Applied cooperative game theory:
Firms and markets

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Overview part E: Extensions and exogenous payoffs

- Firms and markets
- The Size of Government
- A real-estate model

Overview "Firms and markets"

- Introduction
- Extensions of coalition functions
- The model
- The two-player case
- Conclusions

Introduction I

- Boundaries of the firms: What kind of economic activity is conducted through markets and what kind is conducted through firms?
  - Ronald Coase (1937) (Nobel prize winner 1991) and
  - Oliver Williamson (Nobel prize winner 2009)
- In this chapter, a firm is modeled by way of an employment relation between players.
  - Every player has an endowment of 100% of his time.
  - Giving part of his time to other players means that he is a worker for others who are employers.
  - A player can be both a worker and an employer.

Introduction II

- Market inefficiency:
  Gains from trade may remain unexploited because of
  - principal-agent problems of the hidden-information variety (lemons)
  - uncertainty about reservation prices

  Market inefficiencies are reflected by partitions on the player set
  —> chance decides which partition will form.
- Organizational inefficiency (costs):
  principal-agent problems of hidden actions/team-production problems

Introduction III

- Our model: baker (B) and chocolate maker (C) with coalition function

$$
\begin{aligned}
v(B) &= 80, \\
v(C) &= 40, \\
v(B, C) &= 200.
\end{aligned}
$$

- Questions:
  - Will the two agents produce separately and buy or sell chocolate or bread on the market?
  - If a firm turns out to be optimal, will the baker employ the chocolate maker or vice versa?
  - Can an economic situation be imagined where both agents found firms, i.e. where the baker employs the chocolate maker and the chocolate maker employs the baker?

Extensions of coalition functions I

Part-time coalition:
Which worth can the baker and the chocolate maker produce if they work together (in a firm, say) assuming

- the baker spends $\frac{1}{2}$ of his time and
- the chocolate maker $\frac{1}{3}$ of his time.
  $\left(\frac{1}{2}, \frac{1}{3}\right)$ is an example of a part-time coalition.

In case of $s \in \{0, 1\}^N$, we identify s with

$$\mathbf{K}(s) := \{i \in N : s_i = 1\}.$$

Extensions of coalition functions II

- An extension of a coalition function v on N is a function
$$v^{ext}:\mathbb{R}_+^N \to \mathbb{R}$$
obeying
$$v^{ext}(s) = v(\mathbf{K}(s)),\, s \in \{0,1\}^N.$$
- The multilinear extension (MLE) is defined by
$$v^{MLE}(s) := \sum_{T \in 2^N \backslash \{\emptyset\}} d^v(T) \cdot \prod_{i \in T} s_i$$
- The Lovasz extension is given by
$$v^{\ell}(s) := \sum_{T \in 2^N \backslash \{\emptyset\}} d^v(T) \cdot \min_{i \in T} s_i.$$
- Comparisons: $u_{\{1,2\}}$ with $s = \left(\frac{1}{2}, \frac{1}{3}\right)$ or $s' = (2,3)$ yield
  - $u_{\{1,2\}}^{MLE}(s) = \frac{1}{2} \cdot \frac{1}{3}$ and $v^{MLE}(s) = 2 \cdot 3$, respectively, and
  - $u_{\{1,2\}}^{\ell}(s) = \frac{1}{3}$ and $v^{\ell}(s) = 2$, respectively.

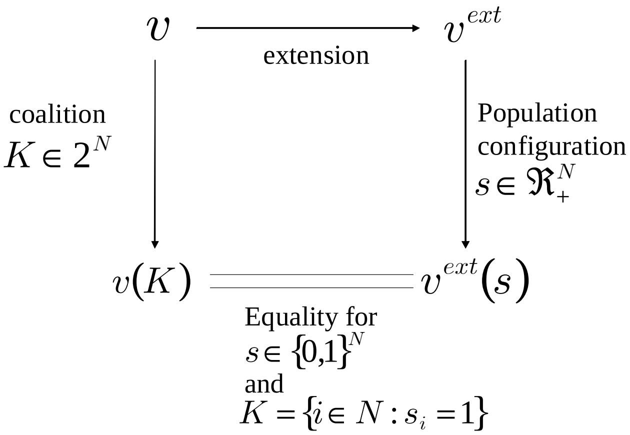


Figure: A coalition function and its extension

Lovasz extension

example apex game

- Assume $s_2 \leq s_3 \leq s_4$ without loss of generality and use the Harsanyi dividends to find
- the apex game's Lovasz extension

