Applied cooperative game theory:
Permission and use values

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Overview “Permission and use values”

- Introduction
- Subordination structures
- Hierarchies
- Autonomous coalitions and the permission game
- Effective coalitions and the use game
- Axioms

Introduction I

Subordination = superior-subordinate relationship

- permission (no action without superior)
- use (by superior)

Introduction II

permission

game v on $N = \{1, 2, 3\}$, 1 needs 2's permission, rank order $(3, 1, 2)$
The marginal contributions are

- the standard one for player 3,
- no contribution for player 1 because player 2 is not present yet to give his permission, and
- the aggregate contribution $v(\{1, 2, 3\}) - v(\{3\})$ for player 2 because he brings to bear both player 1's and his own contribution.

$v = v_{\{1\},\{2,3\}}$ —> permission payoffs $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$

Introduction III

use

game v on $N = \{1, 2, 3\}$, 2 uses 1, rank order $(3, 2, 1)$
The marginal contributions are

- the standard one for player 3,
- the aggregate contribution $v(\{1, 2, 3\}) - v(\{3\})$ for player 2 because he uses both his own and also player 1's productivity, and
- no contribution for player 1.

$v = v_{\{1\},\{2,3\}}$ —> use payoffs $\left(\frac{1}{6}, \frac{2}{3}, \frac{1}{6}\right)$

Subordination structures

definition

Definition

Let N be a set (of players). A function $\mathcal{S} : N \to 2^N$ obeying $i \notin \mathcal{S}(i)$ is called a subordination structure or a subordination relation. i is superior, j subordinate. A subordination structure $\mathcal{S}^C$ gives rise to a clique $C \subseteq N$ if $\mathcal{S}^C$ is defined by

$$\mathcal{S}^C(i) = \begin{cases} C \backslash \{i\}, & i \in C \\ \emptyset, & i \notin C \end{cases}$$

Problem

Define the subordination structure $\mathcal{S}$ *on* $N = \{1, 2, 3\}$ *where player* 1 *is the superior of players* 2 *and* 3 *while player* 3 *is player* 2*'s subordinate.*

Definition (chain of command)

Let $\mathcal{S}$ be a subordination structure on N. The tuple $T(i \rightarrow j) = \langle i = i_0, ..., j = i_k \rangle$ is called a trail in $\mathcal{S}$ from i to j (a $i - j$ trail) if $i_{\ell+1} \in \mathcal{S}(i_\ell)$ holds for all $\ell = 0, \ldots, k-1$. The set of such trails is denoted by $\mathbb{T}(i \rightarrow j)$.

The set of player i's direct or indirect subordinates is denoted by

$$\hat{\mathcal{S}}(i) := \{j \in N \backslash \{i\} : \text{a trail } T(i \rightarrow j) \text{ exists}\}.$$

The set of player j's direct or indirect superiors is denoted by

$$\hat{\mathcal{S}}^{-1}(j) := \{i \in N \backslash \{j\} : \text{a trail } T(i \rightarrow j) \text{ exists}\}.$$

Subordination structures

coalitions rather than individual players

The definitions of $\mathcal{S}$, $\mathcal{S}^{-1}, \hat{\mathcal{S}}$, and $\hat{\mathcal{S}}^{-1}$ can be applied to coalitions rather than individual players in the obvious manner:

$$\begin{aligned}
\mathcal{S}(K) &: = \cup_{i \in K} \mathcal{S}(i), \\
\mathcal{S}^{-1}(K) &: = \cup_{i \in K} \mathcal{S}^{-1}(i), \\
\hat{\mathcal{S}}(K) &: = \cup_{i \in K} \hat{\mathcal{S}}(i), \\
\hat{\mathcal{S}}^{-1}(K) &: = \cup_{i \in K} \hat{\mathcal{S}}^{-1}(i).
\end{aligned}$$

Definition (subordination game)

For any player set N, every coalition function $v \in \mathbb{V}_N$ and any subordination structure $\mathcal{S} \in \mathfrak{S}_N$, $(v, \mathcal{S})$ is called a subordination game.

