Overview part C: The Shapley value on partitions

- The outside option values
- The union value
- Unions and unemployment benefits

Overview "The outside option values"

- Introduction
- Solution functions for partitional games
- Important axioms for partitional values
- The Aumann-Dreze value: formula and axiomatization
- The outside-option value due to Wiese
- The outside-option value due to Casajus
- Contrasting the Casajus and the Wiese values

		number l of left-glove owners				
		0	1	2	3	4
number r	1	0	$\in [0,1]$	1	1	1
of	2	0	0	$\in [0,1]$	1	1
right-glove	3	0	0	0	$\in [0,1]$	1
owners	4	0	0	0	0	$\in [0,1]$

- Shapley/Shubik: violent discontinuity exhibited by ... the core

Introduction

Shapley payoffs for an owner of a right glove

		number l of left-glove owners				
		0	1	2	3	4
number r	1	0	0,500	**0,667**	0,750	0,800
of	2	0	0,167	0,500	0,650	**0,733**
right-glove	3	0	0,083	0,233	0,500	0,638
owners	4	0	0,050	0,133	0,271	0,500

- replication leads to the core
- price of a right glove?

- player set $N = \{1, 2, 3\}$ and the gloves game $v_{\{1\},\{2,3\}}$.
- partition $\mathcal{P} = \{\{1, 2\}, \{3\}\}$
- AD value (due to Aumann and Dreze): every component is an island:

$$AD\left(v_{\{1\},\{2,3\}}, \mathcal{P}\right) = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$$

- Outside-options values (due to Wiese or Casajus): outside options count:

$$\begin{aligned} W\left(v_{\{1\},\{2,3\}}, \mathcal{P}\right) &= \left(\frac{2}{3}, \frac{1}{3}, 0\right), \\ Ca\left(v_{\{1\},\{2,3\}}\right) &= \left(\frac{3}{4}, \frac{1}{4}, 0\right) \end{aligned}$$

- player set $N = \{1, 2, 3\}$ and the unanimity game $u_{\{1,2\}}$
- partition $\mathcal{P}_1 = \{\{1, 3\}, \{2\}\}$
- component efficiency:
  $\sigma_1^{oo}\left(u_{\{1,2\}}, \mathcal{P}_1\right) + \sigma_3^{oo}\left(u_{\{1,2\}}, \mathcal{P}_1\right) = 0 = \sigma_2^{oo}\left(u_{\{1,2\}}, \mathcal{P}_1\right)$
- player 3 is a null player but his payoff cannot be zero under σ^{oo}
  - player 1 has outside options (with player 2 outside player 1's component)
  - player 1 should obtain more than zero and
  - player 3 should get less than zero

Partitions

Example

- Consider the set $\{1, 2, 3, 4\}$.
- The set of subsets

$$\{\{1, 2\}, \{3\}, \{4\}\}$$

  is an example of a partition while
- the sets of subsets

$$\{\{1, 2\}, \{4\}\} \text{ or}$$
$$\{\{1, 2\}, \{2, 3\}, \{4\}\}$$

  are not.

Definition (partition)

Let N be a set (of players). A system of subsets

$$\mathcal{P} = \{C_1, ..., C_k\}$$

is called a partition if

- $\bigcup_{j=1}^{k} C_j = N$,
- $C_j \cap C_{j'} = \emptyset$ for all $j \neq j'$ from $\{1, ..., k\}$ and
- $C_j \neq \emptyset$ for all $j = 1, ..., k$

hold. The subsets $C_j \subseteq N$ are called components. The component hosting player i is denoted by $\mathcal{P}(i)$.

Partitions

finer partitions

Definition

A partition $\mathcal{P}_1$ is called finer than a partition $\mathcal{P}_2$ if $\mathcal{P}_1(i) \subseteq \mathcal{P}_2(i)$ holds for all $i \in N$. In that case, $\mathcal{P}_2$ is called coarser than $\mathcal{P}_1$.

Problem

Is $\mathcal{P}_1$ finer or coarser than $\mathcal{P}_2$?

