Applied cooperative game theory:

The size of government

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Overview "The Size of Government"

- Introduction
- Private-sector and mixed-economy coalition functions
- Introducing inefficiency
- The pay-and-use value: definition and axiomatization
- Suggesting public-service vectors: a noncooperative game
- A simple three-player example

Introduction

- Most economies are mixed economies, with a private sector and a public sector.
- The public sector is often engaged in activities
  - such as education, transport, energy and water supply
  - which can also be undertaken by the private sector
  - in a less costly manner.
- Private agents pay civil servants and benefit from their service. However,
  - payment and benefit may differ between private agents who
  - disagree on the optimal extent of the public sector.
- Can rent seeking be one reason for an excessive public sector?
- Boundaries
  - of the firm: Coase 1937, Wiese 2005
  - of government: ??

Economy with a public sector

Definition

An economy with a public sector is a tuple (N, v, C, π) where

- (N, v) is a TU game (the economy),
- C is a proper subset of N, and
- $\pi = (\pi_c)_{c \in N}$ is a vector specifying an exogenous payoff for every member of C and for all the others, too.

(C, π) is called the public-service vector.

Why $C \subsetneq N$? Payments for the civil servants are exogenous so that budget balancing requires the existence of a private sector.

Private-sector and mixed-sector coalition function

On the basis of an economy with a public sector we define two games that are basically equivalent:

Definition

A mixed-economy game is a TU game $\left(N, m^{v,C,\pi}\right)$ with

$$\begin{aligned} m^{v,C,\pi}(K) &= \begin{cases} \left(v(K \cup C) - \pi_C\right) + \sum_{c \in K \cap C} \pi_c, & K \backslash C \neq \emptyset \\ \sum_{c \in K \cap C} \pi_c, & K \backslash C = \emptyset \end{cases} \\ &= \begin{cases} v(K \cup C) - \sum_{c \in C \backslash K} \pi_c, & K \backslash C \neq \emptyset \\ \sum_{c \in K} \pi_c, & K \backslash C = \emptyset \end{cases} \end{aligned}$$

Definition

A private-sector game is a TU game $\left(N \backslash C, p^{v,C,\pi}\right)$ with

$$p^{v,C,\pi}(S) = \begin{cases} v(S \cup C) - \pi_C, & S \neq \emptyset \\ 0, & S = \emptyset \end{cases}$$

Private-sector and mixed-sector coalition function II

- Equivalence of $m^{v,C,\pi}$ and $p^{v,C,\pi}$:

$$\begin{aligned} m^{v,C,\pi}(K) &= p^{v,C,\pi}(K \backslash C) + \sum_{c \in K \cap C} \pi_c, K \subseteq N, \text{ and} \\ C &= \emptyset \Rightarrow m^{v,C,\pi} = p^{v,C,\pi} = v \text{ and} \\ i &\notin C \Rightarrow Sh_i\left(m^{v,C,\pi}\right) = Sh_i\left(p^{v,C,\pi}\right) \end{aligned}$$

- $p^{v,C,\pi}$ is close to coalition functions defined in Aumann and Dreze (1974) and Peleg (1986).
  - They assume that players from S can choose the players from C they want to use and pay for.
  - We do not allow for this choice because all people have to pay taxes irrespective of whether they do actually use the services.

Incorporating lazyness

- $t \leq 1$ is the work effort exercised by a civil servant.
- A mixed workforce is a function $s : N \rightarrow [0, 1]$ obeying
  - $s_i := s(i) \in \{0, 1\}$ for all $i \notin C$ and
  - $s_i := s(i) \in \{0, t\}$ for all $i \in C$.
- The multilinear extension (MLE) is defined by

$$\begin{aligned} u_T^{MLE}(s) \quad &: \quad = \prod_{i \in T} s_i,\ T \subseteq N,\ T \neq \emptyset \text{ (unanimity games) and} \\ v^{MLE}(s) \quad &: \quad = \sum_{T \in 2^N \backslash \{\emptyset\}} d^v(T) \cdot \prod_{i \in T} s_i \text{ (any games).} \end{aligned}$$

Economy with lazy public servants

Definition

An economy with lazy public servants is a tuple (N, v, C, π, t) where (N, v, C, π) is an economy with a public sector and t is the efficiency parameter for the civil servants obeying $0 \leq t \leq 1$.