Definition (hierarchy)

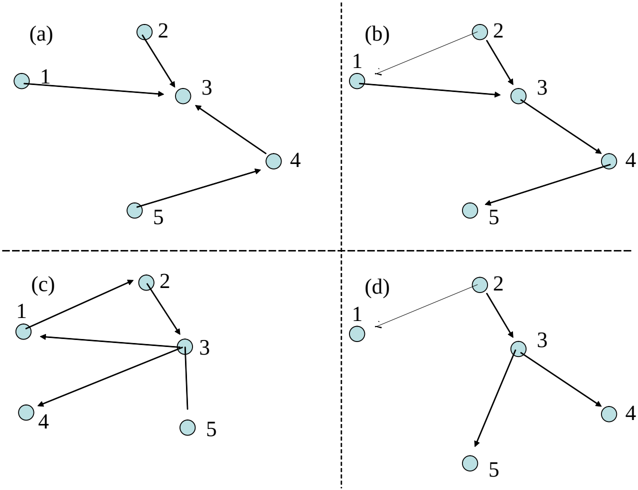
A subordination structure $\mathcal{S} \in \mathfrak{S}_N$ is called a hierarchy on N if

- $\mathcal{S}$ is acyclic, i.e., if $i \notin \hat{\mathcal{S}}(i)$ holds, and
- $\mathcal{S}$ is connected, i.e., there exists a player $i_0 \in N$ with $\hat{\mathcal{S}}(i_0) = N \backslash \{i_0\}$.

If, on top, $\left|\mathcal{S}^{-1}(j)\right| = 1$ for all $j \neq i_0$ holds, too, $\mathcal{S}$ is called a unique hierarchy.

Hierarchies II

hierarchies? unique hierarchies?



Hierarchies III

domination

Definition (domination)

Let $(v, \mathcal{S})$ be a hierarchy game with some player i_0 fulfilling $\mathcal{S}(i_0) = N \backslash \{i_0\}$. A player $i \in N$ dominates another player $j \in N$, $j \neq i$, if i is contained in every trail $T(i_0, j)$. By $\bar{\mathcal{S}}(i)$ we denote the set of all players that player i dominates. $\bar{\mathcal{S}}^{-1}(j) := \left\{ i \in N : j \in \bar{\mathcal{S}}(i) \right\}$ is called the j's set of dominating players.

Problem

If $\mathcal{S}$ is a unique hierarchy, domination of j by i can be expressed by ... .

Hierarchies IV
deleting link

Starting with a hierarchy $\mathcal{S}$ and considering a player j with at least two superiors ($\left|\mathcal{S}^{-1}(j)\right| \geq 2$), the deletion of the directed link between players h and j leads to the subordination structure $\mathcal{S}_{-(h,j)}$ which is defined by

$$\mathcal{S}_{-(h,j)}(i) = \begin{cases} \mathcal{S}(i) \setminus \{j\}, & i = h \\ \mathcal{S}(i), & i \neq h \end{cases}$$

Do you see that $\mathcal{S}_{-(h,j)}$ is a hierarchy if $\mathcal{S}$ is one? How about deleting links from unique hierarchies?

Autonomous coalitions

Definition (autonomous coalition)

Let $\mathcal{S}$ be a subordination structure on N. A coalition $K \subseteq N$ is called autonomous if $\hat{\mathcal{S}}^{-1}(K) \subseteq K$ holds.

Problem

Consider the subordination structure $\mathcal{S}$ *on* $N = \{1, ..., 5\}$ *given by*

$$\begin{aligned} \mathcal{S}(1) &= \{3\}, \mathcal{S}(2) = \emptyset, \mathcal{S}(3) = \{4\}, \\ \mathcal{S}(4) &= \{1\}, \mathcal{S}(5) = \{3\}. \end{aligned}$$

Find all the autonomous coalitions! How about the coalition $\{1, 3, 4\}$*? How about the empty set? Union or intersection of two autonomous sets?*

Autonomous subset

Definition (autonomous subset)

Let $v \in \mathbb{V}_N$ be a coalition function, let $\mathcal{S} \in \mathfrak{S}_N$ be a subordination structure, and $K \subseteq N$ be a coalition. K's autonomous subset $aut(K)$ is defined by

$$aut(K) := \bigcup_{\substack{A \subseteq K, \\ A \text{ autonomous}}} A.$$

A coalition's autonomous subset is its largest autonomous subset.

The permission game

Definition (permission game)

Let $(v, \mathcal{S})$ be a subordination game. The permission game based on this subordination game is the coalition function $v^{\mathcal{S}}$ which is defined by

$$v^{\mathcal{S}}(K) = v(aut(K)).$$

Problem

Let K be an autonomous coalition under the subordination structure $\mathcal{S}$. Determine $v^{\mathcal{S}}(K)$!