1. $\mathcal{P}_1 = \mathcal{P}_2 = \{\{1,2\},\{3,4\},\{5\}\}$,
2. $\mathcal{P}_1 = \{\{1,2\},\{3,4\},\{5\}\}$, $\mathcal{P}_2 = \{\{1,2,3\},\{4,5\}\}$,
3. $\mathcal{P}_1 = \{\{1,2\},\{3,4\},\{5\}\}$, $\mathcal{P}_2 = \{\{1,2\},\{3\},\{4\},\{5\}\}$.

Partitional games

Definition (partitional game)

For any player set N, every coalition function $v \in \mathbb{V}(N)$ and any partition $\mathcal{P} \in \mathfrak{P}(N)$, $(v, \mathcal{P})$ is called a partitional game.

Definition (solution function for partitional games)

A function σ that attributes, for each partitional game $(v, \mathcal{P})$, a payoff to each of v's players,

$$\sigma(v, \mathcal{P}) \in \mathbb{R}^{|N(v)|},$$

is called a solution function (on $\mathbb{V}^{\text{part}}$).

Definition (component-efficiency axiom)

A solution function (on $\mathbb{V}^{\text{part}}$) σ is said to obey the component-efficiency axiom if

$$\sum_{i \in C} \sigma_i (v, \mathcal{P}) = v(C)$$

holds for all partitional games $(v, \mathcal{P}) \in \mathbb{V}^{\text{part}}$ and all $C \in \mathcal{P}$.

Important axioms for partitional values

Definition ($\mathcal{P}$-symmetry)

Two players i and j from N are called $\mathcal{P}$-symmetric if they are symmetric and if $\mathcal{P}(i) = \mathcal{P}(j)$ holds.

Definition (symmetry axiom)

A solution function σ is said to obey the symmetry axiom if we have

$$\sigma_i(v, \mathcal{P}) = \sigma_j(v, \mathcal{P})$$

for all partitional games $(v, \mathcal{P}) \in \mathbb{V}^{\text{part}}$ and for any two $\mathcal{P}$-symmetric players i and j.

Important axioms for partitional values

null players

Definition (null-player axiom)

A solution function σ is said to obey the null-player axiom if we have

$$\sigma_i(v, \mathcal{P}) = 0$$

for all partitional games $(v, \mathcal{P}) \in \mathbb{V}^{\text{part}}$ and for every null player $i \in N$.

Definition (grand-coalition null-player axiom)

A solution function σ is said to obey the grand-coalition null-player axiom if we have

$$\sigma_i(v, \{N\}) = 0$$

for all partitional games $(v, \{N\}) \in \mathbb{V}^{\text{part}}$ and for every null player $i \in N$.

Important axioms for partitional values

additivity

Definition (additivity axiom)

A solution function σ is said to obey the additivity axiom if we have

$$\sigma(v+w,\mathcal{P}) = \sigma(v,\mathcal{P}) + \sigma(w,\mathcal{P})$$

for any two coalition functions $v, w \in \mathbb{V}$ with $N(v) = N(w)$ and any partition $\mathcal{P} \in \mathfrak{P}(N(v))$.

The Aumann-Dreze value

definition

procedure:

1. Restrict the coalition function to the components.
2. Calculate the Shapley value for the restricted function.

Definition (Aumann-Dreze value)

The Aumann-Dreze value on $\mathbb{V}^{\text{part}}$ is the solution function AD given by

$$AD_i(v, \mathcal{P}) := Sh_i\left(v|_{\mathcal{P}(i)}\right)$$

Lemma

We have $AD(v, \{N\}) = Sh(v)$.

Problem

Calculate the Aumann-Dreze payoffs for $\mathcal{P} = \{\{1\}, \{2,3\}\}$ *and the coalition functions*

- $u_{\{1,2\}}$ *and*
- $v_{\{1,2\},\{3\}}$.

