Overview part C: Vector-measure games

- The Solow growth model
- Growth theory without constant returns
- Evolutionary cooperative game theory

Overview "The Solow growth model"

- Introduction
- Growth rates
- Convergence
- Cobb-Douglas production functions
- Dynamics (CD production function)
- Steady state (CD production function)
- Comparative statics and the golden rule (CD production
- function)
- Neoclassical production function
- Dynamics and steady state (neoclassical production function)
- Comparative statics and the golden rule (neoclassical production)

Introduction

The first part of this chapter presents the Solow model on the basis of both

- a Cobb-Douglas production function and
- any neoclassical production function.

We will guide the reader

- to an understanding of discrete and continuous growth rates,
- through the dynamics of the Solow model for both Cobb-Douglas and neoclassical production functions, and
- to the equilibrium concept employed by growth theorists.

By y_t we denote the value of y at time t, $t = 0, 1, ...$

Definition

The discrete-time growth rate of y is defined by

$$\gamma_y^{\langle 1 \rangle} := \frac{y_{t+1} - y_t}{y_t}.$$

Superscript $\langle 1 \rangle$ refers to the full time interval, a year, say.

Problem

What are the growth rates of x_t, y_t, *and* z_t, *given by*

$$
\begin{aligned}
x_t &: = t, \\
y_t &: = t + 4 \text{ and} \\
z_t &: = 100t?
\end{aligned}
$$

Problem

What are the growth rates of x_t, y_t, and z_t, given by

$$\begin{aligned} x_t &:= t, \\ y_t &:= t+4 \text{ and} \\ z_t &:= 100t? \end{aligned}$$

We obtain

$$\begin{aligned} \frac{x_{t+1}-x_t}{x_t} &= \frac{t+1-t}{t} = \frac{1}{t} \\ \frac{y_{t+1}-y_t}{y_t} &= \frac{t+5-(t+4)}{t+4} = \frac{1}{t+4} < \frac{1}{t} \text{ and} \\ \frac{z_{t+1}-z_t}{y_t} &= \frac{100(t+1)-100t}{100t} = \frac{1}{t} \end{aligned}$$

Multiplying

$$y_t$$

by the growth factor

$$1 + \gamma_y^{\langle 1 \rangle} = 1 + \frac{y_{t+1} - y_t}{y_t}$$

yields, at the end of a year,

$$\begin{aligned} & y_t \left(1 + \frac{y_{t+1} - y_t}{y_t} \right) \\ = \; & y_{t+1}. \end{aligned}$$

t years later, a given y_0 (y at time 0) has become

$$y_t = y_0 \left(1 + \gamma_y^{\langle 1 \rangle} \right)^t .$$

Example

If you take Euro 100,- to the bank to earn an interest of $r = \frac{5}{100} = 5\%$, at the end of five years, you collect

$$100\left(1+\frac{5}{100}\right)^5 \approx 100 \cdot 1.276 = 127.6.$$

In growth theory, y_t often denotes income per head at time t, i.e.,

$$y_t = \frac{Y_t}{L_t},$$

where Y_t is the income and L_t the labor force, both at time t. One would, of course, think that the growth rates of y, Y and L are closely connected. Indeed, we obtain

$$\begin{aligned} \frac{y_{t+1} - y_t}{y_t} &= \frac{\frac{Y_{t+1}}{L_{t+1}} - \frac{Y_t}{L_t}}{\frac{Y_t}{L_t}} = \ldots. \\ &= \frac{L_t}{L_{t+1}} \left(\frac{Y_{t+1} - Y_t}{Y_t} - \frac{L_{t+1} - L_t}{L_t} \right) \end{aligned}$$

Thus, the growth rate of

$$y = \frac{Y}{L}$$

is close to the growth rate of Y minus the growth rate of L if L_t is close to L_{t+1},

$$\frac{y_{t+1} - y_t}{y_t} \approx \frac{Y_{t+1} - Y_t}{Y_t} - \frac{L_{t+1} - L_t}{L_t}.$$

Very much the same holds for the product of two variables. Let us consider the production function

$$Y_t = L_t K_t,$$

which supposes that income Y_t is the product (in mathematical terms) of labor L_t and capital K_t. We have

$$\begin{aligned} & \frac{L_{t+1} - L_t}{L_t} + \frac{K_{t+1} - K_t}{K_t} = \ldots \\ = \quad & \frac{Y_{t+1} - Y_t}{Y_t} + \frac{L_t\left(K_{t+1} - K_t\right) - L_{t+1}\left(K_{t+1} - K_t\right)}{L_t K_t} \\ \approx \quad & \frac{Y_{t+1} - Y_t}{Y_t} \end{aligned}$$

If the time intervals are "very small"...

Let us consider half-yearly instead of yearly growth rates. Instead of the growth factor

$$\left(1+\gamma^{\langle 1\rangle}\right)^t$$

for the yearly growth rate $\gamma_y^{\langle 1\rangle}$, we have the growth factor

$$\left(\left(1+\frac{\gamma^{\langle 1\rangle}}{2}\right)^2\right)^t = \left(1+\frac{\gamma^{\langle 1\rangle}}{2}\right)^{2t}.$$

for the half-yearly growth rate $\frac{\gamma^{\langle 1\rangle}}{2}$.

Food for thought Would you prefer an interest payment of $\frac{\gamma^{\langle 1\rangle}}{2}$, two times a year, to an interest rate of $\gamma^{\langle 1\rangle}$, paid out only once a year?

Growth rates

From discrete to continuous time

Food for thought Would you prefer an interest payment of $\frac{\gamma^{\langle 1 \rangle}}{2}$, two times a year, to an interest rate of $\gamma^{\langle 1 \rangle}$, paid out only once a year?

Since we earn interest on the interest, these two factors are not equal:

$$\left(1+\frac{\gamma^{\langle 1 \rangle}}{2}\right)^{2t} > \left(1+\gamma^{\langle 1 \rangle}\right)^{t}.$$

Growth rates

From discrete to continuous time

We now look for a growth rate that makes the investor indifferent between half-yearly payments and yearly payments. That is, we define $\gamma^{\left\langle \frac{1}{2} \right\rangle}$ implicitly by

$$\left(1 + \frac{\gamma^{\left\langle \frac{1}{2} \right\rangle}}{2}\right)^{2t} = \left(1 + \gamma^{\langle 1 \rangle}\right)^{t}.$$

Food for thought Would you expect $\gamma^{\left\langle \frac{1}{2} \right\rangle} > \gamma^{\langle 1 \rangle}$ or $\gamma^{\left\langle \frac{1}{2} \right\rangle} < \gamma^{\langle 1 \rangle}$?