Problem

Determine the permission games $u^{\mathcal{S}_a}_{\{1,2\}}$ and $u^{\mathcal{S}_b}_{\{1,2\}}$ for $N = \{1, 2, 3\}$ and

$$\begin{aligned} \mathcal{S}_a(1) &= \{2\}, \mathcal{S}_a(2) = \{3\}, \mathcal{S}_a(3) = \emptyset \text{ and} \\ \mathcal{S}_b(1) &= \{2\}, \mathcal{S}_b(2) = \emptyset, \mathcal{S}_b(3) = \{1\}. \end{aligned}$$

Lemma

Let v and w be coalition functions on N. The permission game $(v+w)^{\mathcal{S}}$ equals the sum of the permission games $v^{\mathcal{S}} + w^{\mathcal{S}}$.

The proof is not difficult and follows from

$$\begin{aligned} (v+w)^{\mathcal{S}}(K) &= (v+w)(aut(K)) \text{ (definition permission game)} \\ &= v(aut(K)) + w(aut(K)) \text{ (vector sum)} \\ &= v^{\mathcal{S}}(K) + w^{\mathcal{S}}(K) \text{ (definition permission game).} \end{aligned}$$

The permission game

inheritance of monotonicity

Lemma

Let $\mathcal{S}$ be a subordination structure. If v is a monotonic coalition function, so is the permission game $v^{\mathcal{S}}$.

Consider two coalitions E and F with $E \subseteq F$ for a proof. Because of

$$aut(E) = \bigcup_{\substack{A \subseteq E, \\ A \text{ autonomous}}} A \subseteq \bigcup_{\substack{A \subseteq F, \\ A \text{ autonomous}}} A = aut(F)$$

we have $v^{\mathcal{S}}(E) = v(aut(E)) \leq v(aut(F)) = v^{\mathcal{S}}(F)$.

The permission value

Definition (permission value)

The permission value is the solution function *Per* given by

$$Per_i(v, \mathcal{S}) = Sh_i\left(v^{\mathcal{S}}\right), i \in N(v)$$

where $v^{\mathcal{S}}$ is the permission game based on $\mathcal{S}$.

Lemma

We have $Per(v, \mathcal{S}) = Sh(v)$ *for the null subordination structure* $\mathcal{S}$.

Problem

Permission payoffs for $N = \{1, 2, 3\}$, *the subordination structure* $\mathcal{S}$ *given by* $\mathcal{S}(1) = \{2\}$, $\mathcal{S}(2) = \emptyset, \mathcal{S}(3) = \{1\}$ *and the coalition functions*

- $u_{\{1,2\}}$ *and*
- $u_{\{1,3\}}$.

The permission value

clique subordination structures

Lemma

Let $C \subseteq N$ *be a clique and let* $\mathcal{S}^C$ *be the associated subordination structure. Then, we have* $Per_i\left(v, \mathcal{S}^C\right) = Per_j\left(v, \mathcal{S}^C\right)$ *for all* $i, j \in C$.

Effective coalitions

Definition (effective coalition)

Let $\mathcal{S}$ be a subordination structure on N. A coalition $K \subseteq N$ is called effective if $\mathcal{S}(K) \subseteq K$ holds.

Problem

Do you see that a coalition K is effective if and only if $\hat{\mathcal{S}}(K) \subseteq K$ holds?

Problem

Consider the subordination structure $\mathcal{S}$ on $N = \{1, ..., 5\}$ given by

$$\begin{aligned} \mathcal{S}(1) &= \{3\}, \mathcal{S}(2) = \emptyset, \mathcal{S}(3) = \{4\}, \\ \mathcal{S}(4) &= \{1\}, \mathcal{S}(5) = \{3\}. \end{aligned}$$

Find all the effective coalitions! How about the coalition $\{1, 3, 4\}$? How about the empty set? Union or intersection of two autonomous sets?

Effective superset

Definition (effective superset)

Let $v \in \mathbb{V}_N$ be a coalition function, let $\mathcal{S} \in \mathfrak{S}_N$ be a subordination structure, and $K \subseteq$ be a coalition. K's effective superset $\mathit{eff}\,(K)$ is defined by

$$\mathit{eff}\,(K) := K \cup \hat{\mathcal{S}}\,(K)\,.$$

Thus, a coalition's effective superset ist its smallest effective superset.