Theorem

The Aumann-Dreze value is the unique solution function on $\mathbb{V}^{part}$ *that fulfills the symmetry axiom, the component-efficiency axiom, the null-player axiom and the additivity axiom.*

The Aumann-Dreze value rests on the premise that every component is an island. There are not interlinkages between players in a component and those outside.

The outside-option value due to Wiese

the last player in a component

The Wiese outside-option value (IGTR 2007) uses a rank-order definition.

1. assume a partition $\mathcal{P} \in \mathfrak{P}(N)$,
2. a rank order $\rho \in RO_N$ and
3. a player $i \in N$.

- player i belongs to the component $\mathcal{P}(i)$ and also to the set $K_i(\rho)$
- Is i the last player of his component, i.e., have all the other players from $\mathcal{P}(i)$ appeared before him?
- Criterion:

$$\mathcal{P}(i) \subseteq K_i(\rho)$$

The outside-option value due to Wiese

the last player in a component

Problem

Indicate the players that complete their components for the partition $\mathcal{P} = \{\{1,2,3\},\{4,5\},\{6\}\}$ *and the rank order* $\rho = (3,5,6,1,2,4)$*!*

The outside-option value due to Wiese

definition

Definition (Wiese value)

The Wiese value on $\mathbb{V}^{\text{part}}$ is the solution function W given by

$$W_i(v,\mathcal{P}) := \frac{1}{n!}\sum_{\rho\in RO_N}\begin{cases} v(\mathcal{P}(i)) - \sum_{j\in\mathcal{P}(i)\backslash\{i\}} MC_j(v,\rho), & \mathcal{P}(i)\subseteq K_i(\rho) \\ MC_i(v,\rho), & \text{otherwise,}\end{cases}$$

- The payoffs do not depend on the partition $\mathcal{P}$ in general, but on $\mathcal{P}(i)$ for player i.
- Players who are not last in their component obtain their marginal contributions, in general with respect to outside options.
- The last player in his component is the residual claimant.

The outside-option value due to Wiese

properties, generalization

Theorem (properties of the Wiese value)

The Wiese value obeys the symmetry axiom, the component-efficiency axiom, the grand-coalition null-player axiom and the additivity axiom. It violates the null-player axiom.

Lemma

We have $W(v, \{N\}) = Sh(v)$.

Lemma

Let v *be a simple game and* $\mathbb{W}(v)$ *its set of winning coalitions. Let there be a veto player* $i_{veto} \in N$, *i.e.,* $i_{veto} \in W$ *for all* $W \in \mathbb{W}(v)$. *Let* $\mathcal{P}$ *be a partition of* N *such that* $\mathcal{P}(i_{veto}) \in \mathbb{W}(v)$. *Then,* $W_{i_{veto}}(v, \mathcal{P}) = Sh_i(v)$.

Application: the gloves game

Wiese payoffs for an owner of a right glove

		no. of left-glove holders				
		0	1	2	3	4
no. of	1	0	0.500	0.667	0.750	0.800
right-	2	0	0.333	0.500	0.633	0.717
glove	3	0	0.250	0.367	0.500	0.614
holders	4	0	0.200	0.283	0.386	0.500

- Conjecture by Joachim Rosenmüller: the Wiese outside-option value of the gloves game converges to the core
- Corroboration:

replication factor	$n = 3, r = 1$	$n = 4, r = 1$
1	0.6666...	0.75
10	0.8531...	0.9278...
100	0.9734...	0.9904...

Application: the generalized gloves game

burning gloves?

Theorem

Let ω and $\hat{\omega}$ be two endowments and i, j $(i \neq j)$ two players from N. Let $\omega^k = \hat{\omega}^k$ for all $k \neq i$, $\omega_R^i = \hat{\omega}_R^i$ and $\omega_L^i < \hat{\omega}_L^i$. We denote the corresponding endowment games by v^ω and $v^{\hat{\omega}}$, respectively. For any partition $\mathcal{P}$, we get