From

$$\left(1+\frac{\gamma^{\left\langle\frac{1}{2}\right\rangle}}{2}\right)^{2t}=\left(1+\gamma^{\langle1\rangle}\right)^{t}$$

we obtain

$$1+\frac{\gamma^{\left\langle\frac{1}{2}\right\rangle}}{2}=\left(\left(1+\frac{\gamma^{\left\langle\frac{1}{2}\right\rangle}}{2}\right)^{2t}\right)^{\frac{1}{2t}}=\left(\left(1+\gamma^{\langle1\rangle}\right)^{t}\right)^{\frac{1}{2t}}=\left(1+\gamma^{\langle1\rangle}\right)^{\frac{1}{2}}$$

and then

$$\gamma^{\left\langle\frac{1}{2}\right\rangle}=-2+2\sqrt{1+\gamma^{\langle1\rangle}}.$$

We can now conclude

$$
\begin{aligned}
\gamma^{\langle 1\rangle} &> 0 \\
&\Rightarrow \left(\gamma^{\langle 1\rangle}\right)^2 > 0 \\
&\Rightarrow \left(\gamma^{\langle 1\rangle}\right)^2 + 4(1+\gamma) > 4(1+\gamma) \\
&\Rightarrow \left(\gamma^{\langle 1\rangle} + 2\right)^2 > 4(1+\gamma) \\
&\Rightarrow \gamma^{\langle 1\rangle} + 2 > 2\sqrt{1+\gamma} \\
&\Rightarrow \gamma^{\langle 1\rangle} > -2 + 2\sqrt{1+\gamma} = \gamma^{\left\langle \frac{1}{2}\right\rangle}
\end{aligned}
$$

We now decrease the time interval even further. Generally, we consider an interest payment m times a year with interest rate $\gamma^{\langle 1 \rangle}/m$. Then, at the end of t years, we obtain

$$\left(\left(1+\frac{\gamma^{\langle 1 \rangle}}{m}\right)^{m}\right)^{t} = \left(1+\frac{\gamma^{\langle 1 \rangle}}{m}\right)^{mt}.$$

It can be shown (but we will not do that here) that this growth factor is an increasing function of m. The sequence $\left(\left(1+\frac{\gamma^{\langle 1 \rangle}}{m}\right)^{mt}\right)_{m\in N}$ converges (gets closer and closer to some value) and we have

$$\lim_{m\to\infty}\left(1+\frac{\gamma^{\langle 1 \rangle}}{m}\right)^{mt} = e^{\gamma^{\langle 1 \rangle}t}.$$

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From discrete to continuous time

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Growth rates

From discrete to continuous time

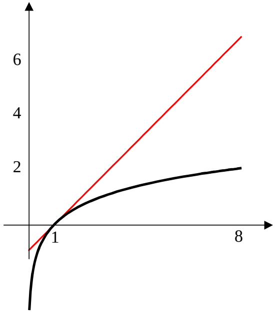
- Again, because of the interest on the interest, one prefers to obtain continuous interest payments.
- Looking for $\gamma^{\langle 0 \rangle}$ – rate at which indifference to a yearly interest rate:

$$e^{\gamma^{\langle 0 \rangle} t} = \left(1 + \gamma^{\langle 1 \rangle}\right)^t .$$

- Applying the natural logarithm on both sides, deviding by t

$$\gamma^{\langle 0 \rangle} = \ln\left(1 + \gamma^{\langle 1 \rangle}\right)$$

We would like to confirm $\gamma^{\langle 0 \rangle} < \gamma^{\langle 1 \rangle}$



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Growth rates

From discrete to continuous time

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- Replacing x by $1 + y$, we obtain

$$\ln (1 + y) < y \text{ for } y > -1, y \neq 0$$

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- $$\gamma^{\langle 0\rangle} = \ln\left(1+\gamma^{\langle 1\rangle}\right) < \gamma^{\langle 1\rangle} \text{ for } \gamma^{\langle 1\rangle} > -1, \gamma^{\langle 1\rangle} \neq 0$$

Growth rates

From discrete to continuous time

The growth rates $\gamma^{\langle 1 \rangle}$ and $\gamma^{\langle 0 \rangle}$ are close for small rates:

$\gamma^{\langle 1 \rangle}$	$\gamma^{\langle 0 \rangle}$ (approximation)
$0,001$ (one-tenth of a percent)	$0,0009995$
$0,01$ (one percent)	$0,0099503$
$0,1$ (10 percent)	$0,09531$
$0,2$ (20 percent)	$0,18232$
$0,3$ (30 percent)	$0,26236$

- In discrete time, the growth rate of y is defined by

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- Taking the limit with respect to Δt yields

$$\begin{aligned} \lim_{\Delta t \to 0} \gamma_y^{\langle \Delta t \rangle} &= \lim_{\Delta t \to 0} \frac{\frac{y_{t+\Delta t} - y_t}{(t+\Delta t) - t}}{y_t} = \lim_{\Delta t \to 0} \frac{\frac{\Delta y_t}{\Delta t}}{y_t} \\ &= \frac{\frac{dy_t}{dt}}{y_t}. \end{aligned}$$

Definition

The continuous-time growth rate of y is defined by

$$\gamma_y := \gamma_{y,t} := \frac{\frac{dy_t}{dt}}{y_t}$$

where the time index is often suppressed.

- Assuming a constant growth rate g

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- solution

$$y_t = y_0 e^{gt}$$

Problem

Calculate $\frac{dy_t}{dt} / y_t$ *for* $y_t = y_0 e^{gt}$. *Hint: the derivative of* e^x *is* e^x, *but do not forget the chain rule.*

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You have found

$$\frac{\frac{dy_t}{dt}}{y_t} = \frac{\frac{d(y_0 e^{gt})}{dt}}{y_0 e^{gt}} = \frac{y_0 e^{gt} g}{y_0 e^{gt}} = g.$$

$\implies \gamma_y = g$ so we can write $y_t = y_0 e^{\gamma_y t}$.

Growth rates

Using the natural logarithm to express growth

- take recourse to

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- By $\frac{d \ln x}{dx} = \frac{1}{x}$
$$\begin{aligned} \frac{d\widehat{y}}{dt} &= \frac{d \ln y_t}{dt} \\ &= \frac{1}{y_t}\frac{dy}{dt} \text{ (chain rule!)} \\ &= \frac{\dot{y}_t}{y_t} \end{aligned}$$

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- If $\widehat{y}$ is plotted against t

$$\text{growth rate of } y = \text{slope of } \widehat{y}\text{-graph}$$

Problem

Try to find the relationship between the (continuous-time) growth rates of Y, K *and* L *for* $Y_t = L_t K_t$. *Hint: apply the product rule of differentiation and use* $\ln(LK) = \ln L + \ln K$.

Problem

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Using the original definition, we obtain

$$\begin{aligned}\gamma_Y &= \frac{\dot{Y}_t}{Y_t} = \frac{\frac{d(K_t L_t)}{dt}}{K_t L_t} = \frac{\frac{dK_t}{dt} L_t + \frac{dL_t}{dt} K_t}{K_t L_t} \\ &= \frac{\frac{dK_t}{dt}}{K_t} + \frac{\frac{dL_t}{dt}}{L_t} = \gamma_K + \gamma_L.\end{aligned}$$

Problem

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Using the logarithm, we have

$$\begin{aligned} \gamma_Y &= \frac{d \ln Y_t}{dt} = \frac{d \ln (K_t L_t)}{dt} \\ &= \frac{d (\ln K_t + \ln L_t)}{dt} = \frac{d \ln K_t}{dt} + \frac{d \ln L_t}{dt} \\ &= \gamma_K + \gamma_L. \end{aligned}$$

Homework:
Also, for

$$y = \frac{Y}{L}$$

we find

$$\gamma_y = \gamma_Y - \gamma_L.$$

Growth rates

Using the natural logarithm to express growth

To sum up, in continuous time we obtain:

- Growth rate of a product = sum of the growth rates of its factors.
- Growth rate of a ratio =
  - growth rates of nominator minus
  - growth rate of denominator.

An application

- r denote the monetary interest rate = growth rate for an asset K_m
- π denotes the rate of inflation

$\Longrightarrow$ real interest rate $= r - \pi$

- Because...