$$
\begin{aligned}
h^{\ell}(s) &= -\min_{i\in\{1,2,3\}} s_i - \min_{i\in\{1,2,4\}} s_i - \min_{i\in\{1,3,4\}} s_i + \min_{i\in\{2,3,4\}} s_i \\
&\quad + \min_{i\in\{1,2\}} s_i + \min_{i\in\{1,3\}} s_i + \min_{i\in\{1,4\}} s_i \\
&= \begin{cases} s_2, & s_1 \leq s_2 \leq s_3 \leq s_4 \\ s_1, & s_2 \leq s_1 \leq s_3 \leq s_4 \\ s_1, & s_2 \leq s_3 \leq s_1 \leq s_4 \\ s_4, & s_2 \leq s_3 \leq s_4 \leq s_1 \end{cases}
\end{aligned}
$$

- First line: the three small players cooperate
- Second line: the apex player cooperates with players 3 or 4
- Third and fourth line: the apex player cooperates with player 4

Lovasz extension versus MLE extension

- v_{MLE} has a probabilistic interpretation:
  - Inside a firm, the players work together only if their time schedules happen to coincide. For the above part-time coalition $\left(\frac{1}{2}, \frac{1}{3}\right)$, chocolate bread will be produced for $\frac{1}{2} \cdot \frac{1}{3}$ time units, only.
  - However, the two agents could show up at the same time. Also, it may be possible that the baker bakes his bread which is coated by chocolate later.
- v^{ℓ} works differently:
  - In case of $\left(\frac{1}{2}, \frac{1}{3}\right)$ chocolate bread will be produced for $\min\left(\frac{1}{2}, \frac{1}{3}\right)$ time units.
  - That is, the baker and the chocolate maker's time are perfect complements in the production of chocolate bread.
  - However, the baker has some time left, $\frac{1}{2} - \min\left(\frac{1}{2}, \frac{1}{3}\right)$, and will spend this time producing bread. Since chocolate bread is more valuable than bread (or chocolate) it is efficient to allocate $\min\left(\frac{1}{2}, \frac{1}{3}\right)$ time units to chocolate-bread production and to use the remainder for bread.

The employment relation

Definition

$\mathcal{A} : N^2 \to [0,1]$ is called an employment matrix or an employment relation if

$$\sum_{i=1}^{n} \mathcal{A}(i,j) = 1 \text{ for any } j = 1, ..., n$$

holds. $\mathcal{A}(i,j)$ is the time spent by agent j in agent i's firm

Example:

$$\mathcal{A} = \begin{pmatrix} 1 & \frac{3}{8} & 1 \\ 0 & 0 & 0 \\ 0 & \frac{5}{8} & 0 \end{pmatrix}$$

- Player 1 uses all his time in his own firm.
- Player 1 employs players 2 and 3, with shares of time $\frac{3}{8}$ and 1.
- Player 2 is employed by player 1 ($\frac{3}{8}$) and player 3 ($\frac{5}{8}$).

The employment relation

Problem

Determine the employment matrix if all players spend all their time in their own one-man firm.

The employment coalition function

On the basis of the employment relation $\mathcal{A}$, we define a part-time coalition

$$s_K^{\mathcal{A}} := \left(\sum_{j \in K} \mathcal{A}(1,j), ..., \sum_{j \in K} \mathcal{A}(n,j) \right).$$

We can now construct the employment coalition function $v_{\text{ext}}^{\mathcal{A}}$ by

$$v_{\text{ext}}^{\mathcal{A}}(K) := v^{ext}\left(s_K^{\mathcal{A}}\right).$$

- The players from K employ themselves and/or other players within and outside K.
- These players are summarized in the part-time coalition $s_K^{\mathcal{A}}$.
- The worth of K is then the worth of this part-time coalition under the given extension.

