Growth theory without constant returns

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Overview "Growth theory without constant returns"

- Introduction
- The static case
  - The population and its structure
  - Factors of production and production function
  - Vector measure game and Shapley value
- Labor and capital immigration

Introduction

Solow-type model with two new features:

- the population of worker-capitalists is divided in m groups which differ in s, δ, n, and k_0
- non-constant returns

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Problem: marginal-product payments do not exhaust the product
Solution: continuous Shapley value introduced by Aumann and Shapley (1974)

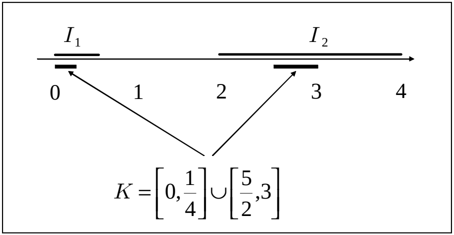
The population and its structure I

- not n players but by m intervals of workers
- $\vec{L} = (L_1, \ldots, L_m) \in \mathbb{R}_+^m$ where L_i
- λ = Lebesgues-Borel measure on $\mathbb{R}$
- m intervals $I_i \subseteq \mathbb{R}$, such that $\lambda(I_i) = L_i$ holds for every $i = 1, \ldots, m$
- $I := \cup_{i=1}^m I_i$ = set of all workers with cardinality $L := \sum_{i=1}^m L_i = \lambda(I)$
- $\mathcal{B}$ = set of Borel sets of I
- define $\mu_i^{\vec{L}}$ by $\mu_i^{\vec{L}}(K) := \lambda(K \cap I_i), K \in \mathcal{B}$
- $\mu_i^{\vec{L}}$ is a measure on $(I, \mathcal{B})$
- $\mu^{\vec{L}} = \prod_{i \in N} \mu_i^{\vec{L}} : \mathcal{B} \to \mathbb{R}^N$, $K \mapsto \left(\mu_i^{\vec{L}}(K)\right)_{i \in N}$ = Cartesian product of these measures

The population and its structure II

$\vec{L} = \left(\frac{1}{2}, 2\right)$ and intervals $I_1 = \left[0, \frac{1}{2}\right]$ and $I_2 = [2, 4]$. For $K := \left[0, \frac{1}{4}\right] \cup \left[\frac{5}{2}, 3\right] \in \mathcal{B}$ we obtain

$$\mu_1(K) = \lambda(K \cap I_1) = \lambda\left(\left[0, \frac{1}{4}\right]\right) = \frac{1}{4} \text{ and}$$
$$\mu_2(K) = \lambda(K \cap I_2) = \lambda\left(\left[\frac{5}{2}, 3\right]\right) = \frac{1}{2}.$$



Factors of production and production function

- k_i = amount of capital owned by a worker of group i
- Thus,

$$K_i \quad : \quad = k_i L_i \text{ for group } i, \text{ and}$$
$$K \quad : \quad = \sum_{i=1}^{m} k_i L_i \text{ for all groups together.}$$

- define

$$g = (g_K, g_L) : \mathbb{R}_+^m \to \mathbb{R}^2,$$
$$(L_1, ..., L_m) \mapsto \left(\sum_{i=1}^{m} k_i L_i, \sum_{i=1}^{m} L_i \right),$$

- For a production function $Y = F(K, L)$,

$$F \circ g \quad : \quad \mathbb{R}_+^m \to \mathbb{R},$$
$$(L_1, ..., L_m) \mapsto F(g_K(L_1, ..., L_m), g_L(L_1, ..., L_m))$$

yields the output producible by the m groups of size $L_1, ..., L_m$.

Vector measure game and Shapley value I

Vector measure game

$$v := F \circ g \circ \mu : \mathcal{B} \to \mathbb{R}$$

Given a coalition $S \in \mathcal{B}$,

- $\mu(S)$ specifies how to devide S among the m groups,
- g shows the labor and capital available to all those groups and
- F yields the product.

Vector measure game and Shapley value II

- If F is continuously differentiable, so is $F \circ g$.
- If $F(0,0) = 0$, we also have $(F \circ g)(0) = 0$.

