Applied cooperative game theory:
The network value

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Overview part D: Shapley values on networks

- The network value
- Granovetter's network theory
- Permission and use values
- Hierarchy and wages

Overview “The network value”

- Introduction
- Links, networks, and subnetworks
- Trails and connectedness
- Networks and their partitions
- The Myerson game
- The network value
- Properties of the communicaton value

Introduction

Networks stand for relationships between players:

- knowing each other
- cooperating

Procedure:

- for a coalition function v and
- a network $\mathcal{L}$

define a new coalition function $v^{\mathcal{L}}$ and
apply the Shapley value

Links, networks, and subnetworks I

Example

On the set $\{1,2,3,4\}$, player 1 may be linked with all the other players who do not have direct links with each other. This network is described by

$$\{12, 13, 14\}.$$

Definition (network)

Let N be a set (of players). The set of all subsets with exactly two elements is called the full network and is denoted by $\mathcal{L}^{\text{full}}$,

$$\mathcal{L}^{\text{full}} = \{\{i,j\} : i,j \in N, i \neq j\}.$$

Elements ℓ from $\mathcal{L}^{\text{full}}$ are called links. $\mathcal{L} \subseteq \mathcal{L}^{\text{full}}$ is called a network on N. The set of all networks on N is denoted by $\mathfrak{L}_N$ or $\mathfrak{L}$. $\mathcal{L}_1 \subseteq \mathcal{L}^{\text{full}}$ is called a subnetwork of $\mathcal{L}_2 \subseteq \mathcal{L}^{\text{full}}$ if $\mathcal{L}_1 \subseteq \mathcal{L}_2$ holds.

Links, networks, and subnetworks II

Definition (network)

The set $\mathcal{L}(i) := \{\ell \in \mathcal{L} : i \in \ell\} \subseteq \mathcal{L}$ is the set of all direct links entertained by player i.
Let R be a subset of N. The links on R induced by a network $\mathcal{L}$ is denoted by $\mathcal{L}(R)$ and defined by

$$\mathcal{L}(R) := \{\{i,j\} : i,j \in R, \{i,j\} \in \mathcal{L}\}.$$

Problem

Consider $N = \{1,2,3,4\}$ *and define the network* $\mathcal{L}$ *where player* 2 *is directly linked to players* 1 *and* 3. *Determine* $\mathcal{L}(1)$, $\mathcal{L}(2)$ *und* $\mathcal{L}(4)$.

Contrasting partitions and networks

partition		network	
symbol	meaning	symbol	meaning
$\mathcal{P}$	partition	$\mathcal{L}$	undirected graph
$\mathcal{P}(i)$	i's component	$\mathcal{L}(i)$	set of i's links
$\mathfrak{P}_N$	set of partitions	$\mathfrak{L}_N$	set of networks on N
$\mathcal{P}(R)$	set of components with R-players	$\mathcal{L}(R)$	set of links on R

Problem

Assume an arbitrary network $\mathcal{L}$ *on* N. *Can you find other expressions for*

- $\mathcal{L}(N)$,
- $\mathcal{L}(\{1,2\})$ *(case distinction!) and*
- $\bigcup_{i \in N} \mathcal{L}(i)$?

Trails and connectedness

Definition

A path in $\mathcal{L}$ from i to j (a $i - j$ path) = network $\{i = i_0 i_1, ..., i_{k-1} i_k = j\} \subseteq \mathcal{L}$.
Players i and j are called connected or linked

- if an $i - j$ path exists or
- if $i = j$ holds.

Definition

A relation on a set M is a subset of $M \times M$.
If a tuple $(a, b) \in M \times M$ is an element of this subset: $a \sim b$.

Definition

A relation $\sim$ on a set M is:

- reflexive if $a \sim a$ holds for all $a \in M$;
- transitive if $a \sim b$ and $b \sim c$ imply $a \sim c$ for all $a, b, c \in M$;
- symmetric if $a \sim b$ implies $b \sim a$ for all $a, b \in M$,
- asymmetric if $a \sim b$ implies $b \nsim a$ (i.e., not $b \sim a$),
- antisymmetric if $a \sim b$ and $b \sim a$ imply $a = b$ for all $a, b \in M$, and
- complete if $a \sim b$ or $b \sim a$ holds for all $a, b \in M$.