Use game

definition

Definition (use game)

Let $(v, \mathcal{S})$ be a subordination game. The use game based on this subordination game is the coalition function $v^{\mathcal{S}}$ which is defined by

$$v^{\mathcal{S}}(K) = v(eff(K)).$$

Problem

Determine the use games $u_{\{1,2\}}^{\mathcal{S}_a}$ *and* $u_{\{1,2\}}^{\mathcal{S}_b}$ *for* $N = \{1, 2, 3\}$ *and*

$$\begin{aligned} \mathcal{S}_a(1) &= \{2\}, \mathcal{S}_a(2) = \{3\}, \mathcal{S}_a(3) = \emptyset \text{ and} \\ \mathcal{S}_b(1) &= \{2\}, \mathcal{S}_b(2) = \emptyset, \mathcal{S}_b(3) = \{1\}. \end{aligned}$$

Use game

adding uses games, inheritance of monotonicity

Problem

Show $(v+w)^{\mathcal{S}} = v^{\mathcal{S}} + w^{\mathcal{S}}$ *for the use game.*

Problem

If v *is a monotonic coalition function, so is the use game* v^{S}.

The use value

Definition (use value)

The use value is the solution function *Use* given by

$$Use_i(v, \mathcal{S}) = Sh_i\left(v^{\mathcal{S}}\right), i \in N(v)$$

where $v^{\mathcal{S}}$ is the use game based on $\mathcal{S}$.

Lemma

We have $Use(v, \mathcal{S}) = Sh(v)$ *for the null subordination structure* $\mathcal{S}$.

Problem

Use payoffs for $N = \{1, 2, 3\}$ *and the subordination structure* $\mathcal{S}$ *given by* $\mathcal{S}(1) = \{2\}$, $\mathcal{S}(2) = \emptyset, \mathcal{S}(3) = \{1\}$ *and the coalition functions*

- $u_{\{1,2\}}$ *and*
- $u_{\{1,3\}}$.

Clique and full subordination structures

Lemma

Let $C \subseteq N$ be a clique and let $\mathcal{S}^C$ be the associated subordination structure. Then, we have $Use_i\left(v, \mathcal{S}^C\right) = Use_j\left(v, \mathcal{S}^C\right)$ for all $i, j \in C$.

Corollary

For the (full) subordination structure $\mathcal{S}^{full} : N \to 2^N$ defined by $\mathcal{S}^{full}(i) = N \backslash \{i\}$ for all $i \in N$, we have

$$Use\left(v, \mathcal{S}^{full}\right) = Per\left(v, \mathcal{S}^{full}\right) = \left(\frac{v(N)}{n}, ..., \frac{v(N)}{n}\right)$$

for all coalition functions $v \in \mathbb{V}_N$.

Important axioms for permissions and use values I

Definition (additivity axiom)

A solution function σ on $\mathbb{V}_N^{\text{sub}}$ is said to obey the additivity axiom if we have

$$\sigma\left(v+w,\mathcal{S}\right)=\sigma\left(v,\mathcal{S}\right)+\sigma\left(w,\mathcal{S}\right)$$

for any two coalition functions $v,w\in\mathbb{V}$ with $N\left(v\right)=N\left(w\right)$ and any subordination $\mathcal{S}\in\mathfrak{S}_{N(v)}$.

Problem

Does the additivity axiom hold for the permission value and/or the use value?

Important axioms for permissions and use values II

Definition (null-player axiom)

A solution function σ on $\mathbb{V}_N^{\text{sub}}$ is said to obey the null-player axiom if we have

$$\sigma_i(v, \mathcal{S}) = 0$$

for all subordination games $(v, \mathcal{L})$ and for every null player $i \in N$.

The null-player axiom does not hold for our two values – we have seen exercises to prove it.

Important axioms for permissions and use values III

inessential-player axiom – definition

Definition (inessential player)

Let $(v, \mathcal{S})$ be a subordination game. A player $i \in N$ is called inessential (with respect to $(v, \mathcal{S})$) if

$$v(K) = v(K \cup \{j\})$$

holds for all $K \subseteq N$ and for all $j \in \{i\} \cup \hat{\mathcal{S}}(i)$.

Definition (inessential-player axiom)

A solution function σ on $\mathbb{V}_N^{\text{sub}}$ is said to obey the inessential-player axiom if

$$\sigma_i(v, \mathcal{S}) = 0$$

holds for all subordination games $(v, \mathcal{S})$ and for every inessential player $i \in N$.

Important axioms for permissions and use values III

inessential-player axiom – claim

Permission value: Let $K \subseteq N$ be any coalition that does not contain i. The set $\Delta K := aut(K \cup \{i\}) \setminus aut(K)$ contains

- player i if i does not have any superiors outside and
- some players from K for whom i is a superior.