Assume these two properties. Then, we can apply the continuous Shapley value as proposed by Aumann and Shapley (1974).
For $S \in \mathcal{B}$, it is given by

$$\begin{aligned} Sh(v)(S) &= \sum_{j=1}^{m} \mu_j(S) \int_0^1 \left. \frac{\partial (F \circ g)}{\partial L_j} \right|_{(\tau\mu_1(I),\ldots,\tau\mu_m(I))} d\tau \\ &= \sum_{j=1}^{m} \lambda(S \cap I_j) \int_0^1 \left. \frac{\partial (F \circ g)}{\partial L_j} \right|_{(\tau L_1,\ldots,\tau L_m)} d\tau. \end{aligned}$$

Diagonal formula!

Vector measure game and Shapley value III

- From the previous chapter, we know

$$\left.\frac{\partial F}{\partial g_K}\right|_{(\tau K,\tau L)} = \tau^{d-1} \left.\frac{\partial F}{\partial g_K}\right|_{(K,L)}$$

- Therefore, we can write the derivative $\left.\frac{\partial(F\circ g)}{\partial L_j}\right|_{(\tau L_1,\ldots,\tau L_m)} d\tau$ as

$$\begin{aligned}
& \left.\frac{\partial F}{\partial g_K}\right|_{(g_K(\tau L_1,\ldots,\tau L_m),g_L(\tau L_1,\ldots,\tau L_m))} \left.\frac{\partial g_K}{\partial L_i}\right|_{(\tau L_1,\ldots,\tau L_m)} + \left.\frac{\partial F}{\partial g_L}\right|_{(g_K(\tau L_1,\ldots,\tau
\\
= & \left.\frac{\partial F}{\partial g_K}\right|_{\left(\tau\sum_{j=1}^m k_j L_j,\tau\sum_{j=1}^m L_j\right)} k_i + \left.\frac{\partial F}{\partial g_L}\right|_{\left(\tau\sum_{j=1}^m k_j L_j,\tau\sum_{j=1}^m L_j\right)}
\\
= & \tau^{d-1} \left.\frac{\partial F}{\partial g_K}\right|_{\left(\sum_{j=1}^m k_j L_j,\sum_{j=1}^m L_j\right)} k_i + \tau^{d-1} \left.\frac{\partial F}{\partial g_L}\right|_{\left(\sum_{j=1}^m k_j L_j,\sum_{j=1}^m L_j\right)}
\\
= & \tau^{d-1}\left[\left.\frac{\partial F}{\partial g_K}\right|_{(K,L)} k_i + \left.\frac{\partial F}{\partial g_L}\right|_{(K,L)}\right].
\end{aligned}$$

Vector measure game and Shapley value IV

- Finally, we calculate the integral

$$\int_0^1 \tau^{d-1} d\tau = \frac{1}{d}\tau^d \Big|_0^1 = \frac{1}{d}.$$

Now, for $S := I_i$, we find

$$\begin{aligned}
\frac{Sh(v)(I_i)}{L_i} &= \frac{1}{L_i}\sum_{j=1}^{m} \lambda(I_i \cap I_j) \int_0^1 \frac{\partial(F \circ g)}{\partial L_j}\Big|_{(\tau L_1,\dots,\tau L_m)} d\tau \\
&= \int_0^1 \tau^{d-1} \left[\frac{\partial F}{\partial g_K}\Big|_{(K,L)} k_i + \frac{\partial F}{\partial g_L}\Big|_{(K,L)} \right] d\tau \\
&= \left[\frac{\partial F}{\partial g_K}\Big|_{(K,L)} k_i + \frac{\partial F}{\partial g_L}\Big|_{(K,L)} \right] \int_0^1 \tau^{d-1} d\tau \\
&= \left[\frac{\partial F}{\partial g_K}\Big|_{(K,L)} k_i + \frac{\partial F}{\partial g_L}\Big|_{(K,L)} \right] \frac{1}{d}
\end{aligned}$$