Growth rates

Using the natural logarithm to express real interest rate

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- Because...
- Notation: price level P, real capital K
- $K := \frac{K_m}{P}$
- $$\gamma_K = \gamma_{K_m} - \gamma_P = r - \pi$$

Growth rates

Using the natural logarithm to express growth

Problem

Apply the natural logarithm to the exponential-growth formula

$$y_t = y_0 e^{\gamma_y t}$$

in order to confirm

$$\gamma_y = \frac{\ln y_t - \ln y_0}{t - 0} = \frac{1}{t} \ln \frac{y_t}{y_0}.$$

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$$\begin{aligned} \ln y_t &= \ln y_0 + \ln e^{\gamma_y t} \\ &= \ln y_0 + \gamma_y t \\ \Rightarrow \quad \gamma_y &= \frac{\ln y_t - \ln y_0}{t - 0} = \frac{\ln \frac{y_t}{y_0}}{t - 0} \end{aligned}$$

Growth rates

Using the natural logarithm to express growth

Rule of thumb

- years to double y is approximately $\frac{70}{\gamma_y \cdot 100}$.

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- achieve a doubling in t years —> growth rate $\frac{70}{t}$ is needed

Growth rates

Using the natural logarithm to express growth

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- years to double y is approximately $\frac{70}{\gamma_y \cdot 100}$.
- Example: interest rate of 2% —> 35 years to double
- achieve a doubling in t years —> growth rate $\frac{70}{t}$ is needed
- double within 10 years, ask for an interest rate of 7%

Confirmation of this rule

- Growth rate γ_y and/or the time span needed to double y,

Growth rates

Using the natural logarithm to express growth

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$$\gamma_y t = \ln \left(e^{\gamma_y t} \right) = \ln 2 \approx 0,69315$$

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- Solving for t or γ_y

$$t \approx \frac{70}{\gamma_y \cdot 100}$$

and

$$\gamma_y \cdot 100 \approx \frac{70}{t}$$

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and

$$\gamma_y \cdot 100 \approx \frac{70}{t}$$

- Two approximations!

Growth rates

Using the natural logarithm to express growth

Growth rate in percentage points	Years needed for doubl. (approximation)	Years needed for doubl. (correct, contin. time)	Years needed for doubl. (correct, yearly interest)
0.1 (one-tenth of a percent)	700	≈ 693.15	≈ 693.49
1 (one percent)	70	≈ 69.31	≈ 69.66
10 (ten percent)	7	≈ 6.93	≈ 7.27
20 (twenty percent)	$3\frac{1}{2}$	≈ 3.46	≈ 3.80
30 (thirty percent)	$2\frac{1}{3}$	≈ 2.31	≈ 2.64

Growth rates

Convergence

Definition

Weak convergence between x_t and y_t is said to hold if, whenever $0 < x_0 < y_0$, the growth rates obey $\gamma_x > \gamma_y$ for all $t \geq 0$.

Lemma

Criterion for weak convergence: $0 < x_0 < y_0$ *implies* $\frac{d\frac{y_t}{x_t}}{dt} < 0$.

Proof:

$$\frac{d\frac{y}{x}}{dt} < 0 \Leftrightarrow \frac{\frac{dy}{dt}x - \frac{dx}{dt}y}{x^2} < 0 \Leftrightarrow \frac{\frac{dy}{dt}}{x} - \frac{\frac{dx}{dt}}{x}\frac{y}{x} < 0$$

$$\Leftrightarrow \quad \frac{\frac{dy}{dt}}{y} - \frac{\frac{dx}{dt}}{x} < 0 \text{ (multiply by } \frac{x}{y}\text{)}$$

$$\Leftrightarrow \quad \gamma_y < \gamma_x$$

Weak convergence may hold even if x and y never get close. For example, weak convergence exists between

$$\begin{aligned} x_t &= t \text{ and} \\ y_t &= 2t + 2. \end{aligned}$$

Problem

Show that weak convergence holds between x_t and y_t.

Obviously, $y_0 > x_0$. Now,

$$\begin{aligned}
\gamma_y &= \frac{2}{2t+2} \\
&= \frac{1}{t+1} \text{ (multiply by } \frac{1/2}{1/2}\text{)} \\
&< \frac{1}{t} \\
&= \gamma_x
\end{aligned}$$

Definition

Strong convergence between x_t and y_t is said to hold if weak convergence between x_t and y_t holds and if

$$\lim_{t \to \infty} \frac{y_t}{x_t} = 1.$$

Problem

Show that strong convergence does not hold between $x_t = t$ and $y_t = 2t + 2$.

Growth rates

Strong convergence

While x and y converge in a weak sense, they do not in a strong sense:

$$
\begin{aligned}
& \lim_{t\to\infty} \frac{2t+2}{t} \\
= \; & \lim_{t\to\infty} \left(2 + \frac{2}{t}\right) \\
= \; & 2 + \lim_{t\to\infty} \frac{2}{t} \\
= \; & 2 > 1
\end{aligned}
$$

$\frac{y_t}{x_t}$ decreases (by weak convergence), but $y_t > 2x_t$ for all t.

Cobb-Douglas production functions

$$Y = F(K, L) = AK^{\alpha}L^{1-\alpha}, \; A > 0, 0 < \alpha < 1$$

A is a technological coefficient
Letting $A := 1$, we work with

$$Y = F(K, L) = K^{\alpha}L^{1-\alpha}, \; 0 < \alpha < 1$$

Cobb-Douglas production functions

Definition

A production function F exhibits constant returns to scale, if we have

$$F(\tau K, \tau L) = \tau F(K, L), K \geq 0, L \geq 0$$

for any $\tau \geq 0$.

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Problem

Can you prove that the CD production function is of constant returns?
Hint: you will use $(a_1 a_2)^b = a_1^b a_2^b$ *and* $a^b a^c = a^{b+c}$.

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$$\begin{aligned} F(\tau K, \tau L) &= (\tau K)^{\alpha} (\tau L)^{1-\alpha} = \tau^{\alpha} K^{\alpha} \tau^{1-\alpha} L^{1-\alpha} \\ &= \tau^{\alpha} \tau^{1-\alpha} K^{\alpha} L^{1-\alpha} = \tau F(K, L). \end{aligned}$$

Cobb-Douglas production functions

- marginal productivity

$$\begin{aligned}\frac{\partial F}{\partial K} &= \alpha K^{\alpha-1} L^{1-\alpha} \\ &= \alpha \frac{L^{1-\alpha}}{K^{1-\alpha}} \\ &= \alpha \left(\frac{L}{K}\right)^{1-\alpha} > 0\end{aligned}$$

—> concavity in K (and L)

- Inada conditions:

$$\lim_{K\to\infty} \frac{\partial F}{\partial K} = 0; \lim_{K\to 0} \frac{\partial F}{\partial K} = \infty$$

Cobb-Douglas production functions

Production elasticity of capital:

$$\varepsilon_{Y,K} = \frac{\frac{\partial Y}{Y}}{\frac{\partial K}{K}} = \frac{\partial Y}{\partial K}\frac{K}{Y}.$$

Problem

Can you confirm that the production elasticity of capital is equal to α?

Cobb-Douglas production functions

Production elasticity of capital:

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Problem

Can you confirm that the production elasticity of capital is equal to α?

