Applied cooperative game theory:

Axiomatizing the Shapley value

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Overview “Axiomatizing the Shapley value”

- Introduction
- The Shapley formula fulfills the four axioms
- ... and is the only solution function to do so
- A second axiomatization via marginalism
- A third axiomatization via balanced contributions
- Balanced contributions and power-over
- The Banzhaf value

Introduction

For any given set of axioms, we have three possibilities:

- There is no solution concept that fulfills all the axioms. That is, the axioms are contradictary.
- The axioms are compatible with several solution concepts.
- There is one and only one solution concept that fulfills the axioms. That is, the solution concept is axiomatized by this set of axioms.

Definition

A solution concept σ is said be axiomatized by a set of axioms if σ fulfills all the axioms and if any solution concept to do so is identical with σ.

Axiomatization of the Shapley value

Definition

Let σ be a solution function σ. σ obeys

- the efficiency (or Pareto) axiom if $\sum_{i \in N} \sigma_i(v) = v(N)$ holds,
- the symmetry axiom if $\sigma_i(v) = \sigma_j(v)$ is true for any two symmetric players i and j,
- the null-player axiom if we have $\sigma_i(v) = 0$ for any null player i and
- the additivity axiom in case of $\sigma(v + w) = \sigma(v) + \sigma(w)$ for any two coalition functions $v, w \in \mathbb{V}_N$.

Theorem (Shapley theorem)

The Shapley formula is axiomatized by the four axioms mentioned in the previous definition.

The marginal contributions fulfill the efficiency axiom

Definition (ρ-solution)

For a player set N and a rank order $\rho \in RO_N$, the ρ-solution is given by

$$\left(MC_1^\rho(v), ..., MC_n^\rho(v)\right).$$

Take any rank order $\rho \in RO_N$. We can savely assume $\rho = (1, ..., n)$.

$$\begin{aligned}
\sum_{i \in N} MC_i^\rho(v) &= \sum_{i \in N} \left[v(K_i(\rho)) - v(K_i(\rho) \setminus \{i\})\right] \\
&= [v(\{\rho_1\}) - v(\emptyset)] \\
&\quad + [v(\{\rho_1, \rho_2\}) - v(\{\rho_1\})] \\
&\quad + [v(\{\rho_1, \rho_2, \rho_3\}) - v(\{\rho_1, \rho_2\})] \\
&\quad + ... \\
&\quad + \left[v(\{\rho_1, ..., \rho_n\}) - v\left(\{\rho_1, ..., \rho_{n-1}\}\right)\right] \\
&= v(N) - v(\emptyset) \\
&= v(N).
\end{aligned}$$

The Shapley formula fulfills the efficiency axiom

Lemma

The ρ-solutions and the Shapley value fulfill the efficiency axiom.

Proof:

$$
\begin{aligned}
\sum_{i\in N} Sh_i(v) &= \sum_{i\in N} \frac{1}{n!} \sum_{\rho\in RO_N} MC_i^{\rho}(v) \\
&= \sum_{\rho\in RO_N} \frac{1}{n!} \sum_{i\in N} MC_i^{\rho}(v) \text{ (rearranging the summands)} \\
&= \sum_{\rho\in RO_N} \frac{1}{n!} v(N) \text{ (ρ-solutions are efficient)} \\
&= n! \frac{1}{n!} v(N) \\
&= v(N).
\end{aligned}
$$

The Shapley formula fulfills the symmetry axiom

Astonishingly, the symmetry axiom is not easy to show. We refer the reader to Osborne and Rubinstein (1994). Intuitively, symmetry is obvious. After all,

- two players are symmetric if they contribute in a similar fashion and
- the Shapley formula's inputs are these marginal contributions.

The Shapley formula fulfills the null-player axiom

A null player contributes nothing, per definition. The average of nothing is nothing.
Therefore, the null-player axiom holds for the Shapley value. Just look at

$$\begin{aligned} Sh_i(v) &= \frac{1}{n!}\sum_{\rho \in RO_N} MC_i^\rho(v) \\ &= \frac{1}{n!}\sum_{\rho \in RO_N} 0 \\ &= 0. \end{aligned}$$

The Shapley formula fulfills the additivity axiom

$$\begin{aligned}
& Sh_i(v+w) \\
= \; & \frac{1}{n!}\sum_{\rho\in RO_N} MC_i^{\rho}(v+w) \\
= \; & \frac{1}{n!}\sum_{\rho\in RO_N} \left[(v+w)(K_i(\rho)) - (v+w)(K_i(\rho)\setminus\{i\})\right] \\
= \; & \frac{1}{n!}\sum_{\rho\in RO_N} \left[v(K_i(\rho)) - v(K_i(\rho)\setminus\{i\})\right] \\
& + \frac{1}{n!}\sum_{\rho\in RO_N} \left[w(K_i(\rho)) - w(K_i(\rho)\setminus\{i\})\right] \\
= \; & Sh_i(v) + Sh_i(w).
\end{aligned}$$

… and is the only solution function to do so I

We remind the reader of two important facts.