Vector measure game and Shapley value V

Special case

$$Y = F(K, L) = K^{\alpha} L^{\beta},\ 0 < \alpha, \beta$$

homogeneous of degree $d := \alpha + \beta$
vector measure game given by

$$v(S) = \left(\sum_{i=1}^{m} k_i \lambda(S \cap I_i) \right)^{\alpha} (\lambda(S))^{\beta},$$

and the Shapley value for group i

$$\begin{aligned} Y_i \quad &:= Sh(v)(I_i) \\ &= L_i \left[\left. \frac{\partial F}{\partial g_K} \right|_{(K,L)} k_i + \left. \frac{\partial F}{\partial g_L} \right|_{(K,L)} \right] \frac{1}{\alpha + \beta} \\ &= L_i \left[\frac{\alpha}{\alpha + \beta} K^{\alpha - 1} L^{\beta} k_i + \frac{\beta}{\alpha + \beta} K^{\alpha} L^{\beta - 1} \right] \end{aligned}$$

Vector measure game and Shapley value VI

Problem

Simplify

$$Y_i = L_i \left[\frac{\alpha}{\alpha+\beta} K^{\alpha-1} L^{\beta} k_i + \frac{\beta}{\alpha+\beta} K^{\alpha} L^{\beta-1} \right]$$

for the one-group case ($m = 1$, dropping the i-index) and comment!

Labor and capital immigration I

Definition

We have labor immigration into an economy if $L_{m+1} > 0$ and $k_{m+1} = 0$ hold. We have capital immigration (capital imports) into an economy if $L_{m+1} = 0$, $k_{m+1} > 0$ hold.

Definition

Group $i = 1, ..., m$ is said to be welcoming towards labor (capital) immigration if $\frac{dY_i}{dL} > 0$ ($\frac{dY_i}{dK} > 0$) holds.

Labor and capital immigration II

Theorem

Considering immigration at time into an economy, we find:

- *Group i benefits from labor immigration in case of*

$$k_i > \frac{1-\beta}{\alpha}\frac{K}{L}.$$

$k_i = \frac{K}{L}$ —> *group i welcomes labor immigration iff increasing returns to scale hold*
$d = 1$ —> *more-than-average capital-rich groups welcome labor immigration*

Labor and capital immigration III

Theorem

- *Group i benefits from capital immigration in case of*

$$\frac{K}{L} > \frac{1-\alpha}{\beta} k_i .$$

  $k_i = \frac{K}{L}$ —> *group i welcomes capital immigration iff increasing returns to scale hold*
  $d = 1$ —> *more-than-average capital-rich groups oppose capital immigration*

- *Increasing returns to scale are a necessary condition for any group to be welcoming to both capital and labor.*

Dynamics of per-head capital endowment I

Taking depreciation into account, the capital stock of group i develops in accordance with

$$\begin{aligned}\frac{d\left(L_{it}k_{it}\right)}{dt} &= \frac{dK_{it}}{dt} = s_i Y_{it} - \delta_i K_{it} \\ &= s_i L_{it}\left[\frac{\alpha}{\alpha+\beta}K_t^{\alpha-1}L_t^{\beta}k_{it} + \frac{\beta}{\alpha+\beta}K_t^{\alpha}L_t^{\beta-1}\right] - \delta_i L_{it}k_{it}.\end{aligned}$$

and we obtain

$$\frac{\dot{k}_{it}}{k_{it}} = s_i\left[\frac{\alpha}{\alpha+\beta}\frac{L_t^{\beta}}{K_t^{1-\alpha}} + \frac{\beta}{\alpha+\beta}\frac{K_t^{\alpha}}{L_t^{1-\beta}}\cdot\frac{1}{k_{it}}\right] - \left(\delta_i + n_i\right).$$

$m = 1 \longrightarrow \frac{\dot{k}_t}{k_t} = s\frac{L_t^{\beta}}{K_t^{1-\alpha}} - (\delta + n)$ and

$\alpha + \beta = 1 \longrightarrow \frac{\dot{k}_t}{k_t} = \frac{s}{k_t^{1-\alpha}} - (\delta + n)$

Dynamics of per-head capital endowment II

Which of two groups i and j grows faster?

$$\frac{\dot{k}_{it}}{k_{it}} - \frac{\dot{k}_{jt}}{k_{jt}} = (s_i - s_j)\frac{\alpha}{\alpha+\beta}\frac{L_t^{\beta}}{K_t^{1-\alpha}} + \left(\frac{s_i}{k_{it}} - \frac{s_j}{k_{jt}}\right)\frac{\beta}{\alpha+\beta}\frac{K_t^{\alpha}}{L_t^{1-\beta}} - (\delta_i + n_i) + (\delta_j + n_j)$$