Definition

Given an economy with lazy public servants (N, v, C, π, t), a private-sector game is a TU game $\left(N \backslash C, p^{v,C,\pi,t}\right)$ given by $p^{v,C,\pi,t} : 2^{N \backslash C} \to \mathbb{R}$ and

$$p^{v,C,\pi,t}(S) = \begin{cases} \sum_{T \in 2^{S \cup C} \backslash \{\emptyset\}} d^v(T)\, t^{|C \cap T|} - \pi_C, & S \neq \emptyset \\ 0, & S = \emptyset \end{cases}$$

Axioms for public-service values I

Axiom X (exogenous payments): For all $i \in C$, we have

$$\varphi_i(N, v, C, \pi, t) = \pi_i.$$

Axiom N (Null player): For any player $i \in N \backslash C$ that is a null player in (N, v),

$$\begin{aligned}
\varphi_i(N, v, C, \pi, t) &= \frac{p^{v,C,\pi,t}(i) - \pi_C}{|N \backslash C|} \\
&= \frac{\sum_{T \in 2^C \backslash \{\emptyset\}} d^v(T) \cdot t^{|C \cap T|} - \pi_C}{|N \backslash C|} \\
&= \frac{v^{MLE}\left(\underbrace{t, ..., t}_{\text{civil servants}}, \underbrace{0, ..., 0}_{\text{private agents}}\right) - \pi_C}{|N \backslash C|}.
\end{aligned}$$

Axioms for public-service values II

Axiom E (efficiency): We have

$$\begin{aligned}
\varphi_N(N, v, C, \pi, t) &= p^{v,C,\pi,t}(N \backslash C) + \pi_C \\
&= \sum_{T \in 2^N \backslash \{\emptyset\}} d^v(T)\, t^{|C \cap T|} \\
&= v^{MLE}\Bigg(\underbrace{t, \ldots, t}_{\text{civil servants}}, \underbrace{1, \ldots, 1}_{\text{private agents}}\Bigg)
\end{aligned}$$

Axiom S (symmetry): For all symmetric players $i, j \in N \backslash C$,

$$\varphi_i(N, v, C, \pi, t) = \varphi_j(N, v, C, \pi, t).$$

Axiom A (additivity): For any coalition functions $v', v'' \in G_N$, any payments π', $\pi'' \in \mathbb{R}^n$ and any player i from N,

$$\varphi_i\left(N, v' + v'', C, \pi' + \pi'', t\right) = \varphi_i\left(N, v', C, \pi', t\right) + \varphi_i\left(N, v'', C, \pi'', t\right).$$

The pay-and-use value

Theorem

There exists one and only one public-service value that satisfies the axioms X, N, E, S, and A. It is called PU-value and is given by

$$PU_i\left(N, v, C, \pi, t\right) = \begin{cases} \pi_i, & i \in C \\ Sh_i\left(N \backslash C, p^{v,C,\pi,t}\right), & i \notin C \end{cases}$$

Rank-order interpretation for $\left(N \backslash C, p^{v,C,\pi}\right)$:

- The civil servants enter first,
- then the private agents.
- The first player j from $N \backslash C$ to join the C-players obtains the marginal contribution

$$p^{v,C,\pi,t}\left(j\right) - \pi_C.$$

Suggesting public-service vectors: a noncooperative game

Definition

The tuple $\left(N, v, (r_i)_{i\in N}\right)$ is called an economy (with emigration) where $r_i \in \mathbb{R}$ is the foreign reservation payoff for agent $i \in N$.