Lemma

On the set of integers $\mathbb{Z}$, the relation $\sim$ defined by

$$a \sim b :\Leftrightarrow a - b \text{ is an even number}$$

is reflexive, transitive, and symmetric, but neither antisymmetric nor complete.

Networks and their partitions

proof

Proof.

- reflexive: $a - a = 0$ for all $a \in \mathbb{Z} \Rightarrow a \sim a$;
- transitive:
  consider a, b, c so that $a \sim b$ and $b \sim c$.
  The sum of two even numbers is even.
  $\Rightarrow (a - b) + (b - c) = a - c$ is even.
  $\Rightarrow a \sim c$;
- symmetric: a number is even iff its negative is even;
- not complete: $0 \nsim 1$ and $1 \nsim 0$;
- not antisymmetric: consider 0 and 2.

□

Networks and their partitions

exercise

Problem

For any two inhabitants from Leipzig, we ask whether:

- *one is the father of the other or*
- *they are of the same sex.*

Which properties have the relations "is the father of" and "is of the same sex as"? Fill in "yes" or "no":

property	*is the father of*	*is of the same sex as*
reflexive		
transitive		
symmetric		
asymmetric		
antisymmetric		
complete		

Networks and their partitions

equivalence relation

Definition

Let $\sim$ be a relation on a set M which obeys reflexivity, transitivity and symmetry. $\Rightarrow$

- equivalent elements: $a, b \in M$ with $a \sim b$;
- equivalence relation: $\sim$;
- equivalence class of $a \in M$: $[a] := \{b \in M : b \sim a\}$.

Example

Our above relation $\sim$ (even difference) on the set of integers $\mathbb{Z}$ is an equivalence relation with

$$\begin{aligned} [0] &= \{b \in M : b \sim 0\} = \{..., -2, 0, 2, 4, ...\} \text{ and} \\ [1] &= \{b \in M : b \sim 1\} = \{..., -3, -1, 1, 3, ...\} \end{aligned}$$

Networks and their partitions

exercise

Problem

- *Find the equivalence classes* $[17]$, $[-23]$, *and* $[100]$.
- *Reconsider the relation "is of the same sex as". Can you describe its equivalence classes?*

Lemma

$a \sim b$ implies $[a] = [b]$.

Proof.

Consider any $a' \in [a]$. We need to show $a' \in [b]$:

- $a' \in [a]$ means $a' \sim a$;
- $a \sim b$, $a' \sim b$ (transitivity) $\Rightarrow a' \in [b] \Rightarrow [a] \subseteq [b]$.

Reversing the roles of a and b, $[b] \subseteq [a]$. □

Networks and their partitions

lemma 2

Lemma

Let $\sim$ be an equivalence relation on a set M. $\Rightarrow$

$$\begin{aligned} \bigcup_{a \in M} [a] &= M \text{ and} \\ [a] &\neq [b] \Rightarrow [a] \cap [b] = \emptyset. \end{aligned}$$

Networks and their partitions

generating partitions from graphs I

Definition (connectedness as a relation)

Let $\mathcal{L}$ be a network on N. If players i and j (not necessarily $i \neq j$) are connected, we write $i \sim^{\mathcal{L}} j$, i.e., $\sim^{\mathcal{L}}$ is a relation on N.

Lemma

$\sim^{\mathcal{L}}$ defines an equivalence relation on N.

Definition

Let $\sim^{\mathcal{L}}$ be the equivalence relation given above. We note the resulting partition by $N/\mathcal{L}$. For any nonempty subset $S \subseteq N$, S is also partitioned (via $\sim^{\mathcal{L}(S)}$) and we define

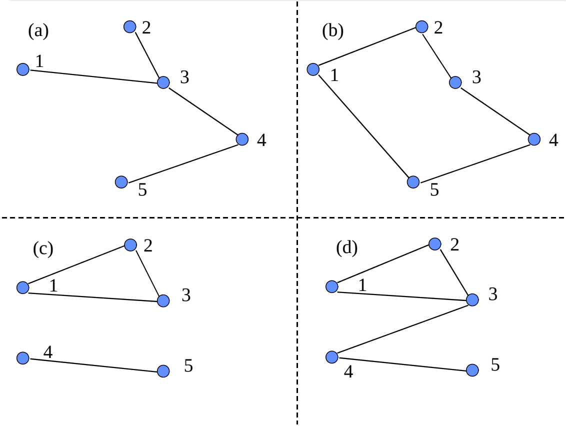
$$S/\mathcal{L} := S/\left(\mathcal{L}\left(S\right)\right).$$