- $$W_i\left(v^\omega, \mathcal{P}\right) \leq W_i\left(v^{\hat{\omega}}, \mathcal{P}\right),$$
- if $\mathcal{P}(i) = \{i, j\}$ and $\omega_L^i + \omega_L^j \geq \omega_R^i + \omega_R^j$,
  $$W_j\left(v^\omega, \mathcal{P}\right) \geq W_j\left(v^{\hat{\omega}}, \mathcal{P}\right),$$
- if $\mathcal{P}(j) \neq \mathcal{P}(i)$, $\omega_R^j \geq \omega_R^k$, and $\omega_L^j \leq \omega_L^k$ for all $k \in \mathcal{P}(j)$,
  $$W_j\left(v^\omega, \mathcal{P}\right) \leq W_j\left(v^{\hat{\omega}}, \mathcal{P}\right),$$

The outside-option value due to Casajus

the splitting axiom

The central axiom of Casajus' outside-option value (GEB 2009) is the splitting axiom:

Definition

Consider two partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $\mathcal{P}_1$ is finer than $\mathcal{P}_2$. If two players i and j belong to the same component of the finer partition ($j \in \mathcal{P}_1(i)$), we have

$$\sigma_i(v, \mathcal{P}_2) - \sigma_i(v, \mathcal{P}_1) = \sigma_j(v, \mathcal{P}_2) - \sigma_j(v, \mathcal{P}_1)$$

for all partitional games $(v, \mathcal{P}) \in \mathbb{V}^{\text{part}}$.

- "Splitting a structural coalition affects all players who remain in the same structural coalition in the same way.
- ... the gains/losses of splitting/separating should be distributed equally within a resulting structural coalition."

The outside-option value due to Casajus

the formula

Definition (Casajus value)

The Casajus value on $\mathbb{V}^{\text{part}}$ is the solution function Ca given by

$$Ca_i(v) := Sh_i(v) + \frac{v(\mathcal{P}(i)) - \sum_{j\in\mathcal{P}(i)} Sh_j(v)}{|\mathcal{P}(i)|}$$

- The players obtain the Shapley value which then has to be made component-efficient.
- If the sum of a component's Shapley values exceed the component's worth, the difference, averaged over all the players in the component, has to be "paid" by every player.

Problem

Determine the Casajus value for $N = \{1, 2, 3\}$ and the unanimity game $u_{\{1,2\}}$. Consider both $\mathcal{P} = \{\{1, 3\}, \{2\}\}$ and $\mathcal{P} = \{\{1, 2\}, \{3\}\}$.

The outside-option value due to Casajus

the axioms

Theorem (axiomatization of Casajus value)

The Casajus formula is axiomatized by

- *the symmetry axiom,*
- *the component-efficiency axiom,*
- *the grand-coalition null-player axiom,*
- *the additivity axiom and*
- *the splitting axiom.*

Application: elections in Germany for the Bundestag 2009

political parties

In 2009, 27 parties were present in one or several or all of the 16 German Länder. Among these, we find

- SPD – Sozialdemokratische Partei Deutschlands (16 lists)
- CDU – Christlich Demokratische Union Deutschlands (15 lists – not in Bavaria)
- FDP – Freie Demokratische Partei (16 lists)
- DIE LINKE – Die Linke (16 lists)
- GRÜNE – Bündnis 90/Die Grünen (16 lists)
- CSU – Christlich-Soziale Union in Bayern (1 list only – Bavaria)
- NPD – Nationaldemokratische Partei Deutschlands (16 lists)
- MLPD – Marxistisch-Leninistische Partei Deutschlands (16 lists)
- PIRATEN – Piratenpartei Deutschland (15 lists, not in Saxony)
- DVU – Deutsche Volksunion (12 lists)
- REP – Die Republikaner (11 lists)
- ödp – Ökologisch-Demokratische Partei (8 lists)

