Applied cooperative game theory:

A real-estate model

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April 2010

Overview "A real-estate model"

- Introduction
- XP-values
- Weighted XP-values
- Application: buying a house in the presence of a realtor

Introduction

- last chapter: *PU*-value
- this chapter: exogenous-payments Shapley value obeying a consistency axiom:
  If the exogenous payments happen to be equal to the payoff determined endogenously (i.e., according to the Shapley value), then the endogenous agents also obtain their Shapley values.

XP values

$X \subseteq N$ —> set of exogenous players (civil servants)
$D := N \backslash X$ —> endogenous players (the private sector)

Definition

XP games are tuples

$$(N, v, X, \pi)$$

where

- (N, v) is a TU game,
- X is a strict subset of N, and
- $\pi \in \mathbb{R}^{|X|}$ is a vector specifying a payoff for every member of X.

Axioms for XP-values

X (exogenous payments): For all $i \in X$, we have $\varphi_i(N, v, X, \pi) = \pi_i$.

E (efficiency): We have $\varphi_N(N, v, X, \pi) = v(N)$.

S (symmetry): For all symmetric players $i, j \in D$,
$\varphi_i(N, v, X, \pi) = \varphi_j(N, v, X, \pi)$.

N-$\emptyset$ (null player for $X = \emptyset$): If $i \in N$ is a null player, then
$\varphi_i(N, v, \emptyset, \pi) = 0$.

A (additivity): For any coalition functions $v', v'' \in \mathbb{V}_N$, any payments $\pi', \pi'' \in \mathbb{R}^{|X|}$ and any player i from N, we obtain

$$\varphi_i(N, v' + v'', X, \pi' + \pi'') = \varphi_i(N, v', X, \pi') + \varphi_i(N, v'', X, \pi'').$$

Axioms for XP-values II

M (marginalism): Assume two coalition functions v and z from $\mathbb{V}_N$. Let i be a player from D obeying

$$v\left(S\cup\{i\}\right)-v\left(S\right)=z\left(S\cup\{i\}\right)-z\left(S\right)$$

for all $S\subseteq N\backslash\{i\}$. Then

$$\varphi_i\left(N,v,X,\pi\right)=\varphi_i\left(N,z,X,\pi\right).$$

Our value does not fulfill axiom M:

- The players from D pay π to the players from X but
- enjoy the contributions made by these exogenous players by efficiency.

BF (Brink fairness): Let i and j be players from D that are symmetric in (N,z). Then

$$\varphi_i\left(N,v+z,X,\pi\right)-\varphi_i\left(N,v,X,\pi\right)=\varphi_j\left(N,v+z,X,\pi\right)-\varphi_j\left(N,v,X,\pi\right).$$

Axioms for XP-values III

Lawyer or civil servant is responsible for an increase (or a decrease) of the social product and his renumeration is changed by the very same amount:

SH (shifting): For all $i \in D$, we have

$$\varphi_i \left(N, v + \pi_X, X, \pi\right) = \varphi_i \left(N, v + \pi'_X, X, \pi'\right)$$

for all $\pi, \pi' \in \mathbb{R}^{|X|}$.

It is not difficult to show that axioms X, S, E, and A imply axiom SH. The final axiom is a very important one:

C (consistency): For any player $i \in D$,

$$\varphi_i \left(N, v, X, \left(\varphi_x \left(N, v, \emptyset, \pi\right)\right)_{x \in X}\right) = \varphi_i \left(N, v, \emptyset, \pi\right).$$

If the players in X (happen to) obtain the value dictated by the axioms for games without exogenous players, so do the other players.

Axiomatization I

Theorem

Assuming $X = \emptyset$ (in which case N-$\emptyset$ and N are equivalent) and ignoring π in that case, the Shapley value is characterized by the following sets of axioms for solution φ:

- *E, S, N, and A (Shapley 1953)*
- *E, S, and M (Young 1985)*
- *E, N, and BF (Van den Brink 2001)*

The Shapley value with exogenous payments is denoted by $Sh^{X,\pi}$ and given by

$$Sh_i^{X,\pi}(N,v) = \begin{cases} \pi_i, & i \in X \\ Sh_i(N,v) + \frac{1}{|D|}\left(Sh_X(N,v) - \pi_X\right), & i \in D \end{cases}$$

Axiomatization II

Lemma

Assuming axiom C and any of the two following axiom sets

- *E, S, N-∅, and A or*
- *E, N-∅, and BF*

we obtain

$$Sh_i(N,v) = \varphi_i\left(N, v, X, (Sh_x(N,v))_{x \in X}\right)$$

for all players $i \in D$.

Proof. Either one of the set of axioms obviously imply

$$\varphi_i(N, v, \emptyset, \pi) = Sh_i(N,v).$$

$$\begin{aligned} Sh_i(N,v) &= \varphi_i(N, v, \emptyset, \pi) \text{ (above equation)} \\ &= \varphi_i\left(N, v, X, (\varphi_x(N, v, \emptyset, \pi))_{x \in X}\right) \text{ (axiom C)} \\ &= \varphi_i\left(N, v, X, (Sh_x(N,v))_{x \in X}\right) \text{ (above equation)} \end{aligned}$$

Axiomatization III

Theorem

The Shapley value with exogenous payments is characterized by the axioms X, E, S, N-∅, A, and C.