$\mathcal{A}$ represents the market —> $v_{\text{ext}}^{\mathcal{A}}(K) = v(K)$
Notation: $v(s)$ instead of $v^{\ell}(s)$, $v^{\mathcal{A}}(K)$ instead of $v_{\min}^{\mathcal{A}}(K)$

Organizational inefficiency
definition

We build the team production costs, t $(0 \leq t \leq 1)$, into the employment coalition function. For $K \subseteq N$, we define

$$s^K \;\; : \;\; = \left((1-t) \sum_{j \in K, j \neq 1} \mathcal{A}(1,j) + \mathcal{A}(1,1) \cdot \begin{cases} 1, & 1 \in K \\ 0, & 1 \notin K \end{cases}, \right.$$
$$\left. ..., (1-t) \sum_{j \in K, j \neq n} \mathcal{A}(n,j) + \mathcal{A}(n,n) \cdot \begin{cases} 1, & n \in K \\ 0, & n \notin K \end{cases} \right)$$

and

$$v^{\mathcal{A},t}(K) := v\left(s^K\right)$$

Our measure of welfare that incorporates organizational inefficiencies is given by

$$v^{\mathcal{A},t}(N).$$

Organizational inefficiency

the baker's worth I

$$v^{\mathcal{A},t}(B) =$$

$$\underbrace{v(B,C)}_{\substack{\text{worth of}\\ \text{chocolate bread}}} \underbrace{\min\left(\underbrace{\mathcal{A}(B,B)}_{\substack{\text{time spent by B}\\ \text{in his own firm}}}, \underbrace{(1-t)\,\mathcal{A}(B,C)}_{\substack{\text{effective time spent by C}\\ \text{in B's firm}}}\right)}_{\substack{\text{effective time use of both players}\\ \text{for chocolate-bread production}}}$$

$$+ \underbrace{v(B)}_{\substack{\text{worth of}\\ \text{bread}}} \underbrace{\left[\mathcal{A}(B,B) - \min\left(\mathcal{A}(B,B), (1-t)\,\mathcal{A}(B,C)\right)\right]}_{\substack{\text{time spent by B in his own firm}\\ \text{not used for chocolate-bread production}}}$$

Organizational inefficiency

the baker's worth II

$$+ \underbrace{v(C)}_{\substack{\text{worth of}\\ \text{chocolate}}} \left[(1-t) \underbrace{\left(\mathcal{A}(B,C) - \frac{1}{1-t} \min\left(\mathcal{A}(B,B), (1-t)\mathcal{A}(B,C)\right) \right.}_{\substack{\text{time spent by C in B's firm}\\ \text{not used for chocolate-bread production}}} \right.$$

Organizational inefficiency

effective time for chocolate-bread production

$$T^{CB} := \min\left(\underbrace{\mathcal{A}(B,B)}_{\substack{\text{time spent by B}\\ \text{in his own firm}}} + \underbrace{(1-t)\,\mathcal{A}(C,B)}_{\substack{\text{effective time spent by B}\\ \text{in C's firm}}}, \right.$$

$$\left. \underbrace{\mathcal{A}(C,C)}_{\substack{\text{time spent by C}\\ \text{in his own firm}}} + \underbrace{(1-t)\,\mathcal{A}(B,C)}_{\substack{\text{effective time spent by C}\\ \text{in B's firm}}} \right)$$

Organizational inefficiency

effective time for bread/for chocolate production

Now, the baker is effective T^{CB} time units contributing to chocolate bread. He spoils $t\mathcal{A}(C,B)$ time units. Therefore, he will produce

$$T^{B}=1-t\mathcal{A}(C,B)-T^{CB}$$

units of bread. Similarly, the chocolate maker will produce

$$T^{C}=1-t\mathcal{A}(B,C)-T^{CB}$$

units of chocolate.

Organizational inefficiency

two-player case

Summarizing, we obtain

$$v^{\mathcal{A},t}(B,C) = \underbrace{v(B,C)}_{\substack{\text{worth of}\\ \text{chocolate bread}}} \quad T^{CB}$$
$$+ \underbrace{v(B)}_{\substack{\text{worth of}\\ \text{simple bread}}} \quad \underbrace{T^{B}}_{\substack{\text{effective time spent by B}\\ \text{producing simple bread}}}$$
$$+ \underbrace{v(C)}_{\substack{\text{worth of}\\ \text{chocolate}}} \quad \underbrace{T^{C}}_{\substack{\text{effective time spent by C}\\ \text{producing chocolate}}}$$

