Overview "The union value"

- Introduction
- Partitions and rank orders
- Union formula and axiomatization
- Examples

Introduction

Does it pay to unionize?

- Chances of a German citizen to become a EU commissioner versus those of an Irish citizen
- Productive players in a unanimity game profit when they dissociate themselves from other productive players.
- Left-glove owners may benefit from forming a cartel of left-gloves holders.

Partitions and rank orders

subpartitions

$\mathcal{P}(i) \in 2^N$ = component that contains player i
$\mathcal{P}(R) \subseteq 2^N$ = subpartition of $\mathcal{P}$ fulfilling $C \cap R \neq 0 \Leftrightarrow C \in \mathcal{P}(R)$ for all $C \in \mathcal{P}$

Problem

Express $\mathcal{P}(T)$ and $\mathcal{P}(i) \cap T$ in your own words.

Partitions and rank orders

union of components

Definition (union of components)

Let $\mathcal{P} = \{C_1, ..., C_k\}$ be a partition of N. We denote the union of R-components by

$$\bigcup \mathcal{P}(R) := \bigcup_{i \in R} \mathcal{P}(i).$$

Thus, $\bigcup \mathcal{P}(R)$ is the set with partition $\mathcal{P}(R)$.

Problem

Consider $\mathcal{P} = \{\{1\}, \{2\}, \{3,4\}, \{5,6,7\}\}$ *and find* $\mathcal{P}(\{2,5\})$ *and* $\bigcup \mathcal{P}(\{2,5\})$.

Partitions and rank orders

rank orders consistent with partitions

Consider the partition $\mathcal{P} = \{\{1\}, \{2\}, \{3,4\}\}$.

- The rank order $\rho = (3,1,2,4)$ tears the component $\{3,4\}$ apart while
- the rank order $\rho = (3,4,1,2)$ does not.

Problem

Which of the following rank oders are consistent with the partition $\mathcal{P} = \{\{1\}, \{2\}, \{3,4\}, \{5,6,7\}\}$?

- $\rho = (1,2,3,4,5,6,7)$
- $\rho = (2,1,4,5,6,7,3)$
- $\rho = (1,5,2,3,4,6,7)$
- $\rho = (1,4,3,7,5,6,2)$

Partitions and rank orders

rank orders consistent with partitions

Problem

Which rank orders from RO_7 are consistent with

- $\mathcal{P} = \{\{1,2,3,4,5,6,7\}\}$ *or*
- $\mathcal{P} = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\}\}$?

Partitions and rank orders

how many consistent rank orders are there

How many rank orders are consistent with a partition

$$\mathcal{P} = \{S_1, ..., S_k\}?$$

Note:

- We have $k!$ possibilities to rank the components S_1 through S_k.
- Within component S_j, there are $|S_j|!$ possibilities to rank its players.

Thus, we find

$$\left|RO_n^{\mathcal{P}}\right| = k! \cdot |S_1|! \cdot ... \cdot |S_k|!$$

Union formula

definition

Definition

The union, or Owen, formula is given by

$$Ow_i(v, \mathcal{P}) = \frac{1}{|RO_n^{\mathcal{P}}|} \sum_{\rho \in RO_n^{\mathcal{P}}} [v(K_i(\rho)) - v(K_i(\rho) \setminus \{i\})], i \in N.$$

Do you see that $\mathcal{P} = \{\{1, 2, ..., n\}\}$ and $\mathcal{P} = \{\{1\}, \{2\}, ..., \{n\}\}$ lead to the same Owen values?

Union formula

calculating the Owen payoffs

Example

$N = \{1, 2, 3\}$, $v_{\{1,2\},\{3\}}$,
$\mathcal{P} = \{\{1, 2\}, \{3\}\}$ with consistent rank orders

$$(1, 2, 3), (2, 1, 3), (3, 1, 2) \text{ and } (3, 2, 1).$$

We obtain player 1's Owen payoff:

$$Ow_1 \left(u_{\{1,2\}}, \mathcal{P}\right)$$
$$= \frac{1}{4}\left(\underbrace{0}_{(1,2,3)} + \underbrace{/}_{(1,3,2)} + \underbrace{0}_{(2,1,3)} + \underbrace{/}_{(2,3,1)} + \underbrace{1}_{(3,1,2)} + \underbrace{0}_{(3,2,1)}\right) = \frac{1}{4},$$

Union value axiomatization I

The Owen value is a solution function σ on $(N, \mathfrak{P}_N)$ that obeys

- the efficiency axiom,
- the symmetry axiom (payoff equality for $\mathcal{P}$-symmetric players),
- the null-player axiom, and
- the additivity axiom.

