an element of $\mathbb{R}^N$ or a function $N \to \mathbb{R}$. We define

- $\mathbb{R}^N_+ := \left\{x \in \mathbb{R}^N : x_i \geq 0 \text{ for all } i \in N\right\}$,
- $\mathbb{R}^N_{++} := \left\{x \in \mathbb{R}^N : x_i > 0 \text{ for all } i \in N\right\}$,
- $\Delta := \Delta(N) := \left\{x \in \mathbb{R}^N_+ : \sum x_i = 1\right\}$ and
- $int(\Delta) := int(\Delta(N)) = \left\{x \in \mathbb{R}^N_{++} : \sum x_i = 1\right\}$.

Agents and measures

intervals of agents

- $s = (s_1, \ldots, s_n) \in \mathbb{R}_+^N$
- In case of $s \in \{0,1\}^N$, we identify s with the coalition

$$\mathbf{K}(s) := \{i \in N : s_i = 1\}.$$

- λ = Lebesgues-Borel measure on $\mathbb{R}$
- Choose n non-intersecting intervals $I_i \subseteq \mathbb{R}$ with $\lambda(I_i) = s_i$
- $I := \cup_{i \in N} I_i$ = set of all agents
- $\mathcal{B}$ = set of Borel sets of I
- μ_i^s defined by

$$\mu_i^s(K) := \lambda(K \cap I_i), K \in \mathcal{B},$$

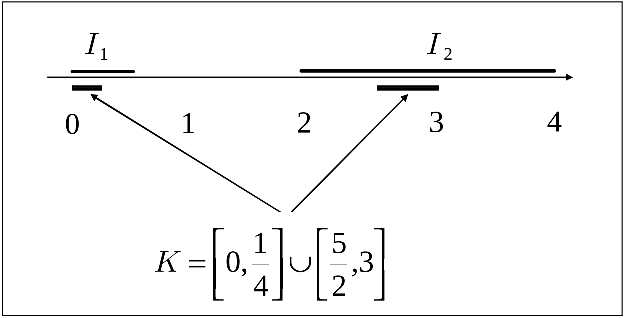
  is a measure on $(I, \mathcal{B})$.
- $\mu^s = \prod_{i \in N} \mu_i^s : \mathcal{B} \to \mathbb{R}^N$, $K \mapsto (\mu_i^s(K))_{i \in N}$ = Cartesian product of these measures
- $\mu^s(K)$ distributes the agents in K among the n players (player types) and attributes a measure to each player.

Agents and measures

an example

- $N = \{1, 2\}$
- $s = \left(\frac{1}{2}, 2\right)$ and intervals $I_1 = \left[0, \frac{1}{2}\right]$ and $I_2 = [2, 4]$.
- For $K := \left[0, \frac{1}{4}\right] \cup \left[\frac{5}{2}, 3\right] \in \mathcal{B}$ we obtain

$$\mu_1^s(K) = \lambda(K \cap I_1) = \lambda\left(\left[0, \frac{1}{4}\right]\right) = \frac{1}{4} \text{ and similarly for } \mu_2^s$$



- Let $s_- := \min_T(s) := \min_{i \in T} s_i$ be the minimum player size of the T-players and
- let $T_- := \{j \in N | s_j = s_-\}$ be the set of T-players with minimal size.

For $\emptyset \neq T \subseteq N$ and $m \in \mathbb{N}$, we define $u_T^{\ell,m} = \min_T^m : \mathbb{R}_+^N \to \mathbb{R}$ by

$$\min_T^m(s) := \begin{cases} 0, & s_- = 0 \\ \frac{|T|^{\frac{1}{m}}}{\left(\sum_{i \in T} \frac{1}{s_i^m}\right)^{\frac{1}{m}}}, & \text{else.} \end{cases}$$

$s \in \mathbb{R}_+^N$, and have

$$\lim_{m \to \infty} \min_T^m(s) = \min_T(s)$$

Define $v^{\ell,m}$ by

$$v^{\ell,m}(s) := \sum_{\emptyset \neq T \subseteq N} m_v(T) \cdot \min_T^m(s), \qquad m \in \mathbb{N}$$