$\frac{\partial Y}{\partial K}$ is just another expression of $\frac{\partial F}{\partial K}$, therefore

$$\begin{aligned}\varepsilon_{Y,K} &= \frac{\partial F}{\partial K}\frac{K}{Y} \\ &= \alpha\left(\frac{L}{K}\right)^{1-\alpha}\frac{K}{K^{\alpha}L^{1-\alpha}} \\ &= \alpha.\end{aligned}$$

Cobb-Douglas production functions

Problem

Prove Euler's theorem for CD production functions:

$$\frac{\partial F}{\partial K} \cdot K + \frac{\partial F}{\partial L} \cdot L = F(K, L).$$

Problem

Prove Euler's theorem for CD production functions:

$$\frac{\partial F}{\partial K} \cdot K + \frac{\partial F}{\partial L} \cdot L = F(K, L).$$

$$\begin{aligned}
\frac{\partial F}{\partial K} \cdot K + \frac{\partial F}{\partial L} \cdot L &= \alpha \left(\frac{L}{K}\right)^{1-\alpha} \cdot K + (1-\alpha)\left(\frac{K}{L}\right)^{\alpha} \cdot L \\
&= \alpha \frac{L^{1-\alpha}}{K^{1-\alpha}} \cdot K + (1-\alpha)\frac{K^{\alpha}}{L^{\alpha}} \cdot L \\
&= \alpha K^{\alpha} L^{1-\alpha} + (1-\alpha) K^{\alpha} L^{1-\alpha} \\
&= F(K, L).
\end{aligned}$$

Cobb-Douglas production functions

Problem

Assuming a CD production function, show how the growth rate of output depends on the growth rates of capital and labor. Hint: you will use the product and chain rule of differentiation (first growth-rate definition) or the rules for manipulating the natural logarithm (second growth-rate definition).

Cobb-Douglas production functions

For $Y_t = F(K_t, L_t) = K_t^\alpha L_t^{1-\alpha}$,

$$
\begin{aligned}
\gamma_Y &= \frac{\frac{dY_t}{dt}}{Y_t} \\
&= \frac{\frac{d\left(K_t^\alpha L_t^{1-\alpha}\right)}{dt}}{K_t^\alpha L_t^{1-\alpha}} \\
&= \frac{\alpha K_t^{\alpha-1} \frac{dK}{dt} L_t^{1-\alpha} + K_t^\alpha (1-\alpha) L_t^{-\alpha} \frac{dL}{dt}}{K_t^\alpha L_t^{1-\alpha}} \text{ (product rule and chain rule)} \\
&= \alpha \frac{\frac{dK}{dt}}{K_t} + (1-\alpha) \frac{\frac{dL}{dt}}{L_t} \\
&= \alpha \gamma_K + (1-\alpha) \gamma_L .
\end{aligned}
$$

Cobb-Douglas production functions

Alternatively:

$$\begin{aligned}
\gamma_Y &= \frac{d \ln Y_t}{dt} \\
&= \frac{d \ln \left(K_t^{\alpha} L_t^{1-\alpha}\right)}{dt} \\
&= \frac{d \left(\alpha \ln K_t + (1-\alpha) \ln L_t\right)}{dt} \\
&= \frac{d \left(\alpha \ln K_t\right)}{dt} + \frac{d \left((1-\alpha) \ln L_t\right)}{dt} \\
&= \alpha \frac{d \ln K_t}{dt} + (1-\alpha) \frac{d \ln L_t}{dt} \\
&= \alpha \gamma_K + (1-\alpha) \gamma_L.
\end{aligned}$$

Cobb-Douglas production functions

$$y := \frac{Y}{L}$$

$$k := \frac{K}{L}$$

$$y = \frac{K^{\alpha} L^{1-\alpha}}{L} = \frac{K^{\alpha}}{L^{\alpha}} = k^{\alpha} =: f(k)$$

f — production function in intensive form

Dynamics (CD production function)

- Consumption function

$$C := (1 - s)\, Y$$

$s \geq 0$ — constant saving rate

Dynamics (CD production function)

- Consumption function

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- Per-head consumption

$$c := \frac{C}{L} = (1 - s)\, \frac{Y}{L} = (1 - s)\, y.$$

Dynamics (CD production function)

- Consumption function

$$C := (1-s)\,Y$$

  $s \geq 0$ — constant saving rate

- Per-head consumption

$$c := \frac{C}{L} = (1-s)\,\frac{Y}{L} = (1-s)\,y.$$

- Savings = investments

$$\dot{K} = sY - \delta K$$

Dynamics (CD production function)

$$\begin{aligned}
\dot{k} &= \dot{\left(\frac{K}{L}\right)} = \frac{\dot{K}L - \dot{L}K}{L^2} \\
&= \frac{\dot{K}}{L} - \frac{\dot{L}}{L}\frac{K}{L} \\
&= \frac{\dot{K}}{L} - nk \; (n := \gamma_L) \\
&= \frac{sY - \delta K}{L} - nk \\
&= s\frac{Y}{L} - \delta\frac{K}{L} - nk \\
&= sk^{\alpha} - (\delta + n)\,k
\end{aligned}$$

Steady state (CD production function)

Definition

A steady state is a tuple of relevant economic variables that grow at constant rates.

- Solow model: (Y, K, L) or (Y, y, k, L)

Steady state (CD production function)

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- constant $\frac{\dot{k}}{k}$

Steady state (CD production function)

Definition

A steady state is a tuple of relevant economic variables that grow at constant rates.

- Solow model: (Y, K, L) or (Y, y, k, L)
- constant $\frac{\dot{k}}{k}$
- $= \frac{s}{k^{1-\alpha}} - (\delta + n)$

Steady state (CD production function)

$\frac{\dot{k}}{k}$ constant —>

$$
\begin{aligned}
0 &= \frac{d\left(\frac{s}{k^{1-\alpha}} - (\delta + n)\right)}{dt} = \frac{d\left(sk^{-1+\alpha}\right)}{dt} \\
&= s\left(-1+\alpha\right) k^{-2+\alpha} \frac{dk}{dt} \\
&= s\left(-1+\alpha\right) \frac{1}{k^{2-\alpha}} \frac{dk}{dt}
\end{aligned}
$$

—> $s = 0$ or

—> $\frac{dk}{dt} = 0$ —> $\frac{\dot{k}}{k} = 0$ —>

$$
\begin{aligned}
\frac{s}{\left(k^*\right)^{1-\alpha}} &= \delta + n, \\
s\left(k^*\right)^{\alpha} &= \left(\delta + n\right) k^*, \text{ or} \\
k^* &= \left(\frac{s}{\delta + n}\right)^{\frac{1}{1-\alpha}}
\end{aligned}
$$

Steady state (CD production function)

k^* is constant; so are

$$\begin{aligned} y^* &= f(k^*) = (k^*)^\alpha = \left(\frac{s}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}} \text{ and} \\ c^* &= (1-s)\left(\frac{s}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}. \end{aligned}$$

Problem

Show K, Y, and C grow at rate n. Hint: remember $K = kL$, $Y = yL$, and $C = cL$.