These axioms do not suffice to pin down the Owen value.

Union value axiomatization II

Definition (component symmetry)

Consider a partition $\mathcal{P} \in \mathfrak{P}_N$. Two components C and C' from $\mathcal{P}$ are called symmetric if

$$v\left(\bigcup \mathcal{P}(K) \cup C\right) = v\left(\bigcup \mathcal{P}(K) \cup C'\right)$$

holds for all $K \subseteq N \backslash (C \cup C')$.

Definition (symmetry axiom for components)

A solution function (on $\mathbb{V}^{\text{part}}$) σ is said to obey symmetry between components if

$$\sigma_C(v, \mathcal{P}) = \sigma_{C'}(v, \mathcal{P})$$

holds for all symmetric components C and C' from $\mathcal{P}$.

Union value axiomatization III

Theorem (Axiomatization of the Owen value)

The Owen formula is the unique solution function that fulfills the symmetry axiom, the symmetry axiom for components, the efficiency axiom, the null-player axiom and the additivity axiom.

Union value axiomatization IV

Again: gloves game $v_{\{1,2\},\{3\}}$, partition $\mathcal{P} = \{\{1,2\},\{3\}\}$

- Both components are needed to produce the worth of 1. Therefore, the symmetry axiom for components yields

$$Ow_1\left(v_{\{1,2\},\{3\}},\mathcal{P}\right) + Ow_2\left(v_{\{1,2\},\{3\}},\mathcal{P}\right) = Ow_3\left(v_{\{1,2\},\{3\}},\mathcal{P}\right).$$

- Efficiency then leads to

$$\begin{aligned} Ow_3\left(v_{\{1,2\},\{3\}},\mathcal{P}\right) &= 1 - \left(Ow_1\left(v_{\{1,2\},\{3\}},\mathcal{P}\right) + Ow_2\left(v_{\{1,2\},\{3\}},\mathcal{P}\right)\right) \\ &= 1 - Ow_3\left(v_{\{1,2\},\{3\}},\mathcal{P}\right) \end{aligned}$$

  and hence to $Ow_3\left(v_{\{1,2\},\{3\}},\mathcal{P}\right) = \frac{1}{2}$.
- The symmetry between players 1 and 2 produces $Ow_1\left(v_{\{1,2\},\{3\}},\mathcal{P}\right) = Ow_2\left(v_{\{1,2\},\{3\}},\mathcal{P}\right) = \frac{1}{4}$.

Examples

unanimity games

- Disregard null components $C \subseteq N \backslash T$.
- Each component in $\mathcal{P}(T)$ has the same probability $\frac{1}{|\mathcal{P}(T)|}$ to be the last component.
- Within each of these components, every $i \in T$ player has the same probability $\frac{1}{|\mathcal{P}(i) \cap T|}$ to complete T.

Thus, the Owen value yields the following payoffs for a unanimity game u_T, $T \neq \emptyset$:

$$Ow_i(u_T, \mathcal{P}) = \begin{cases} \frac{1}{|\mathcal{P}(T)|} \frac{1}{|\mathcal{P}(i) \cap T|}, & i \in T \\ 0, & \text{otherwise} \end{cases}$$

Breaking off pays.

Examples

symmetric games

Symmetric players need not be $\mathcal{P}$-symmetric.
Consider $N = \{1,2,3\}$, $\mathcal{P} = \{\{1,2\},\{3\}\}$ and the coalition function v given by $\alpha \in \mathbb{R}$ and

$$v(S) = \begin{cases} 0, & |S| \leq 1 \\ \alpha, & |S| = 2 \\ 1, & |S| = 3 \end{cases}$$

rank order	marginal contribution for player 1
1-2-3	0
2-1-3	α
3-1-2	α
3-2-1	$1-\alpha$
sum	$1+\alpha$
Owen payoff	$\frac{1+\alpha}{4}$

Since players 1 and 2 are $\mathcal{P}$-symmetric, we have $Ow_2(v,\mathcal{P}) = Ow_1(v,\mathcal{P}) = \frac{1+\alpha}{4}$. Efficiency yields

$$\begin{aligned} Ow_3(v,\mathcal{P}) &= 1 - Ow_1(v,\mathcal{P}) - Ow_2(v,\mathcal{P}) \\ &= 1 - 2 \cdot \frac{1+\alpha}{4} = \frac{1}{2} - \frac{1}{2}\alpha. \end{aligned}$$

Thus, we obtain $Ow_3(v,\mathcal{P}) \neq Ow_1(v,\mathcal{P})$ unless $\alpha = \frac{1}{3}$ happens to hold.