Convergence in the special case: $n := n_i = n_j$, $\delta := \delta_i = \delta_j$, $s := s_i = s_j$ —>

$$\frac{\dot{k}_{it}}{k_{it}} - \frac{\dot{k}_{jt}}{k_{jt}} = \left(\frac{s}{k_{it}} - \frac{s}{k_{jt}}\right)\frac{\beta}{\alpha+\beta}\frac{K_t^{\alpha}}{L_t^{1-\beta}}$$

- Noting

$$\frac{L^{\beta}}{K^{1-\alpha}} = \frac{(L_0 e^{nt})^{\beta}}{(L_0 e^{nt} k)^{1-\alpha}} = \frac{(L_0 e^{nt})^{\beta}}{(L_0 e^{nt})^{1-\alpha} k^{1-\alpha}}$$

$$= L_0^{\alpha+b-1} \frac{(e^{nt})^{\alpha+\beta-1}}{k^{1-\alpha}}$$

$$= L_0^{\alpha+b-1} \frac{e^{(\alpha+\beta-1)nt}}{k^{1-\alpha}},$$

- normalizing $L_0 = 1$, we can work with

$$\frac{\dot{k}}{k} = s\frac{e^{(\alpha+\beta-1)nt}}{k^{1-\alpha}} - (\delta + n)$$

The steady state for a one-group economy II

If (!) a steady state exists, we find

$$
\begin{aligned}
0 &= \frac{\partial \frac{\dot{k}}{k}}{\partial t} = \frac{\partial \left(\frac{se^{(\alpha+\beta-1)nt}}{k^{1-\alpha}} - (\delta + n) \right)}{\partial t} \\
&= \frac{se^{(\alpha+\beta-1)nt} (\alpha + \beta - 1) n \cdot k^{1-\alpha} - (1-\alpha) k^{-\alpha} \dot{k} \cdot se^{(\alpha+\beta-1)nt}}{k^{2(1-\alpha)}},
\end{aligned}
$$

hence

$$\frac{\dot{k}^c}{k^c} = \frac{\alpha + \beta - 1}{1 - \alpha} n \text{ (c=candidate).}$$

—>

$$\frac{se^{(\alpha+\beta-1)nt}}{k^{1-\alpha}} - (\delta + n) = \frac{\alpha + \beta - 1}{1 - \alpha} n$$

The steady state for a one-group economy III

... candidate equilibrium path

$$k^c = \left(\frac{s e^{(\alpha+\beta-1)nt}}{\frac{(\alpha+\beta-1)n}{1-\alpha} + (\delta + n)} \right)^{\frac{1}{1-\alpha}}.$$

with growth rate

$$\frac{\frac{d\left(\frac{s e^{(\alpha+\beta-1)nt}}{\frac{(\alpha+\beta-1)n}{1-\alpha} + (\delta+n)} \right)^{\frac{1}{1-\alpha}}}{dt}}{\left(\frac{s e^{(\alpha+\beta-1)nt}}{\frac{(\alpha+\beta-1)n}{1-\alpha} + (\delta+n)} \right)^{\frac{1}{1-\alpha}}} = \frac{\alpha + \beta - 1}{1 - \alpha} n \ (!)$$

$\longrightarrow k^* := k^c$

The steady state for a one-group economy IV

- constant returns to scale —> growth rate 0
- increasing returns to scale —> per-head capital endowment grows at a constant rate
- decreasing returns —> per-head capital endowment grows at a constant shrinks

both growth and shrinkage is leveraged by the growth of the population

The steady state for a one-group economy V

For $m = 1$, steady-state per-head consumption is given by

$$\begin{aligned} c &: = (1-s)\, K^{\alpha} L^{\beta-1} \\ &= (1-s)\left(e^{nt}k\right)^{\alpha}\left(e^{nt}\right)^{\beta-1} \;\; (L_0 = 1) \\ &= (1-s)\, e^{nt(\alpha+\beta-1)} k^{\alpha}. \end{aligned}$$

$\alpha + \beta - 1 < 0 =$ decreasing returns to scale (possibly due to shortage of land)
—$>$ consumption tends to zero which makes a positive growth rate of the population unsustainable and gives rise to a Malthusian interpretation