Steady state (CD production function)

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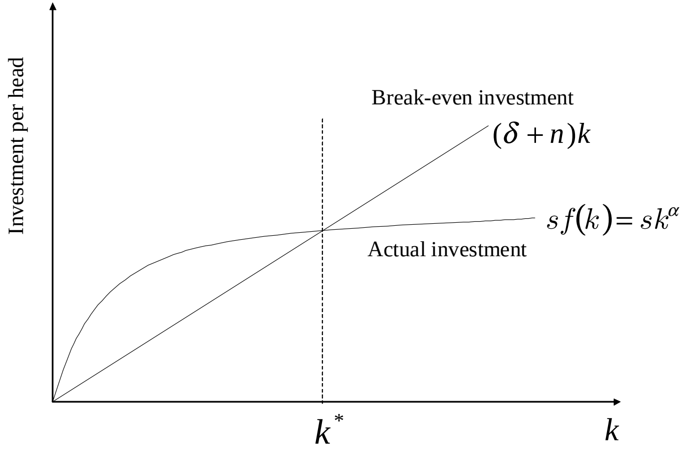
Problem

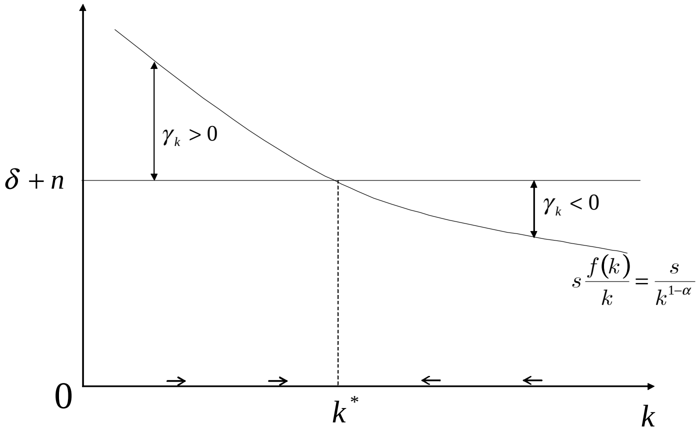
Show K, Y, and C grow at rate n. Hint: remember $K = kL$, $Y = yL$, and $C = cL$.

$$\gamma_K = \gamma_k + \gamma_L = 0 + n = n$$
$$\gamma_Y = \gamma_y + \gamma_L = 0 + n = n \text{ and}$$
$$\gamma_C = \gamma_c + \gamma_L = 0 + n = n$$

Steady state (CD production function)



Steady state (CD production function)



Steady state (CD production function)

- Algebraically: $0 < k < k^* = \left(\frac{s}{\delta+n}\right)^{\frac{1}{1-\alpha}}$ implies

$$\begin{aligned}
\gamma_k &= \frac{s}{k^{1-\alpha}} - (\delta + n) \\
&> \frac{s}{(k^*)^{1-\alpha}} - (\delta + n) \\
&= \frac{s}{\left(\left(\frac{s}{\delta+n}\right)^{\frac{1}{1-\alpha}}\right)^{1-\alpha}} - (\delta + n) \\
&= 0
\end{aligned}$$

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\end{aligned}$$

- Alternative way: differential equation

$$\dot{k} = sk^{\alpha} - (n+\delta)k$$

with solution

$$k_t = \left(\frac{s}{n+\delta} + \left(k_0^{1-\alpha} - \frac{s}{n+\delta}\right) e^{-(1-\alpha)(n+\delta)t}\right)^{\frac{1}{1-\alpha}}$$

Steady state (CD production function)

- $$\lim_{t\to\infty} e^{-(1-\alpha)(n+\delta)t} = \lim_{t\to\infty} \frac{1}{e^{(1-\alpha)(n+\delta)t}} = 0$$

Steady state (CD production function)

- $$\lim_{t\to\infty} e^{-(1-\alpha)(n+\delta)t} = \lim_{t\to\infty} \frac{1}{e^{(1-\alpha)(n+\delta)t}} = 0$$

- $$\begin{aligned}
& \lim_{t\to\infty} k_t \\
= & \left(\frac{s}{n+\delta} + \left(k_0^{1-\alpha} - \frac{s}{n+\delta}\right) \lim_{t\to\infty} e^{-(1-\alpha)(n+\delta)t}\right)^{\frac{1}{1-\alpha}} \\
= & \left(\frac{s}{n+\delta} + \left(k_0^{1-\alpha} - \frac{s}{n+\delta}\right) \cdot 0\right)^{\frac{1}{1-\alpha}} \\
= & \left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\alpha}} \\
= & k^*.
\end{aligned}$$

Comparative statics and the golden rule (CD production function)

How do the (exogenous) parameters of our model influence the (endogenous) variables?

- $$k^* = \left(\frac{s}{\delta + n}\right)^{\frac{1}{1-\alpha}}$$

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    - α
    - δ
    - n

Comparative statics and the golden rule (CD production function)

Towards the golden rule

- max

$$\begin{aligned} c^* &= (1-s)\,y^* \\ &= (k^*(s))^\alpha - s\,(k^*(s))^\alpha \\ &= (k^*(s))^\alpha - (\delta+n)\,k^* \end{aligned}$$

Comparative statics and the golden rule (CD production function)

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- w.r.t. s

$$\alpha\,(k^*(s))^{\alpha-1}\frac{dk^*}{ds} - (\delta+n)\frac{dk^*}{ds} = 0$$

Comparative statics and the golden rule (CD production function)

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- w.r.t. s

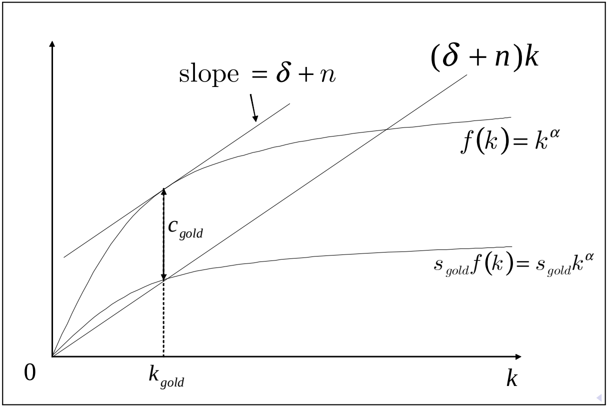
$$\alpha\,(k^*(s))^{\alpha-1}\frac{dk^*}{ds} - (\delta+n)\frac{dk^*}{ds} = 0$$

- k_{gold}

$$k_{gold} \overset{!}{=} \left(\frac{\alpha}{\delta+n}\right)^{\frac{1}{1-\alpha}}.$$

Comparative statics and the golden rule (CD production function)

$$k_{gold} \overset{!}{=} \left(\frac{\alpha}{\delta + n}\right)^{\frac{1}{1-\alpha}} \text{ and } k^*(s) = \left(\frac{s}{\delta + n}\right)^{\frac{1}{1-\alpha}} \text{ yields } s_{gold} \overset{!}{=} \alpha.$$



Neoclassical production function

Constant returns to scale

A production function $Y = F(K, L)$ is called neoclassical if F has two properties:

1. constant returns to scale and
2. decreasing marginal productivities obeying the Inada conditions.

Neoclassical production function

Constant returns to scale

Definition

A production function F is homogeneous of degree d, if we have

$$F(\tau K, \tau L) = \tau^d F(K, L), K \geq 0, L \geq 0$$

for any $\tau \geq 0$. A production function F exhibits constant returns to scale if it is homogeneous of degree 1.

Neoclassical production function

Constant returns to scale

Problem

Prove that the production function given by

$$F(K,L) = \left[\alpha K^{-\rho} + (1-\alpha) L^{-\rho}\right]^{-1/\rho}, 0 < \alpha < 1, \rho > -1, \rho \neq 0$$

exhibits constant returns to scale.