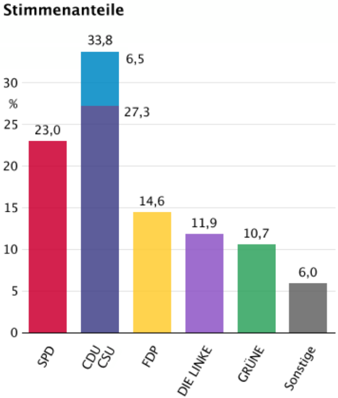
Application: elections in Germany for the Bundestag 2009

results I

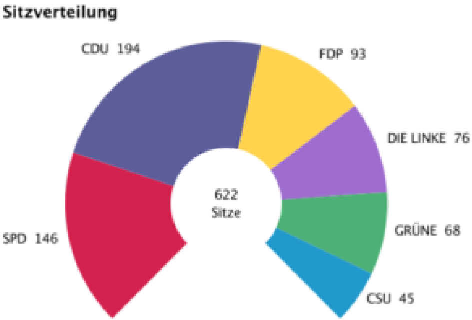
The election for the 17^{th} German Bundestag took playe on September, 27^{th}, 2009 and brought forth some extreme results:

- The participation rate (70.78%) was the lowest ever recorded in the Federal Republic of Germany.
- The Christian democrats and the liberals collected the number of votes necessary to form a government coalition.
- The liberals, the lefts and the greens obtained the best results in their party histories.
- The parties of the ruling grand coalition (Christian democrats, social democrtes) lost in big way:
    - The social democrats witnessed their worst result in any election for the Bundestag.
    - The Christian democrats saw their worst election result since 1949.

vote distribution



seat distribution



Coalitions functions and actual political outcome

Which parties can form government coalitions?

- The Christian democrats and the liberales ruled out a coalition with the leftist party.
- So did Frank-Walter Steinmeier on behalf of the social democrats.
- The liberals excluded a coalition with the greens and the social democrats (traffic-light coalition: red - yellow - green).
- The green party excluded the Jamaica coalition (black - yellow - green).

Coalitions functions and actual political outcome

three assumptions I

We suggest to consider three assumptions:

- assumption 1: Black - yellow and black - red are possible coalitions, only.
- assumption 2: Apart from the two coalitions mentioned in assumption 1, red - yellow -green and black - yellow - green are also possible
- assumption 3: All government coalitions are feasible except that the left party will not be seen in a coalition with the christian democrats or the liberals.

Coalitions functions and actual political outcome

three assumptions II

- assumption 1:

$$v(K) = \begin{cases} 1, & \text{CDU} \in K, \text{SPD} \in K \\ 1, & \text{CDU} \in K, \text{FDP} \in K \\ 0, & \text{otherwise} \end{cases}$$

with the Shapley payoffs

$$\text{Sh}_{\text{CDU}} = \frac{2}{3}, \text{Sh}_{\text{SPD}} = \frac{1}{6}, \text{Sh}_{\text{FDP}} = \frac{1}{6},$$

the Casajus payoffs for the black - yellow coalition

$$\chi_{\text{CDU}} = \frac{3}{4}, \chi_{\text{SPD}} = 0, \chi_{\text{FDP}} = \frac{1}{4}$$

and the Casajus payoffs for the black - red coalition

$$\chi_{\text{CDU}} = \frac{3}{4}, \chi_{\text{SPD}} = \frac{1}{4}, \chi_{\text{FDP}} = 0$$

Coalitions functions and actual political outcome

three assumptions III

- assumption 1 = assumption 2 = assumption 3:
  - The green party is a null player within a Jamaica (black - yellow - green) coalition.
  - The traffic-light (red - yellow - green) coalition does not avail of 50% of the seats in the Bundestag.

Coalitions functions and actual political outcome

power and portfolios

The actual government coalition has the Christian democrats form a government coalition with the liberal party.

- 11 portfolios are in the hands of CDU/CSU and
- 5 in the hands of the liberals
- with $\frac{5}{16}$ being slightly above $\frac{4}{16} = \frac{1}{4}$.

Coalitions functions and the Sonntagsfrage

results

On February, 19th, 2010, a few months after the 2009 elections, Infratest dimap reported these results:

	distribution of votes	... of seats
SPD	27	28
CDU	34	36
Left	10	10
FDP	10	10
Green	15	16

After the elections, Oskar Lafontaine

- a very prominent member of the left party and
- a former social democrat disliked by many social democrats

withdraws from politics, some social democrats are ready to review their willingness to form a coalition with the left party.