Vector measure games

In our setting, vector measures games are given by

$$\begin{aligned} v^{\ell,s} &: = v^{\ell} \circ \mu^{s} : \mathcal{B} \rightarrow \mathbb{R} \text{ and} \\ v^{\ell,m,s} &: = v^{\ell,m} \circ \mu^{s} : \mathcal{B} \rightarrow \mathbb{R} \end{aligned}$$

Given a coalition $K \in \mathcal{B}$,

- $\mu^{s}(K)$ specifies how to devide K among the n groups and how to measure these subgroups.
- v^{ℓ} or $v^{\ell,m}$ then yield the worth in accordance with the underlying TU game v.

Shapley value for vector measure games

Aumann/Shapley 1974 (Theorem B) = diagonal formula:
Let $K \in \mathcal{B}$ be any continuous coalition of agents. They receive

$$\mathrm{Sh}\, v^{\ell,m,s}(K) = \sum_{j=1}^{n} \mu_j^s(K) \int_0^1 \left. \frac{\partial v^{\ell,m}}{\partial s_j} \right|_{\tau s} d\tau$$

- Aanalogue of player j's marginal contribution in the discrete Shapley formula
  = derivative of the coalition's worth with respect to the measure of agents of player j.
- Evaluation at $\tau s = (\tau s_1, ..., \tau s_n)$ (diagonal formula):
  - Draw a subset of agents by chance.
  - More likely than not, the composition in this subset (how many agents of player 1, player 2 etc.) will not deviate much from the composition in the overall population.

Shapley value for the agents of player i

Lemma

We have

$$\mathrm{Sh}\, u_T^{\ell,m,s}(I_i) = \begin{cases} 0, & i \notin T \\ 0, & s_- = 0 \\ |T|^{\frac{1}{m}} s_i^{-m} \left(\sum_{j \in T} s_j^{-m}\right)^{-\frac{m+1}{m}}, & i \in T \text{ and } s_- \neq 0 \end{cases}$$

and

$$\mathrm{Sh}\, u_T^{\ell,s}(I_i) := \lim_{m \to \infty} \mathrm{Sh}\, u_T^{\ell,m,s}(I_i) = \begin{cases} \frac{s_-}{|T_-|}, & i \in T_-, s_- \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

Definition

The averge payoff accruing to agents from I_i is also called agent i's payoff and is given by $\mathrm{Sh}_i\left(v^{\ell,s}\right) := \frac{\mathrm{Sh}\, v^{\ell,s}(I_i)}{s_i}$.

Shapley value for the agents of player i

example apex game

$$\left(\text{Sh}_1\left(h^{\ell,s}\right),\ \text{Sh}_2\left(h^{\ell,s}\right),\ \text{Sh}_3\left(h^{\ell,s}\right),\ \text{Sh}_4\left(h^{\ell,s}\right)\right)$$

$$= \begin{cases} (0,1,0,0), & s_1 < s_2 < s_3 < s_4 \\ \left(0,\frac{1}{2},\frac{1}{2},0\right) & s_1 < s_2 = s_3 < s_4 \\ \left(0,\frac{1}{3},\frac{1}{3},\frac{1}{3}\right) & s_1 < s_2 = s_3 = s_4 \\ \left(\frac{1}{2},\frac{1}{2},0,0\right) & s_1 = s_2 < s_3 < s_4 \\ \left(\frac{2}{3},\frac{1}{6},\frac{1}{6},0\right) & s_1 = s_2 = s_3 < s_4 \\ \ldots & \\ (0,0,0,1) & s_2 < s_3 < s_4 < s_1 \\ \left(0,0,\frac{1}{2},\frac{1}{2}\right) & s_2 < s_3 = s_4 < s_1 \\ \left(0,\frac{1}{3},\frac{1}{3},\frac{1}{3}\right) & s_2 = s_3 = s_4 < s_1 \\ (0,0,0,1) & s_2 = s_3 < s_4 < s_1 \end{cases}$$