Problem

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exhibits constant returns to scale.

$$\begin{aligned} F(\tau K, \tau L) &= \left[\alpha (\tau K)^{-\rho} + (1-\alpha)(\tau L)^{-\rho}\right]^{-1/\rho} \\ &= \left[\alpha \tau^{-\rho} K^{-\rho} + (1-\alpha) \tau^{-\rho} L^{-\rho}\right]^{-1/\rho} \\ &= \left(\tau^{-\rho}\left[\left(\alpha K^{-\rho} + (1-\alpha) L^{-\rho}\right)\right]\right)^{-1/\rho} \\ &= \left(\tau^{-\rho}\right)^{-1/\rho}\left[\left(\alpha K^{-\rho} + (1-\alpha) L^{-\rho}\right)\right]^{-1/\rho} \\ &= \tau^{-\rho\cdot(-1/\rho)} F(K,L) = \tau F(K,L) \end{aligned}$$

Neoclassical production function

Constant returns to scale

Problem

Can you show that the Leontief production function, given by

$$Y = F(K, L) = \min(AK, BL),$$

also obeys constant returns to scale?

Problem

Can you show that the Leontief production function, given by

$$Y = F(K, L) = \min(AK, BL),$$

also obeys constant returns to scale?

First, we note $F(0 \cdot K, 0 \cdot L) = 0 \cdot F(K, L)$, so that the equality holds for $\tau = 0$. For $\tau > 0$, we have

$$AK \leq BL \Leftrightarrow \tau(AK) \leq \tau(BL)$$

and hence

$$\begin{aligned} F(\tau K, \tau L) &= \min(A(\tau K), B(\tau L)) \\ &= \min(\tau(AK), \tau(BL)) \\ &= \tau \min(AK, BL). \end{aligned}$$

Problem

Can you prove $F(0,0) = 0$ *for any constant-returns production function* F*?*

Problem

Can you prove $F(0,0) = 0$ for any constant-returns production function F?

For $\tau := 0$, the desired equation follows easily:

$$F(0,0) = F(0 \cdot K, 0 \cdot L) = 0 \cdot F(K,L) = 0$$

- For $\tau := \frac{1}{L}$, we obtain

$$F\left(\frac{K}{L},1\right)=F\left(\frac{1}{L}K,\frac{1}{L}L\right)=\frac{1}{L}F\left(K,L\right).$$

Neoclassical production function

Constant returns to scale

- For $\tau := \frac{1}{L}$, we obtain

$$F\left(\frac{K}{L}, 1\right) = F\left(\frac{1}{L}K, \frac{1}{L}L\right) = \frac{1}{L}F(K, L).$$

- Defining

$$\begin{aligned} k &:= \frac{K}{L}, \\ y &:= \frac{Y}{L}, \text{ and} \\ f(k) &:= F(k, 1) \end{aligned}$$

Neoclassical production function

Constant returns to scale

- For $\tau := \frac{1}{L}$, we obtain

$$F\left(\frac{K}{L}, 1\right) = F\left(\frac{1}{L}K, \frac{1}{L}L\right) = \frac{1}{L}F(K, L).$$

- Defining

$$\begin{aligned} k &:= \frac{K}{L}, \\ y &:= \frac{Y}{L}, \text{ and} \\ f(k) &:= F(k, 1) \end{aligned}$$

- yields

$$y = \frac{F(K, L)}{L} = F\left(\frac{K}{L}, 1\right) = f(k)$$

Problem

Determine the intensive form of the CES production function.

Neoclassical production function

Constant returns to scale

Problem

Determine the intensive form of the CES production function.

$$
\begin{aligned}
f(k) &= F\left(\frac{K}{L}, 1\right) \\
&= \left[\alpha\left(\frac{K}{L}\right)^{-\rho} + (1-\alpha)\cdot 1^{-\rho}\right]^{-1/\rho} \\
&= \left[\alpha k^{-\rho} + (1-\alpha)\right]^{-1/\rho}
\end{aligned}
$$

- Which are equal?

$$\frac{\partial^2 F}{(\partial K)^2} \quad : \quad = \frac{\partial \frac{\partial F}{\partial K}}{\partial K},$$

$$\frac{\partial^2 F}{(\partial L)^2} \quad : \quad = \frac{\partial \frac{\partial F}{\partial L}}{\partial L},$$

$$\frac{\partial^2 F}{\partial K \partial L} \quad : \quad = \frac{\partial \frac{\partial F}{\partial L}}{\partial K}, \text{ and}$$

$$\frac{\partial^2 F}{\partial L \partial K} \quad : \quad = \frac{\partial \frac{\partial F}{\partial K}}{\partial L}$$

- Which are equal?

$$\frac{\partial^2 F}{(\partial K)^2} := \frac{\partial \frac{\partial F}{\partial K}}{\partial K},$$

$$\frac{\partial^2 F}{(\partial L)^2} := \frac{\partial \frac{\partial F}{\partial L}}{\partial L},$$

$$\frac{\partial^2 F}{\partial K \partial L} := \frac{\partial \frac{\partial F}{\partial L}}{\partial K}, \text{ and}$$

$$\frac{\partial^2 F}{\partial L \partial K} := \frac{\partial \frac{\partial F}{\partial K}}{\partial L}$$

- Notation:

$$\frac{\partial F}{\partial K} = \left.\frac{\partial F}{\partial K}\right|_{(K,L)}$$

Neoclassical production function

Constant returns to scale

Lemma

Let F be homogeneous of degree 1. Then,

1. *the marginal productivities are homogeneous of degree 0 :*

$$\left.\frac{\partial F}{\partial K}\right|_{(\tau K,\tau L)} = \left.\frac{\partial F}{\partial K}\right|_{(K,L)} \text{ and}$$

$$\left.\frac{\partial F}{\partial L}\right|_{(\tau K,\tau L)} = \left.\frac{\partial F}{\partial L}\right|_{(K,L)}$$

(Generalization: Let F be homogeneous of degree d. Then,

$$\left.\frac{\partial F}{\partial K}\right|_{(\tau K,\tau L)} = \tau^{d-1} \left.\frac{\partial F}{\partial K}\right|_{(K,L)} \text{)}$$

Proof: Derivative of $F(\tau K, \tau L) = \tau F(K, L)$ with respect to K :

$$\begin{aligned}
& \frac{\partial F(\tau K, \tau L)}{\partial K} = \frac{\partial [\tau F(K, L)]}{\partial K} \\
\Leftrightarrow \quad & \frac{\partial F(\tau K, \tau L)}{\partial (\tau K)} \frac{d(\tau K)}{dK} = \tau \frac{\partial [F(K, L)]}{\partial K} \\
\Leftrightarrow \quad & \frac{\partial F(\tau K, \tau L)}{\partial (\tau K)} = \frac{\partial [F(K, L)]}{\partial K} \\
\Leftrightarrow \quad & \left. \frac{\partial F}{\partial K} \right|_{(\tau K, \tau L)} = \left. \frac{\partial F}{\partial K} \right|_{(K, L)} .
\end{aligned}$$

Analogously, forming the derivative with respect to L leads to

$$\frac{\partial F(\tau K, \tau L)}{\partial L} = \frac{\partial [\tau F(K, L)]}{\partial L} \Leftrightarrow \left. \frac{\partial F}{\partial L} \right|_{(\tau K, \tau L)} = \left. \frac{\partial F}{\partial L} \right|_{(K, L)} .$$

Neoclassical production function

Constant returns to scale

Lemma

Let F be homogeneous of degree 1.