Examples

apex games

Problem

Find the Owen payoffs for the n-player apex game h_1 *and the partition* $\mathcal{P} = \{\{1\}, \{2, ..., n\}\}$.

If the unimportant players form several components, the apex player obtains a positive payoff. For example, if the players 2 to n form two components, the apex player obtains the marginal payoff 1 in one out of three cases – therefore, we have $Ow_1(v, \mathcal{P}) = \frac{1}{3}$.

Problem

Can you find a partition $\mathcal{P} = \{\{1\}, C_1, C_2\}$ *such that a player* $j \in \{2, ..., n\}$ *obtains a higher payoff than* $\frac{1}{n-1}$*?*

Definition (probability distribution)

Let M be a nonempty set. A probability distribution on M is a function

$$prob : 2^M \rightarrow [0, 1]$$

such that

- $prob\left(\emptyset\right) = 0$,
- $prob\left(A \cup B\right) = prob\left(A\right) + prob\left(B\right)$ for all $A, B \in 2^M$ obeying $A \cap B = \emptyset$ and
- $prob\left(M\right) = 1$.

The Shapley value is an average of Owen values

probability distribution II

Problem

Probability for the events

- *"the number of pips (spots) is* 2*",*
- *"the number of pips is odd", and, applying the definition*
- *"the number of pips is* 1, 2, 3 *or* 5*".*

The Shapley value is an average of Owen values

symmetric probability distribution I

prob on $M = \mathfrak{P}_N$

- not symmetric:

$$prob_1\left(\{\{1,2\},\{3\}\}\right) = \frac{1}{2} = prob_1\left(\{\{1\},\{2,3\}\}\right)$$

- not symmetric:

$$prob_2\left(\{\{1,2\},\{3\}\}\right) = 1$$

- symmetric:

$$\begin{aligned} prob_1\left(\{\{1,2\},\{3\}\}\right) &= prob_1\left(\{\{1\},\{2,3\}\}\right) \\ &= prob_1\left(\{\{2\},\{1,3\}\}\right) = \frac{1}{3}, \\ prob_2\left(\{\{1,2,...,n\}\}\right) &= 1, \text{ and} \\ prob_3\left(\{\{1\},\{2\},...,\{n\}\}\right) &= 1 \end{aligned}$$

The Shapley value is an average of Owen values

symmetric probability distribution II

- bijection $\pi : N \to N$. For example, for $N = \{1, 2, 3\}$, a bijection π is defined by

$$\begin{aligned} \pi(1) &= 3, \\ \pi(2) &= 1, \text{ and} \\ \pi(3) &= 2. \end{aligned}$$

- For a partition $\mathcal{P}$, $\pi(\mathcal{P})$ is the partition $\{\pi(C) : C \in \mathcal{P}\}$.

Problem

Let $\mathcal{P} = \{\{1, 2\}, \{3\}\}$. Find $\pi(\mathcal{P})$ for the above bijection π!

The Shapley value is an average of Owen values

symmetric probability distribution III

Definition

Let *prob* be a probability distribution on $\mathfrak{P}_N$. *prob* is called symmetric if every bijection $\pi : N \to N$ yields

$$prob\left(\mathcal{P}\right) = prob\left(\pi\left(\mathcal{P}\right)\right).$$

Do you see:

- $\pi\left(\{1, 2, ..., n\}\right) = \{1, 2, ..., n\}$ for every bijection π.
- Every partition π keeps the atomic partition intact.

Definition (probabilistic Owen value)

The probabilistic Owen value on $\mathbb{V}^{\text{part}}$ is the solution function Ow given by

$$Ow_i(v, prob) = \sum_{\mathcal{P} \in \mathfrak{P}_N} prob(\mathcal{P})\, Ow_i(v, \mathcal{P}), i \in N,$$

where $prob \in Prob(\mathfrak{P}_N)$ is a probability distribution on the set of partitions of N.

Theorem (Casajus in IGTR)

For any symmetric probability distribution prob on $\mathfrak{P}_N$, *we have*

$$Ow(v, prob) = Sh(v).$$

Further exercises: Problem 1

Assume two men, Max (M) and Onno (O), who both love Ada (A). Their coalition function is given

$$v(K) = \begin{cases} 0, & |K| \leq 1 \\ 6, & K = \{M, A\} \\ 4, & K = \{O, A\} \\ 1, & K = \{M, O\} \\ 2, & K = \{M, O, A\} \end{cases}$$

1. Calculate the Owen payoffs for the partition $\mathcal{P} = \{\{M, O\}, \{A\}\}$!
2. Comment!