Networks and their partitions

generating partitions from graphs II

Problem

Determine the partitions of the player subset $\{1, 3, 4\}$ *resulting from the four networks:*



Networks and their partitions

generating partitions from graphs III

For any subset $S \subseteq N$, we find

- that $S/\emptyset$ equals the atomic partition of S – every player is an island and
- that $S/\mathcal{L}^{\text{full}}$ equals the trivial partition $\{S\}$.

This observation can be generalized:

Lemma

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be networks on N such that $\mathcal{L}_1$ is a subnetwork of $\mathcal{L}_2$, $\mathcal{L}_1 \subseteq \mathcal{L}_2$. Then $S/\mathcal{L}_1$ is finer than $S/\mathcal{L}_2$ for every subset $S \subseteq N$.

The network value: definition

Definition (Myerson game)

Let $(v, \mathcal{L})$ be a network game. The Myerson game based on this network game is the coalition function $v^{\mathcal{L}}$ which is defined by

$$v^{\mathcal{L}}(S) = \sum_{K \in S/\mathcal{L}} v(K).$$

Definition (network value)

The network, or Myerson, value on $\mathbb{V}^{net}$ is given by

$$My_i(v, \mathcal{L}) = Sh_i\left(v^{\mathcal{L}}\right), i \in N(v)$$

The Myerson game: an example

- A symmetric coalition function v

$$v(S) = \begin{cases} 0, & |S| \leq 1 \\ 60, & |S| = 2 \\ 72 & S = N \end{cases}$$

- and the network $\mathcal{L} = \{12, 23\}$.
- While v is symmetrc, $v^{\mathcal{L}}$ is not. We obtain

$$v^{\mathcal{L}}(S) = \begin{cases} 0, & |S| \leq 1, S = \{1,3\} \\ 60, & S = \{1,2\}, S = \{2,3\} \\ 72 & S = N \end{cases}$$

The Myerson game: a second example

Let us determine the Myerson game for $N = \{1, 2, 3, 4\}$, the unanimity game $u_{\{1,3\}}$ and the network $\mathcal{L} = \{12, 23, 34\}$.

- The productive players 1 and 3 need player 2 in order to link up. Player 4 is of no help.
- Thus, we find

$$u^{\mathcal{L}}_{\{1,3\}}(K) = \begin{cases} 1, & K \supseteq \{1, 2, 3\} \\ 0, & \text{otherwise.} \end{cases}$$

  and hence
- $u^{\mathcal{L}_2}_{\{1,3\}} = u_{\{1,2,3\}}$.

The Myerson game: exercises

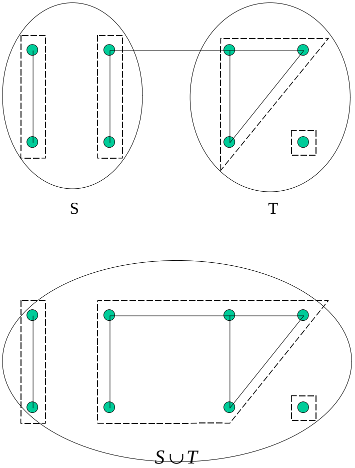
Problem

Given any coaliton function $v \in \mathbb{V}_N$, *determine the Myerson game* $v^{\mathcal{L}}$ *for* $\mathcal{L} = \mathcal{L}^{full}$ *and for* $\mathcal{L} = \emptyset$.

Problem

Given $N = \{1, 2, 3, 4\}$ *and the coalition function* $u_{\{1,3\}}$, *determine the Myerson game for* $\mathcal{L} = \{12, 23, 34, 41\}$.

The Myerson game: inheritance of superadditivity I



The Myerson game: inheritance of superadditivity II

Lemma

Let $\mathcal{L}$ be a network on N. If $v \in \mathbb{V}_N$ is superadditive, so is $v^{\mathcal{L}}$.