Thus, we find $\Delta K \setminus \{i\} = K \cap \hat{\mathcal{S}}(i)$ and

$$\begin{aligned} & v^{\mathcal{S}}(K \cup \{i\}) - v^{\mathcal{S}}(K) \\ = \; & v(aut(K \cup \{i\})) - v(aut(K)) \\ = \; & \sum_{j \in \Delta K} MC_j^{K_j}(v) = \sum_{j \in \{i\} \cup \left(K \cap \hat{\mathcal{S}}(i)\right)} MC_j^{K_j}(v) \end{aligned}$$

with suitably chosen $K_j \subseteq N$. Since i is inessential, all these marginal contributions are zero so that i is indeed a null palyer with respect to $v^{\mathcal{S}}$.

Proof.

Show that the use value obeys the inessential-player axiom. □

Important axioms for permissions and use values IV

necessary-player axiom – definition

Definition (necessary player)

Let $(v, \mathcal{S})$ be a subordination game. A player $i \in N$ is called necessary (with respect to $(v, \mathcal{S})$) if

$$v(K) = 0$$

holds for all $K \subseteq N \backslash \{i\}$.

Definition (necessary-player axiom)

A solution function σ on $\mathbb{V}_N^{\text{sub}}$ is said to obey the necessary-player axiom if

$$\sigma_i(v, \mathcal{S}) \geq \sigma_j(v, \mathcal{S})$$

holds for every monotonic coalition function v and for every necessary player $i \in N$.

Important axioms for permissions and use values IV

necessary-player axiom – claim

According to van den Brink (??), the permission value fulfills the necessary player axiom.
However, the use value does not. Consider $N = \{1, 2, 3\}$, the unanimity game $u_{\{2,3\}}$ and the hierarchy $\mathcal{S}$ given by $\mathcal{S}(1) = \{2\}$ and $\mathcal{S}(2) = \{3\}$. The productive player 3 is a necessary player (as is player 2). But his payoff is zero which you can see by a rank-order argument. If player 3 is first, the productive player 2 is still missing so that player 3's marginal contribution is 0. If players 1 or 2 are first, both their marginal contributions are 1. Therefore, we find the use payoffs $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$.

Important axioms for permissions and use values V

Definition (efficiency axiom)

A solution function σ on $\mathbb{V}_N^{\text{sub}}$ is said to obey the efficiency axiom if

$$\sum_{i \in N} \sigma_i (v, \mathcal{S}) = v (N)$$

holds for all subordination games $(v, \mathcal{S})$.

Problem

Does the efficiency axiom hold for the permission value and/or the use value?

Important axioms for permissions and use values VII

dominant players and superior players

Definition (dominant player)

A solution function σ on $\mathbb{V}_N^h$ (!) is said to obey the dominant-player axiom if we have

$$\sigma_i(v, \mathcal{S}) \geq \sigma_j(v, \mathcal{S})$$

for every monotonic coalition function v whenever player i dominates j.

Definition (superior player)

A solution function σ on $\mathbb{V}_N^{sub}$ is said to obey the superior-player axiom if we have

$$\sigma_i(v, \mathcal{S}) \geq \sigma_j(v, \mathcal{S})$$

for every monotonic coalition function v and for $j \in \mathcal{S}(i)$.

Hints: which axiom is stronger? consider the marginal contributions with respect to coalitions $E \subseteq N \backslash \{i, j\}$

Important axioms for permissions and use values VIII

Definition (balanced contributions)

A solution function σ on $\mathbb{V}_N^{\mathrm{h}}$ (!) is said to obey the balanced-contribution axiom if, for all players $h, j, g \in N$ with $h \neq g$ and $j \in S(g) \cap S(h)$, we have

$$\sigma_j(v, \mathcal{S}) - \sigma_j\left(v, \mathcal{S}_{-(h,j)}\right) = \sigma_i(v, \mathcal{S}) - \sigma_i\left(v, \mathcal{S}_{-(h,j)}\right) \text{ for all } i \in \{g\} \cup \bar{\mathcal{S}}^{-1}$$

holds for all subordination games $(v, \mathcal{S})$.

Note that the equality does not only apply to g himself but also to all players that dominate g (the players from $\bar{\mathcal{S}}^{-1}(g)$).

Problem

Does the balanced-contribution axiom hold for the permission value and/or the use value?

Axiomatizations of the permission value

Theorem (first axiomatization of the permission value)

The permission value on hierarchies is axiomatized by the additivity axiom, the inessential-player axiom, the necessary-player axiom, the efficiency axiom, the dominant-player axiom, and the balanced-contribution axiom.

Theorem (second axiomatization of the permission value)

The permission value on subordination structures is axiomatized by the additivity axiom, the inessential-player axiom, the necessary-player axiom, the efficiency axiom, and the superior-player axiom.