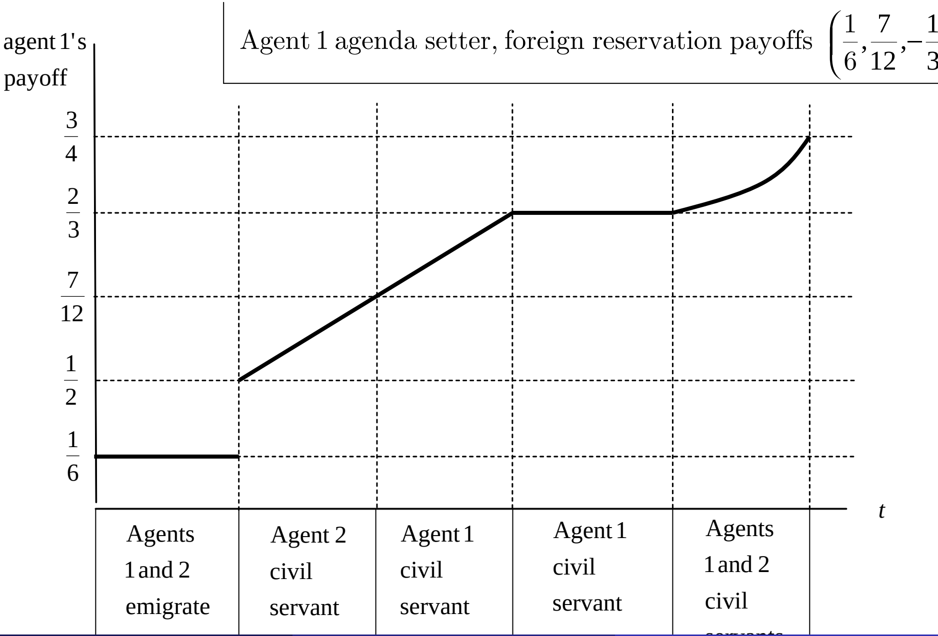
In rough terms, we define a five-stage game:

- Nature picks an agenda setter $\hat{\imath} \in N$.
- $\hat{\imath}$ suggests a public-service vector (C, π).
- All agents $i \in N$ stay (not emigrate) or go (emigrate), $e(i) \in \{s, g\}$. Let $E := \{i \in N : e(i) = g\}$. In case of $C = \emptyset$, $E \cap C \neq \emptyset$, or $E = N \backslash C$ the game is over and $C^* := \emptyset$ (together with any π^*) and $E^* := E$ the outcome.
- All agents $i \in N \backslash E$ cast a vote on (C, π), $a(i) \in \{yes, no\}$. in case of less than 50% yes-votes, the game is over.
- All agents $c \in C$ accept or decline the offer, $p(c) \in \{acc, dec\}$. If any of them declines, the game is over.

A simple three-player example

$$N = \{1,2,3\} \text{ and unanimity game } u_{\{1,2\}}$$

- A: no civil servants: $PU\left(\{1,2,3\}, u_{\{1,2\}}, \emptyset, \pi, t\right) = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$.
- B: productive player 1 is civil servant:
  $PU\left(\{1,2,3\}, u_{\{1,2\}}, \{1\}, \pi, t\right) = \left(\pi_1, t - \frac{\pi_1}{2}, -\frac{\pi_1}{2}\right)$.
- C: productive player 2 is civil servant:
  $PU\left(\{1,2,3\}, u_{\{1,2\}}, \{2\}, \pi, t\right) = \left(t - \frac{\pi_2}{2}, \pi_2, -\frac{\pi_2}{2}\right)$.
- D: unproductive player 3 is civil servant:
  $PU\left(\{1,2,3\}, u_{\{1,2\}}, \{3\}, \pi, t\right) = \left(\frac{1}{2} - \frac{\pi_3}{2}, \frac{1}{2} - \frac{\pi_3}{2}, \pi_3\right)$.
- E: two productive players 1 and 2 are civil servants:
  $PU\left(\{1,2,3\}, u_{\{1,2\}}, \{1,2\}, \pi, t\right) = \left(\pi_1, \pi_2, t^2 - \pi_1 - \pi_2\right)$.
- F: productive player 1 and unproductive player 3 are civil servants:
  $PU\left(\{1,2,3\}, u_{\{1,2\}}, \{1,3\}, \pi, t\right) = \left(\pi_1, t - \pi_1 - \pi_3, \pi_3\right)$.
- G: productive player 2 and unproductive player 3 are civil servants:
  $PU\left(\{1,2,3\}, u_{\{1,2\}}, \{2,3\}, \pi, t\right) = \left(t - \pi_2 - \pi_3, \pi_2, , \pi_3\right)$.


Agent 1 agenda setter, foreign reservation payoffs
agent 1's payoff
3/4
2/3
7/12
1/2
1/6
t
Agents 1 and 2 emigrate
Agent 2 civil servant
Agent 1 civil servant
Agent 1 civil servant
Agents 1 and 2 civil