Market inefficiency I

Market —> $\mathcal{A}(i, i) = 1$, $i \in N$
Market efficiency —> realizing any gains from trade without employing each other

- trusting and knowing: trivial partition
- not trusting or not knowing: atomic partiton

Market inefficiency II

Let $prob(\mathcal{P})$ be the probability of a partition $\mathcal{P}$. Welfare without organizational inefficiencies:

$$\sum_{\mathcal{P}\in\mathfrak{P}} prob(\mathcal{P}) \sum_{C\in\mathcal{P}} v^{\mathcal{A}}(C).$$

Development of a one-parameter measure for market inefficiency by looking at rank orders

$$\rho = (\rho_1, \rho_2, ..., \rho_n).$$

- $b_0 = 1$ signifies a "break" before player ρ_1.
- $b_n = 1$ signifies a break after player ρ_n.
- $K = \left\{\rho_l, ..., \rho_{j-1}, \rho_j\right\}$ is called effective if $b_{l-1} = 1 = b_j$ and $b_l = ... = b_{j-1} = 0$.
- $(b_1, ..., b_{n-1})$ specifies a partition of the players,

$$\mathcal{P} := \{C_1, ..., C_m\},$$

into effective coalitions.

Market inefficiency II

Assume constant probability p for the b_j ($j = 1, ..., n-1$) being equal to 0. $\mathcal{P}'s$ probability under p, $prob\left(\mathcal{P}, p\right)$, is given by

$$prob\left(\mathcal{P}, p\right) = \frac{1}{n!} \cdot \prod_{j=1}^{m} |C_j|! \cdot m! \cdot (1-p)^{m-1} \cdot p^{n-m}$$

Two players:

$$\begin{aligned} prob\left(\{\{1,2\}\}, p\right) &= p, \\ prob\left(\{\{1\}, \{2\}\}, p\right) &= 1-p \end{aligned}$$

Three players:

$$\begin{aligned} prob\left(\{\{1,2,3\}\}, p\right) &= \frac{1}{6} \cdot 6 \cdot 1 \cdot 1 \cdot p^2 = p^2, \\ prob\left(\{\{1\}, \{2,3\}\}, p\right) &= \frac{1}{6} \cdot 2 \cdot 2 \cdot (1-p) \cdot p = \frac{2}{3}(1-p)\, p, \\ prob\left(\{\{1\}, \{2\}, \{3\}\}, p\right) &= \frac{1}{6} \cdot 1 \cdot 6 \cdot (1-p)^2 \cdot 1 = (1-p)^2. \end{aligned}$$

Putting market and organizational ineffiencies together

Welfare with organizational inefficiencies:

$$\sum_{\mathcal{P}\in\mathfrak{P}} prob\left(\mathcal{P}\right)\sum_{C\in\mathcal{P}} v^{\mathcal{A}}\left(C\right).$$

If we have only two players (baker and chocolate maker), we obtain

$$\begin{aligned}\pi &= prob\left(\left\{\left\{B\right\},\left\{C\right\}\right\}\right)\left[v^{\mathcal{A},t}\left(B\right)+v^{\mathcal{A},t}\left(C\right)\right]\\ &\quad +prob\left(\left\{\left\{B,C\right\}\right\}\right)v^{\mathcal{A},t}\left(B,C\right).\end{aligned}$$

Welfare maximization

We assume that employers and workers will agree on wages that exploit all welfare potential while organizational and market inefficiencies persist. That is, we look for

$$\arg\max_{\mathcal{A}} \sum_{\mathcal{P}\in\mathfrak{P}} prob\left(\mathcal{P}\right) \sum_{C\in\mathcal{P}} v^{\mathcal{A},t}\left(C\right).$$

The two-player case I

Assuming superadditivity, $u := v(B,C) - v(B) - v(C) \geq 0$, we find:

- Total cross employment ($\mathcal{A}(B,C) = 1 = \mathcal{A}(C,B)$) is never the unique best outcome.
- If v is inessential (i.e. $u = 0$), the market outcome ($\mathcal{A}(B,B) = 1$, $\mathcal{A}(B,C) = 0$) is the unique best solution for $t > 0$, and $v(C) > 0$.
- If the market is efficient (i.e., $p = 1$), the market outcome ($\mathcal{A}(B,B) = 1$, $\mathcal{A}(B,C) = 0$) is the unique best solution for $t > 0$, and one of the following conditions:
  - $v(C) > 0$,
  - $u > 0$.
- $t = 0$, $p < 1$, and $u > 0$ —> the set of best solutions is the continuum defined by