Replicator dynamics

- the agents' Shapley payoffs = fitness
- a constant birthrate β
- a constant death rate δ

—> evolution of s_i is defined by

$$\dot{s}_i = \left[\beta + \mathrm{Sh}_i\left(v^{\ell,s}\right) - \delta\right] s_i.$$

In terms of population shares

$$x_i := \frac{s_i}{\sum_{j=1}^{n} s_j}$$

we obtain the replicator dynamics

$$\dot{x}_i = \left(\mathrm{Sh}_i\left(v^{\ell,s}\right) - \sum_{j=1}^{n} \mathrm{Sh}_j\left(v^{\ell,s}\right) x_j\right) x_i$$

Replicator dynamics

from discrete to continuous I

Existence not guaranteed by standard methods. Therefore:

$$x_i(t) = x_i(t-1) + x_i(t-1)\left[\mathrm{Sh}_i\left(v^{\ell,x(t-1)}\right) - \sum_{j=1}^{n}\mathrm{Sh}_j\left(v^{\ell,x(t-1)}\right)x_j\right]$$

In order to smooth out the solution orbit, we introduce a (very small) step length $\sigma > 0$ and work with the replicator dynamics

$$x_i(t) = x_i(t-1) + x_i(t-1)\,\sigma\left[\mathrm{Sh}_i\left(v^{\ell,x(t-1)}\right) - \sum_{j=1}^{n}\mathrm{Sh}_j\left(v^{\ell,x(t-1)}\right)x_j\right],$$

In a continuous case, σ would affect the velocity of change but not the solution orbit.

We use the formula

number of time periods = number of steps times step length

Definition

The Euler replicator dynamic for T time periods is defined by the discrete replicator dynamics obeying $0 \leq t \leq S$, $\sigma = \frac{T}{S}$ and $S \to \infty$.

Steady states, asymptotic stability

Definition

A vector of population shares $\hat{x} = (\hat{x}_1, ..., \hat{x}_n) \in \Delta$ is a steady state if there exists a population share vector $x(0) = (x_1(0), ..., x_n(0)) \in \Delta$ such that the Euler replicator dynamics yields

$$\lim_{T \to \infty} x_i(t) = \hat{x}_i$$

for all $i = 1, ..., n$.

Definition

A steady state $\hat{x} = (\hat{x}_1, ..., \hat{x}_n)$ is called asymptotically stable if there exists some $\varepsilon > 0$ such that for all population vectors $x(0)$ obeying $\|x(0) - \hat{x}\|_2 < \varepsilon$ we have

$$\lim_{T \to \infty} x(t) = \hat{x}.$$

Definition

Player $i \in N$ strictly dominates player $j \in N$ if $v(K \cup \{i\}) > v(K \cup \{j\})$ holds for all $K \subseteq N \setminus \{i, j\}$.

Example: $N = \{1, 2\}$, $v(1) = 1$, $v(2) = 0$ and $v(1, 2) = 3$:

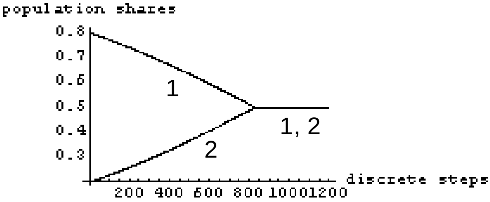


Figure: Player 2 is dominated but does not vanish.

Simple games

Definition

- A game $v \in V(N)$ is called simple if it is
  - monotonic,
  - obeys $v(K) = 0$ or $v(K) = 1$ for every coalition $K \subseteq N$ and
  - $v(N) = 1$.
- Set of minimal winning coalitions $\mathbb{M}$
- Examples: Unanimity games, apex games, contradictory games (where there is a coalition K such that both K and $N \backslash K$ are winning coalitions)

Simple games

Lovasz extension and derived simple game

- For $s \in \mathbb{R}_+^N$, we have

$$v^\ell(s) = \max_{K \in \mathbb{M}} \min_{i \in K} s_i.$$

  where the arguments are

  - $\mathbb{M}^{\max\min} \subseteq \mathbb{M}$ and
  - $N^{\max\min} \subseteq N$.