Proof. It is not difficult to show that $Sh^{X,\pi}$ fulfills all the axioms mentioned in the theorem. Let φ be an XP value. For $i \in X$, axiom X guarantees $\varphi_i(N, v, X, \pi) = \pi_i$. For $i \in D$, we obtain the desired result by

$$\begin{aligned}
& \varphi_i(N, v, X, \pi) \\
= \; & \varphi_i\left(N, v, X, (Sh_x(N, v))_{x \in X}\right) \\
& + \varphi_i\left(N, 0, X, (\pi_x)_{x \in X} - (Sh_x(N, v))_{x \in X}\right) \text{ (axiom A)} \\
= \; & Sh_i(N, v) + \varphi_i\left(N, 0, X, (\pi_x)_{x \in X} - (Sh_x(N, v))_{x \in X}\right) \text{ (lemma 3)} \\
= \; & Sh_i(N, v) + \frac{1}{|D|}\left(Sh_X(N, v) - \pi_X\right) \text{ (axioms E, S)}
\end{aligned}$$

□

Axiomatization IV

Theorem

The Shapley value with exogenous payments is characterized by the axioms X, E, BF, N-∅, SH, and C.

Proof. $Sh^{X,\pi}$ also fulfills the axioms BF and SH. Consider the coalition function $z := \pi_X - Sh_X(N, v)$. Then any two players i and j from D are symmetric in (N, z) and Brink fairness implies

$$\varphi_i(N, v+z, X, \pi) - \varphi_i(N, v, X, \pi) = \varphi_j(N, v+z, X, \pi) - \varphi_j(N, v, X, \pi).$$

Axiomatization V

Fix $i \in D$ and sum this equation for all $j \in D$. Using axioms X and E and hence $\varphi_D(N, v, X, \pi) = v(N) - \pi_X$, we find

$$\varphi_i(N, v, X, \pi) = \varphi_i(N, v + z, X, \pi) + \frac{1}{|D|}(Sh_X(N, v) - \pi_X).$$

The equations

$$\begin{aligned} Sh_i(N, v) &= \varphi_i\left(N, v, X, (Sh_x(N, v))_{x \in X}\right) \text{ (above lemma)} \\ &= \varphi_i\left(N, v - Sh_X(N, v) + \pi_X, X, (\pi_x)_{x \in X}\right) \text{ (axiom SH)} \\ &= \varphi_i(N, v + z, X, \pi) \end{aligned}$$

provide the final bit of our proof.

Application: Basic income I

Suggestion Andre Casajus:
Duplicate a TU game (N, v) (which stands for the economy) in the following manner.

- On the basis of player set $N = \{1, ..., n\}$, we define a set $N' := \{1', ..., n'\}$ with $|N| = |N'|$ and a player set $\hat{N} := N \cup N'$.
- We define a TU game $(\hat{N}, \hat{v})$ by $\hat{v}(K) = v(K \cap N)$. Thus, every player from N' is a null player in $(\hat{N}, \hat{v})$ and we have $Sh_i(N, v) = Sh_i(\hat{N}, \hat{v})$ for all players $i \in N$.
- Every player $i' \in N'$ is an exogenous player and obtains the payoff (the basic income) $\pi_{i'}$.

Application: Basic income II

Obviously, the dash-player is just a copy of a player from N invented for the purpose of collecting the basic income. We find the payoffs

$$Sh_i^{N',\pi}\left(\hat{N},\hat{v}\right) = \begin{cases} \pi_i, & i \in N' \\ Sh_i\left(N,v\right) - \frac{\pi_{N'}}{|N|}, & i \in N \end{cases}$$

Thus, the overall payoff for a player $i \in N$ and his clone $i' \in N$ is

$$\underbrace{Sh_i\left(N,v\right)}_{\text{market income}} + \underbrace{\pi_{i'}}_{\text{basic income}} - \underbrace{\frac{\pi_{N'}}{|N|}}_{\text{tax}}.$$

Therefore, the introduction of a basic-income system makes an agent better off iff his basic payoff is greater than the average basic payoff.

Weighted XP values

definition of weighted Shapley value with exogenous payments

A weighted XP game is a tuple (N, v, X, π, w) where (N, v, X, π) is an XP game and $w = (w_i)_{i \in D}$ a tuple of strictly positive numbers.
The weighted Shapley value with exogenous payments is given by

$$Sh_i^{X,\pi,w}(N,v) = \begin{cases} \pi_i, & i \in X \\ Sh_i(N,v) + \frac{w_i}{\sum_{d \in D} w_d}\left(Sh_X(N,v) - \pi_X\right), & i \in D \end{cases}$$

Weighted XP values

axiomatization I

The value $Sh_i^{X,\pi,w}$ can be axiomatized:

X (exogenous payments): For all $i \in X$, we have $\varphi_i(N,v,X,\pi,w) = \pi_i$.