Proof:

$$\begin{aligned} v^{\mathcal{L}}(S \cup T) &= \sum_{C \in (S \cup T)/\mathcal{L}} v(C) \\ &\geq \sum_{C \in S/\mathcal{L}} v(C) + \sum_{C \in T/\mathcal{L}} v(C) \\ &= v^{\mathcal{L}}(R) + v^{\mathcal{L}}(S). \end{aligned}$$

The Myerson game: no inheritance of convexity I

Consider the network game on $N = \{1, 2, 3, 4\}$, defined by the "cycle" $L = \{12, 23, 34, 41\}$ and the coalition function v given by

$$v(S) = |S| - 1, S \neq \emptyset.$$

v is convex, because

- the marginal contribution is zero for any player who joins the empty set,
$$v(\emptyset \cup \{i\}) - v(\emptyset) = [|\{i\}| - 1] - 0 = 0,$$
- while the marginal contribution with respect to any nonempty coalition is 1.

The Myerson game: no inheritance of convexity II

However, $v^{\mathcal{L}}$ is not convex:

- The sets $\{1,2,3\}$, $\{1,3,4\}$ and $\{1,2,3,4\}$ are internally connected while $\{1,3\}$ is not.
- Therefore, we obtain

$$\begin{aligned} v^{\mathcal{L}}(\{1,2,3\}) &= v(\{1,2,3\}) = 2, \\ v^{\mathcal{L}}(\{1,3,4\}) &= v(\{1,3,4\}) = 2, \\ v^{\mathcal{L}}(\{1,2,3,4\}) &= v(\{1,2,3,4\}) = 3 \text{ und} \\ v^{\mathcal{L}}(\{1,3\}) &= v(\{1\}) + v(\{3\}) = 0 + 0 = 0. \end{aligned}$$

- and player 2's marginal contributions to coalitions $\{1,3\}$ and $\{1,3,4\}$

$$\begin{aligned} MC_2^{\{1,3\}}\left(v^{\mathcal{L}}\right) &= v^{\mathcal{L}}(\{1,2,3\}) - v^{\mathcal{L}}(\{1,3\}) = 2 - 0 \\ &> 3 - 2 = v^{\mathcal{L}}(\{1,2,3,4\}) - v^{\mathcal{L}}(\{1,3,4\}) \\ &= MC_2^{\{1,3,4\}}\left(v^{\mathcal{L}}\right). \end{aligned}$$

Generalization of the Shapley value

The network value is a generalization of the Shapley value:

Lemma

We have $My\left(v, \mathcal{L}^{full}\right) = Sh\left(v\right)$.

Problem

Calculate the network payoffs for $N = \{1, 2, 3\}$, $\mathcal{L} = \{12, 23\}$ *and the coalition functions*

- $u_{\{1,2\}}$ *and*
- $u_{\{1,3\}}$*!*

Components are islands: component decomposability

- We define $C_i = (N/\mathcal{L})(i)$ for networks $\mathcal{L}$ on N.
- Very close the AD value, the network value treats components as islands.

Definition (component-decomposability axiom)

A solution function σ on $\mathbb{V}^{\text{net}}$ is said to obey component decomposability if

$$\sigma_i(v, \mathcal{L}) = \sigma_i\left(v|_{C_i}, \mathcal{L}(C_i)\right)$$

holds for all $i \in N$.

Thus, the payoff for a player does not depend on how the graph $\mathcal{L}$ is structured outside player i's component. The payoff depends only on the coalition function restricted to C_i and on the network restricted to C_i.

Components are islands: component efficiency

Definition (component-efficiency axiom)

A solution function σ on $\mathbb{V}^{\text{net}}$ is said to obey the component-efficiency axiom if

$$\sum_{i \in C_i} \sigma_i (v, \mathcal{L}) = v (C_i)$$

holds for all components $C_i \in N/\mathcal{L}$.

Problem

We may conjecture the equality of the Myerson and the Aumann-Dreze value whenever both deal with the same partition, $\mathcal{P} = N/\mathcal{L}$. *That is, do we have*

$$\mu (v, \mathcal{L}) = \varphi^{AD} (v, N/\mathcal{L}) \qquad ??$$

Components are islands, but

- For $N = \{1, 2, 3, 4\}$, $\mathcal{L} = \{12, 23, 34, 41\}$ and $u_{\{1,3\}}$, we find

$$u^{\mathcal{L}}_{\{1,3\}}(K) = \begin{cases} 1, & K \supseteq \{1,2,3\} \text{ or } K \supseteq \{1,3,4\} \\ 0, & \text{otherwise.} \end{cases}$$

- You can confirm or believe the author that the Shapley payoffs are

$$\left(\frac{5}{12}, \frac{1}{12}, \frac{5}{12}, \frac{1}{12}\right).$$

Problem

Determine $N/\mathcal{L}$ and $\varphi^{AD}(v, N/\mathcal{L})$.