- Given $\mathbb{M}$ and s, we define the derived simple game $v(\mathbb{M}, s)$
  - on the player set $N^{\max\min}$
  - by specifying:
    $W \subseteq N^{\max\min}$ is a winning coalition if there exists a coalition $K \in \mathbb{M}^{\max\min}$ s.t. $W = K \cap N^{\max\min}$

Simple games

An agent's Shapley payoff = his player's Shapley payoff in derived game

- $v \in V(N)$ —> $v(\mathbb{M}, s)$ derived simple game
- The agents' Shapley values for players $i \in N$ are given by

$$\mathrm{Sh}_i\left(v^{\ell,s}\right) = \begin{cases} \mathrm{Sh}_i\left(v\left(\mathbb{M}, s\right)\right), & i \in N^{\max\min} \\ 0, & \text{otherwise} \end{cases},$$

- Thus, in a simple game, a player obtains a non-zero payoff zero if and only if
  - he belongs to minimal winning coalition,
  - his size is minimal within at least one minimal winning coalition, and
  - this minimal size is at least as large as the minimal sizes found in any other winning coalition.

- $v \in V(N)$ simple game with $\mathbb{M}$.
- Asymptotically stable states $\hat{x} = (\hat{x}_1, ..., \hat{x}_n)$ are characterized by minimal wining coalitions $W \in \mathbb{M}$ and

$$\hat{x}_i = \begin{cases} \frac{1}{|W|}, & i \in W \\ 0, & \text{otherwise} \end{cases}$$

- Example: apex game

ENGT's basic model:

- pairwise contests
- monomorphic population playing a symmetric game
- selection of equilibrium strategies

ECGT' basic model (as presented here):

- playing the field and
- polymorphic
- selection of players and coalitions

In our model, the agents' shares change.
Alternatively, players themselves could grow:

- Depending on their profits, firms grow in an organic fashion (rather than grow by mergers and acquisitions).
- Filar and Petrosjan (2000) present dynamic cooperative games where they define a sequence of games (in discrete or in continuous time) so that one TU game is determined
  - by the previous one and
  - by the payoffs achieved under some solution concept.

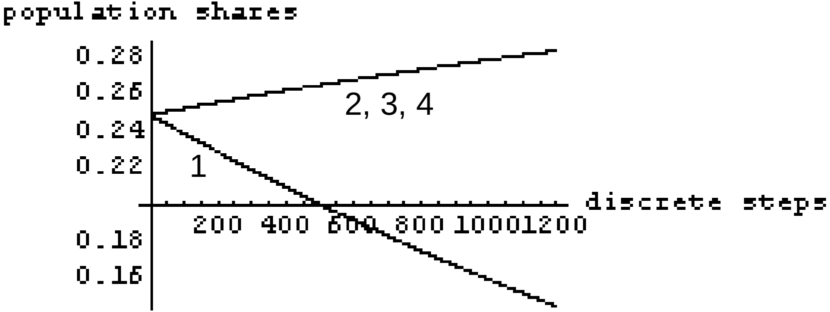
Conclusions

Future work

Selection = evolution for a given set of parameters
Mutation = change of parameters

- We may consider small changes of the coalition function v.
- Other players could be added with very small sizes such that the worths for the other players stays the same for a zero size of the new arrival.

2 time periods with step length $\frac{1}{600}$
$(x_1(0), x_2(0), x_3(0), x_4(0)) = \left(\frac{1}{4}+\varepsilon, \frac{1}{4}-\frac{\varepsilon}{3}, \frac{1}{4}-\frac{\varepsilon}{3}, \frac{1}{4}-\frac{\varepsilon}{3}\right)$
$\to \left(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$
discreteness problem!



Two different non-zero shares

Consider $N = \{1, 2, 3\}$, $v \in V(N)$ given by

- $v(1) = v(2) = v(3) = 0$,
- $v(1, 3) = 2$,
- $v(1, 2) = v(2, 3) = 1$ and
- $v(1, 2, 3) = 3$.