E (efficiency): We have $\varphi_N(N,v,X,\pi,w) = v(N)$.

N-$\emptyset$ (null player for $X = \emptyset$): If $i \in N$ is a null player, then $\varphi_i(N,v,\emptyset,\pi,w) = 0$.

A (additivity): For any coalition functions $v', v'' \in \mathbb{V}_N$, any payments $\pi', \pi'' \in \mathbb{R}^{|X|}$ and any player i from N, we obtain

$$\varphi_i\left(N, v'+v'', X, \pi'+\pi'', w\right) = \varphi_i\left(N, v', X, \pi', w\right) + \varphi_i\left(N, v'', X, \pi'', w\right)$$

C (consistency): For any player $i \in D$,

$$\varphi_i\left(N, v, X, \left(\varphi_x(N,v,\emptyset,\pi,w)\right)_{x\in X}, w\right) = \varphi_i(N,v,\emptyset,\pi,w).$$

Weighted XP values

axiomatization II

The symmetry axiom has to take the weights into account:

S (symmetry): For all symmetric players $i, j \in D$ obeying $w_i = w_j$,

$$\varphi_i(N, v, X, \pi, w) = \varphi_j(N, v, X, \pi, w).$$

IR (irrelevance): For all $i \in D$ and all $\pi, \pi' \in \mathbb{R}^{|X|}$, $w, w' \in \mathbb{R}^{|D|}$, we have

$$\varphi_i(N, v, \emptyset, \pi, w) = \varphi_i\left(N, v, \emptyset, \pi', w'\right).$$

W (weighing): For all players $i, j \in D$,

$$w_i \varphi_j(N, 0, X, \pi, w) = w_j \varphi_i(N, 0, X, \pi, w).$$

Theorem

The weighted Shapley value with exogenous payments is characterized by the axioms (given in **this** *section) X, E, S, N-$\emptyset$, A, C, IR, and W.*

Application: buying a house in the presence of a realtor I

- A seller 's reservation price for a house r is below the buyer's willingness to pay w. Thus, the gains from trade are positive, $w - r > 0$.
- The seller and the buyer need the realtor to come into contact. Therefore, the coalition function v is given by $N = \{S, B, A\}$ and

$$v(K) = \begin{cases} w - r, & K = N, \\ 0, & \text{otherwise} \end{cases}$$

- The realtor charges a fee π which is a fraction f of the house price p for his service, $\pi = fp$
- This payoff to the realtor π is payable by the buyer and the seller in proportions $g_S = 0$ and $g_B = 1$.

Application: buying a house in the presence of a realtor II

- At the first stage, the realtor decides on f.
- At the second stage, the seller and the buyer decide whether they will indeed do business with each other. If not, the game ends with payoffs 0 for every player.
- At the third stage, the seller and the buyer engage in a bargaining process, the outcome of which is determined by the weighted XP value.

Application: buying a house in the presence of a realtor III

third stage: bargaining

Abbreviating $Sh^{\{A\},\pi,(0,1)}(N,v)$ by ξ, we find

$$\begin{aligned}
\xi &= (\xi_S, \xi_B, \xi_A) \\
&= \left(\frac{w-r}{3}, \frac{w-r}{3} + 1 \cdot \left(\frac{w-r}{3} - \pi\right), \pi\right) \\
&= \left(\frac{w-r}{3}, \frac{2}{3}(w-r) - \pi, \pi\right)
\end{aligned}$$

So far, the realtor's fee π is exogenous so that we could apply our formula.

Application: buying a house in the presence of a realtor IV

third stage: bargaining

However, the model allows to calculate the "equilibrium" house price p^* so that payments to the realtor are now endogenous at fp^*. Indeed, the seller's rent is $p - r = \xi_S$ so that we obtain

$$\begin{aligned} p^* &= \xi_S^*(f) + r = \frac{w-r}{3} + r \\ &= \frac{2}{3}r + \frac{1}{3}w \end{aligned}$$

and

$$\begin{aligned} \xi_B^*(f) &= w - p^* - fp^*, \\ \pi^*(f) &= fp^* \end{aligned}$$

Application: buying a house in the presence of a realtor V

second stage: do they have a deal

- The seller is willing to sell his house if $\xi_S \geq 0$ holds which is true by $w - r > 0$.
- The buyer will buy if $w - p^* - fp^* \geq 0$ or

$$f \leq \frac{w - \left(\frac{w-r}{3} + r\right)}{\frac{w-r}{3} + r} = \frac{2\left(w - r\right)}{2r + w}$$

  hold.
- For any $f \geq 0$, the realtor is happy to help in the deal.
  Thus, the deal can be struck for any fee percentage f obeying

$$0 \leq f \leq \frac{2\left(w - r\right)}{2r + w}.$$

Application: buying a house in the presence of a realtor VI

first stage: setting the realtor's fraction

The real-estate agent maximizes her profit by letting

$$f^* = \frac{2(w - r)}{2r + w}$$

As expected, we find $\frac{df^*}{dw} > 0$ and $\frac{df^*}{dr} < 0$.