Coalitions functions and the Sonntagsfrage

assumption 3

assumption 3 yields the coalition function

$$v(K) = \begin{cases} 1, & \text{CDU} \in K, \text{SPD} \in K \\ 1, & \text{CDU} \in K, \text{Green} \in K \\ 1, & \text{SPD} \in K, \text{Green} \in K, \text{FDP} \in K \\ 1, & \text{SPD} \in K, \text{Green} \in K, \text{Left} \in K \\ 0, & \text{otherwise} \end{cases}$$

and the Shapley payoffs

$$\text{Sh}_{\text{CDU}} = \frac{22}{60}, \text{Sh}_{\text{SPD}} = \frac{17}{60}, \text{Sh}_{\text{FDP}} = \frac{2}{60}, \text{Sh}_{\text{Linke}} = \frac{2}{60}, \text{Sh}_{\text{Green}} = \frac{17}{60}$$

Coalitions functions and the Sonntagsfrage

Casajus payoffs

- grand coalition:

$$\chi_{\text{CDU}} = \frac{39}{72}, \chi_{\text{SPD}} = \frac{33}{72},$$

- black-green coalition:

$$\chi_{\text{CDU}} = \frac{39}{72}, \chi_{\text{Gr}} = \frac{33}{72},$$

- black-green-liberal coalition:

$$\chi_{\text{SDP}} = \frac{30}{72}, \chi_{\text{Gr}} = \frac{30}{72}, \chi_{\text{FDP}} = \frac{12}{72}$$

- red-red-green coalition:

$$\chi_{\text{SDP}} = \frac{30}{72}, \chi_{\text{Gr}} = \frac{30}{72}, \chi_{\text{Left}} = \frac{12}{72}$$

- Jamaica coalition

$$\chi_{\text{CDU}} = \frac{34}{72}, \chi_{\text{Gr}} = \frac{28}{72}, \chi_{\text{FDP}} = \frac{10}{72}.$$

Thus, the Christian democrats are free to choose the social democrats or the green party as a coalition partner.

- Consider the game on $N = \{1, 2, 3\}$ partly given by

$$\begin{aligned} v(i) &= 0, i = 1, 2, 3, \\ v(N) &= 1. \end{aligned}$$

- Consider the grand coalition $N = \{1, 2, 3\}$ and the partition

$$\mathcal{P} = \{\{1, 2\}, \{3\}\}$$

with the Wiese payoffs

$$\begin{aligned} W_1(v, \mathcal{P}) &= \frac{-2 + 2v(1,2) + v(2,3)}{6}, \\ W_2(v, \mathcal{P}) &= \frac{-2 + 2v(1,2) + v(1,3)}{6}. \end{aligned}$$

Contrasting the Casajus and the Wiese values

the Wiese value violates the splitting axiom

- We have

$$W_1(v, \{N\}) - W_1(v, \mathcal{P}) < W_2(v, \{N\}) - W_2(v, \mathcal{P})$$

  if an only if

$$v(1,3) - v(3) < v(2,3) - v(3)$$

  holds.

- Thus, splitting away from player 3 hurts player 1 less than player 2 iff player 1's marginal contribution with respect to player 3 is less than player 2's marginal contribution.
- Are outside options as important as inside opportunities?
  - The Casajus value says "yes" while
  - the Wiese value says "not quite".

Further exercises: Problem 1

Assume two men, Max (M) and Onno (O), who both love Ada (A). Their coalition function is given

$$v(K) = \begin{cases} 0, & |K| \leq 1 \\ 6, & K = \{M, A\} \\ 4, & K = \{O, A\} \\ 1, & K = \{M, O\} \\ 2, & K = \{M, O, A\} \end{cases}$$

1. Calculate the AD payoffs and the outside options values due both to Casajus and Wiese for the partition $\mathcal{P} = \{\{M, A\}, \{O\}\}$!
2. Comment!