Superfluous players

Definition (superfluous player)

Let $(v, \mathcal{L})$ be a network game. A player $i \in N$ is called superfluous if

$$v^{\mathcal{L}}(S) = v^{\mathcal{L}}(S \cup i)$$

holds for all $S \subseteq N$ gilt.

Definition (superfluous-player axiom)

A solution function σ on $\mathbb{V}^{\text{net}}$ is said to obey the superfluous-player axiom if

$$\sigma(v, \mathcal{L}) = \sigma(v, \mathcal{L} \backslash \mathcal{L}(i))$$

holds for every superfluous player $i \in N$.

Superfluous links

Definition (superfluous link)

Let $(v, \mathcal{L})$ be a network game. A link $\ell \in \mathcal{L}$ is called superfluous if

$$v^{\mathcal{L}} = v^{\mathcal{L} \backslash \ell}$$

holds.

Problem

Superfluous link: $N = \{1, 2, 3\}$, $v = u_{\{1,2\}}$ *and* $\mathcal{L} = \{12, 13\}$?

Definition (superfluous-link axiom)

A solution function σ on $\mathbb{V}^{\text{net}}$ is said to obey the superfluous-link axiom if

$$\sigma(v, \mathcal{L}) = \sigma(v, \mathcal{L} \backslash \ell)$$

holds for every superfluous link $\ell \in \mathcal{L}$.

Balanced contributions

Definition (axiom of balanced contributions, one link)

A solution function σ on $\mathbb{V}^{\text{net}}$ is said to obey the axiom of balanced contributions if, for any coalition function v and any two players $i, j \in N$,

$$\sigma_i(v, \mathcal{L}) - \sigma_i(v, \mathcal{L} \backslash \{ij\}) = \sigma_j(v, \mathcal{L}) - \sigma_j(v, \mathcal{L} \backslash \{ij\})$$

holds.

Definition (axiom of balanced contributions, all links)

A solution function σ on $\mathbb{V}^{\text{net}}$ is said to obey the axiom of balanced contributions if, for any coalition function v and any two players $i, j \in N$,

$$\sigma_i(v, \mathcal{L}) - \sigma_i(v, \mathcal{L} \backslash \mathcal{L}(j)) = \sigma_j(v, \mathcal{L}) - \sigma_j(v, \mathcal{L} \backslash \mathcal{L}(i))$$

holds.

Properties of the network value

Theorem (properties of the communication value)

The network value obeys

- *the component-decomposability axiom,*
- *the component-efficiency axiom,*
- *the superfluous-player axiom,*
- *the superfluous-link axioms,*
- *the additivity axiom,*
- *and the balanced-contributions axiom.*

Axiomatization of the network value

Among the several known axiomatizations of the Myerson value, we like to highlight the two that make use of balanced contributions:

Theorem

A solution concept σ *on* $\mathbb{V}^{net}$ *fulfills the two axioms of*

- *component efficiency and*
- *balanced contributions (for one link or for all links) for all player sets* $N \subseteq \mathbb{N}$,

if and only if σ *is the network value* My.

Further exercises: Problem 1

Consider the coalition function v given by $N = \{1, 2, 3, 4\}$ and

$$v(K) = \begin{cases} 0, & |K| \leq 1 \\ 2, & K \in \{\{1,2\}, \{1,3\}, \{1,4\}\} \\ 3 & K \in \{\{2,3\}, \{2,4\}\} \\ 5 & K \in \{\{3,4\}, \{1,2,3\}, \{1,2,4\}\} \\ 7, & K \in \{\{1,3,4\}, \{2,3,4\}, N\} \end{cases}$$

1. Consider three networks $\mathcal{L}_a = \{12, 14, 34\}$, $\mathcal{L}_b = \{12, 14, 24, 34\}$, $\mathcal{L}_c = \{12, 13, 24, 34\}$. Determine the three Myerson games associated with these networks. Determine the Shapley values of these games.
2. Comment!