1. ...
2. *the second-order derivatives are homogenous of degree* -1 :

$$\tau \left.\frac{\partial^2 F}{(\partial K)^2}\right|_{(\tau K,\tau L)} = \left.\frac{\partial^2 F}{(\partial K)^2}\right|_{(K,L)} \text{ and}$$

$$\tau \left.\frac{\partial^2 F}{(\partial L)^2}\right|_{(\tau K,\tau L)} = \left.\frac{\partial^2 F}{(\partial L)^2}\right|_{(K,L)}$$

Proof similar to 1. (form the derivative of $\left.\frac{\partial F}{\partial K}\right|_{(\tau K,\tau L)} = \left.\frac{\partial F}{\partial K}\right|_{(K,L)}$)

Neoclassical production function

Constant returns to scale

Lemma

Let F be homogeneous of degree 1.

1. ...
2. ...
3. *the marginal productivities can be expressed as functions of capital per head,* k :

$$\frac{\partial F}{\partial K} = \frac{df}{dk} \text{ and} \tag{1}$$

$$\frac{\partial F}{\partial L} = f(k) - k\frac{df}{dk} =: \omega(k) \tag{2}$$

Neoclassical production function

Constant returns to scale

$$\frac{\partial F}{\partial K} = \frac{df}{dk} \text{ and}$$
$$\frac{\partial F}{\partial L} = f(k) - k\frac{df}{dk} =: \omega(k)$$

Proof:

$$\frac{\partial F(K,L)}{\partial K} = \frac{\partial\left[Lf\left(\frac{K}{L}\right)\right]}{\partial K} = L\frac{df}{d\left(\frac{K}{L}\right)}\frac{\partial\left(\frac{K}{L}\right)}{\partial K}$$
$$= L\frac{df}{dk}\frac{1}{L} = \frac{df}{dk}.$$

Neoclassical production function

Constant returns to scale

Problem

Show $\frac{\partial F}{\partial L} = f(k) - k\frac{df}{dk}$. *Hint: Beginn with* $\frac{\partial F(K,L)}{\partial L} = \frac{\partial\left(Lf\left(KL^{-1}\right)\right)}{\partial L}$ *and apply the product rule of differentiation.*

Neoclassical production function

Constant returns to scale

Problem

Show $\frac{\partial F}{\partial L} = f(k) - k\frac{df}{dk}$. *Hint: Beginn with* $\frac{\partial F(K,L)}{\partial L} = \frac{\partial\left(Lf\left(KL^{-1}\right)\right)}{\partial L}$ *and apply the product rule of differentiation.*

$$\begin{aligned} \frac{\partial F(K,L)}{\partial L} &= \frac{\partial\left(Lf\left(KL^{-1}\right)\right)}{\partial L} \\ &= f\left(KL^{-1}\right) + L\frac{\partial f}{\partial\left(KL^{-1}\right)}\frac{d\left(KL^{-1}\right)}{dL} \\ &= f(k) + L\frac{\partial f}{\partial k}(-1)KL^{-2} \\ &= f(k) - \frac{df}{dk}k. \end{aligned}$$

Here, ω is reminiscent of w as in *wage* rate.

$$\omega(k) = \underbrace{f(k)}_{\substack{\text{output by one worker} \\ \text{with capital } k}} - \underbrace{\underbrace{k}_{\substack{\text{capital} \\ \text{used by worker}}} \underbrace{\frac{df}{dk}}_{\substack{\text{marginal-product} \\ \text{price for capital}}}}_{\substack{\text{payments for capital} \\ \text{used by worker}}}$$

Lemma

Let F be homogeneous of degree 1. Then,

1. ...
2. ...
3. $$\frac{\partial F}{\partial K}=\frac{df}{dk} \text{ and } \frac{\partial F}{\partial L}=f(k)-k\frac{df}{dk}=:\omega(k)$$
4. *Euler's theorem holds:*
$$F(K,L)=\frac{\partial F}{\partial K}K+\frac{\partial F}{\partial L}L$$

Problem

Prove! Hint: You need the results from item 3.

Euler's theorem:

$$\begin{aligned}\frac{\partial F}{\partial K}K+\frac{\partial F}{\partial L}L &= \frac{df}{dk}K+\left(f\left(k\right)-k\frac{df}{dk}\right)L \\ &= \frac{df}{dk}K+f\left(k\right)L-\frac{K}{L}\frac{df}{dk}L \\ &= Lf\left(k\right) \\ &= F\left(K,L\right).\end{aligned}$$

Lemma

Let F be homogeneous of degree 1*. Then,*

5. *the second-order derivatives relate to each other in a simple manner:*

$$
\begin{aligned}
\frac{\partial^2 F}{\partial K \partial L} &= -k \frac{\partial^2 F}{(\partial K)^2}, \\
\frac{\partial^2 F}{\partial K \partial L} &= -\frac{1}{k} \frac{\partial^2 F}{(\partial L)^2}, \text{ and} \\
\frac{\partial^2 F}{(\partial K)^2} \frac{\partial^2 F}{(\partial L)^2} &= \left(\frac{\partial^2 F}{\partial K \partial L} \right)^2
\end{aligned}
$$

Proof of 5: Differentiate Euler's equation to find

$$\frac{\partial F}{\partial K} = \left(\frac{\partial^2 F}{(\partial K)^2}K + \frac{\partial F}{\partial K}\right) + \frac{\partial^2 F}{\partial K \partial L}L \text{ and}$$
$$\frac{\partial F}{\partial L} = \frac{\partial^2 F}{\partial K \partial L}K + \left(\frac{\partial^2 F}{(\partial L)^2}L + \frac{\partial F}{\partial L}\right)$$

hence

$$\frac{\partial^2 F}{\partial K \partial L} = -k\frac{\partial^2 F}{(\partial K)^2} \text{ and } \frac{\partial^2 F}{\partial K \partial L} = -\frac{1}{k}\frac{\partial^2 F}{(\partial L)^2}$$

implying the third one,

$$\frac{\partial^2 F}{(\partial K)^2}\frac{\partial^2 F}{(\partial L)^2} = \left(\frac{\partial^2 F}{\partial K \partial L}\right)^2.$$

Decreasing marginal productivity

$$\frac{\partial F}{\partial K} > 0 \text{ for } L > 0$$
$$\frac{\partial^2 F}{(\partial K)^2} < 0$$

Inada conditions

$$\lim_{K \to \infty} \frac{\partial F}{\partial K} = 0$$
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& for labor...