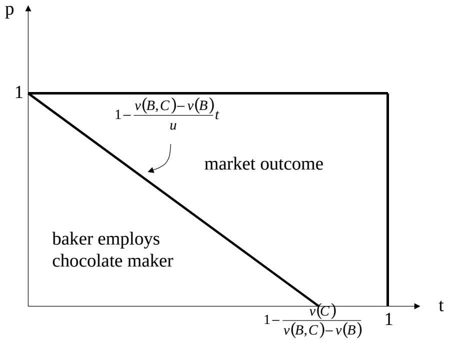
$$\mathcal{A}(B,B) = \mathcal{A}(B,C).$$

- $p = 1$ and $t = 0$ imply that every employment matrix is optimal.
- $t = 0$ and $u = 0$ imply that every employment matrix is optimal.

The two-player case II

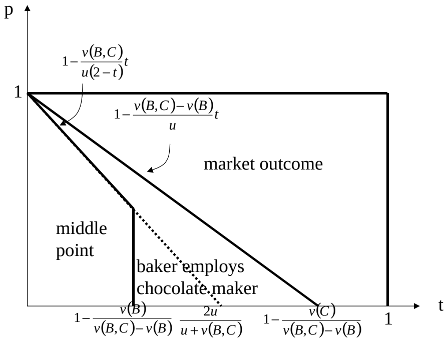
$u > 0$, $p < 1$, and $t > 0$ —> two cases:

- Either, we have $v(B, C) \leq 2v(B)$ —> market is the unique optimal outcome for sufficiently high values of p and t. Otherwise, the baker employs the chocolate maker (note that $u > 0$ and $v(B, C) \leq 2v(B)$ imply $v(B) > v(C)$).



The two-player case III

- ...or, we have $v(B,C) > 2v(B)$ —> partial cross-employment at $\mathcal{A}(B,B) = \frac{1-t}{2-t}$ and $\mathcal{A}(B,C) = \frac{1}{2-t}$ is the unique optimal outcome for $p < 1 - \frac{tv(B,C)}{u(2-t)}$ and $0 < t < 1 - \frac{v(B)}{v(B,C)-v(B)}$. Otherwise, result similar to previous slide.



The two-player case IV

The middle point is given by $\mathcal{A}(B,B) = \frac{1-t}{2-t}$ and $\mathcal{A}(B,C) = \frac{1}{2-t}$. For this employment matrix, we have

- $$\underbrace{\mathcal{A}(B,B)}_{\substack{\text{time spent by B}\\ \text{in his own firm}}} = \underbrace{(1-t)\,\mathcal{A}(B,C)}_{\substack{\text{effective time spent by C}\\ \text{in B's firm}}},$$
- $$\underbrace{1-\mathcal{A}(B,C)}_{\substack{\text{time spent by C}\\ \text{in his own firm}}} = \underbrace{(1-t)\,(1-\mathcal{A}(B,B))}_{\substack{\text{effective time spent by B}\\ \text{in C's firm}}} \quad \text{and}$$
- $$\underbrace{\mathcal{A}(B,B)}_{\substack{\text{time spent by B}\\ \text{in his own firm}}} + \underbrace{(1-t)\,(1-\mathcal{A}(B,B))}_{\substack{\text{effective time spent by B}\\ \text{in C's firm}}}$$
  $$= \underbrace{1-\mathcal{A}(B,C)}_{\substack{\text{time spent by C}\\ \text{in his own firm}}} + \underbrace{(1-t)\,\mathcal{A}(B,C)}_{\substack{\text{effective time spent by C}\\ \text{in B's firm}}}$$

The two-player case IV

For the production of chocolate bread one needs both the baker's and the chocolate maker's effective time.

- By concentrating production in the baker's firm $(\mathcal{A}(B, B) = 1 = \mathcal{A}(B, C))$ there is a relative shortage of chocolate production time.
- Similarly, if the chocolate maker employs the baker, there is a shortage of baking time.

The middle point avoids these shortages thus ensuring that chocolate bread is produced.