Neoclassical production function

f inherits from F

- the marginal product per head

$$\frac{df}{dk} = \frac{\partial F(k,1)}{\partial k} > 0$$

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- Barro & Sala-i-Martin (1999, p. 52)
$$F(0,L) = F(K,0) = 0, \text{ hence } f(0) = F(0,1) = 0$$

Dynamics and steady state (neoclassical production function)

$$\dot{K}_t = sY_t - \delta K_t$$

- K_t — economy's stock of capital,
- $\dot{K}_t$ — change of this stock
- s — saving rate of income Y_t
- δ — depreciation rate

Dynamics and steady state (neoclassical production function)

Dynamics of k :

$$
\begin{aligned}
\dot{k} &= \left(\frac{\dot{K}}{L}\right) \\
&= \frac{\dot{K}L - \dot{L}K}{L^2} \\
&= \frac{\dot{K}}{L} - nk \\
&= \frac{sY - \delta K}{L} - nk \\
&= sf(k) - (n+\delta)k
\end{aligned}
$$

n — (working) population's growth rate

Dynamics and steady state (neoclassical production function)

- Rate

$$\gamma_k = \frac{\dot{k}}{k} = s\frac{f(k)}{k} - (\delta + n), k > 0$$

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  constant
- $0 = \frac{d\left[\frac{sf(k)}{k} - (\delta + n)\right]}{dt}$

Dynamics and steady state (neoclassical production function)

$$
\begin{aligned}
0 &= \frac{d\left[\frac{sf(k)}{k}-(\delta+n)\right]}{dt} \\
&= s\frac{d\frac{f(k)}{k}}{dt} \\
&= s\frac{\frac{df}{dk}\frac{dk}{dt}k-\frac{dk}{dt}f(k)}{k^2} \\
&= s\frac{\frac{df}{dk}k-f(k)}{k}\frac{\frac{dk}{dt}}{k} \\
&= -s\frac{f(k)-\frac{df}{dk}k}{k}\gamma_k \\
&= -s\frac{\frac{\partial F}{\partial L}}{k}\gamma_k \implies 0=\gamma_k
\end{aligned}
$$

Dynamics and steady state (neoclassical production function)

- $0 = \gamma_k$

$$\implies sf(k^*) = (\delta + n) k^*$$

Dynamics and steady state (neoclassical production function)

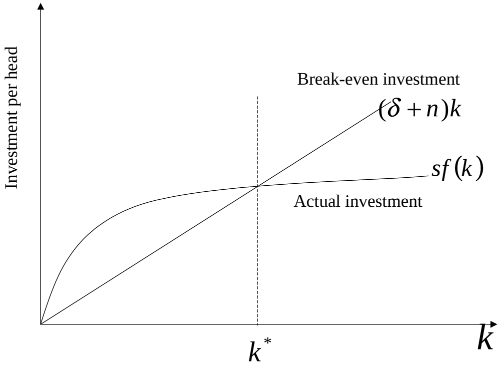
- $0 = \gamma_k$

$$\Longrightarrow sf(k^*) = (\delta + n) k^*$$

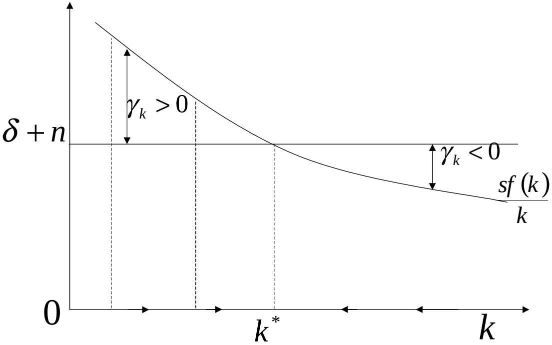
- Output & consumption per head

$$\begin{aligned} y^* &= f(k^*) \text{ and} \\ c^* &= (1-s) y^*, \end{aligned}$$

Dynamics and steady state (neoclassical production function)



Dynamics and steady state (neoclassical production function)



Dynamics and steady state (neoclassical production function)

Problem

Draw the equivalents of the above figures for $s = 0$.

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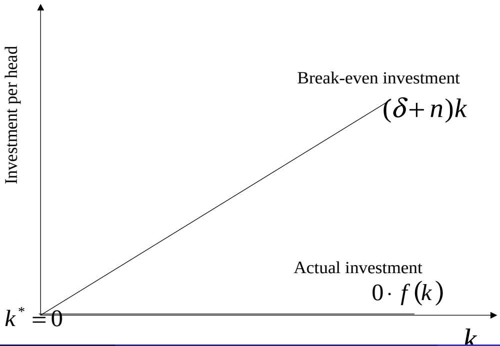
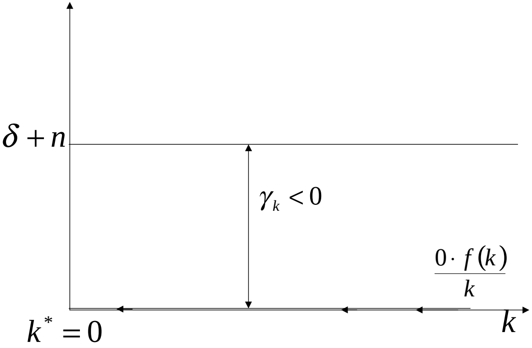


Dynamics and steady state (neoclassical production function)



Dynamics and steady state

$k = 0$ and $k^* > 0$ is where investment = break-even investment.

- For sufficiently small endowments of capital per head $k > 0$, actual investment per head $sf(k)$ is greater than the break-even investment $(\delta + n)k$ by the Inada condition. Hence, $\dot{k} = sf(k) - (n + \delta)k$ is positive and capital per head increases.

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- Summarizing, $sf(k) - (n + \delta)k$ is positive for small k and negative for large ones. Therefore, we should find a k^* in between where $sf(k^*) - (n + \delta)k^*$ is zero. This follows from the so-called intermediate-value theorem which holds for continuous functions. $(sf(k) - (n + \delta)k$ is continuous for $k > 0$.)

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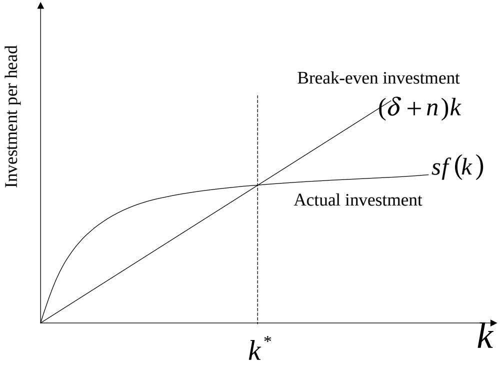
- $k = 0$ is a steady state by $f(0) = 0$.
- Finally, f and hence $sf(k) - (n + \delta)k$ is concave by so that further nulls are excluded.

Comparative statics and the golden rule (neoclassical production function)

$dk^* / ds > 0$



Comparative statics and the golden rule (neoclassical production function)

- Starting from

$$c^*(s) = (1-s) f(k^*(s)) = f(k^*(s)) - (\delta + n) k^*(s)$$

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- Starting from

$$c^* (s) = (1 - s) f (k^* (s)) = f (k^* (s)) - (\delta + n) k^* (s)$$

- Maximize consumption per head

$$\begin{aligned} & f' (k^* (s)) \frac{dk^*}{ds} - (\delta + n) \frac{dk^*}{ds} \overset{!}{=} 0 \\ \Leftrightarrow \quad & f' (k^* (s)) \overset{!}{=} (\delta + n) \ \left(\text{note } \frac{dk^*}{ds} > 0\right). \end{aligned}$$

Comparative statics and the golden rule (neoclassical production function)

- Starting from

$$c^*(s) = (1-s)\, f(k^*(s)) = f(k^*(s)) - (\delta + n)\, k^*(s)$$

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$$\begin{aligned} & f'(k^*(s))\,\frac{dk^*}{ds} - (\delta+n)\,\frac{dk^*}{ds} \overset{!}{=} 0 \\ \Leftrightarrow \quad & f'(k^*(s)) \overset{!}{=} (\delta+n) \ \left(\text{note } \frac{dk^*}{ds} > 0\right). \end{aligned}$$

- Golden rule of capital accumulation

$$f'(k_{gold}) \overset{!}{=} \delta + n$$

Comparative statics and the golden rule (neoclassical production function)