- The unanimity games u_T, $T \neq \emptyset$, form a basis of the vector space $\mathbb{V}_N$ so that every coalition function v is a linear combination of these games:
$$v = \sum_{T \in 2^N \backslash \{\emptyset\}} \lambda_T(v)\, u_T.$$
- For any game γu_T, $\gamma \in \mathbb{R}$, the players from $N \backslash T$ are the null players.

Consider, now, any solution function σ that obeys the four axioms. We obtain

$$\begin{aligned}\sum_{i\in T}\sigma_i(\gamma u_T) &= \sum_{i\in T}\sigma_i(\gamma u_T)+\sum_{i\in N\backslash T}\sigma_i(\gamma u_T) \text{ (null-player axiom)}\\ &= (\gamma u_T)(N) \text{ (Pareto axiom)}\\ &= \gamma u_T(N)\\ &= \gamma.\end{aligned}$$

The null players (from $N\backslash T$) get zero payoff, the (symmetric!) T-players share γ :

$$\sigma_i(\gamma u_T)=\begin{cases}\frac{\gamma}{|T|}, & i\in T\\ 0, & i\notin T.\end{cases}$$

Let now v be any coalition function on N. Using the above results and applying the additivity axiom several times, we find

$$\begin{aligned}
\sigma_i(v) &= \sigma_i\left(\sum_{T\in 2^N\backslash\{\emptyset\}} \lambda_T(v)\,u_T\right) \\
&= \sum_{T\in 2^N\backslash\{\emptyset\}} \sigma_i\left(\lambda_T(v)\,u_T\right) \text{ (additivity axiom)} \\
&= \sum_{T\in 2^N\backslash\{\emptyset\}} \begin{cases} \frac{\lambda_T(v)}{|T|}, & i\in T \\ 0, & i\notin T. \end{cases} \quad \left(\text{with } \gamma := \lambda_T(v)\right)
\end{aligned}$$

Thus, the axioms determine the payoffs. Since the Shapley formula fulfills the axioms, we obtain the desired result

$$\sigma = Sh.$$

And we are done.

A second axiomatization via marginalism

Definition (marginalism axiom)

A solution function σ is said to obey the marginalism axiom if, for any player $i \in N$ and any two coalition functions v, w with $N(v) = N(w)$,

$$MC_i^K(v) = MC_i^K(w), K \subseteq N(v)$$

implies

$$\sigma_i(v) = \sigma_i(w).$$

Theorem (Shapley theorem)

The Shapley formula is axiomatized by the symmetry axiom, the marginalism axiom and the efficiency axiom.

A third axiomatization via balanced contributions

Definition (restriction)

The restriction of v onto S is the coalition function

$$v|_S : 2^S \to \mathbb{R}, K \mapsto v|_S (K) = v(K).$$

Definition (axiom of balanced contributions)

A solution function σ is said to obey the axiom of balanced contributions if we have

$$\sigma_i (v) - \sigma_i \left(v|_{N \backslash \{j\}} \right) = \sigma_j (v) - \sigma_j \left(v|_{N \backslash \{i\}} \right), i, j \in N.$$

Theorem (Shapley theorem)

The Shapley formula is axiomatized by the efficiency axiom and the axiom of balanced contributions.

Balanced contributions and power-over: overview

- Definitions of power and of power-over
- Payoff differences
    - market power (gloves game)
    - emotional dependence
- Where would you be without me?
    - market power (gloves game)
    - emotional dependence
    - the robber game

Definitions of power and of power-over: actions and payoffs

- Felsenthal and Machover (1998) distinguish between
  - I-power (with I standing for "influence") and
  - P-power (with P denoting "prize" or "payoff").
- I- and P-power can be defined absolutely or in relation to other people (power-over).
- Max Weber's famous definition of power:
  Power is the probability that one actor within a social relationship will be in a position to carry out his own will despite resistance ... ."
- Literature: Harald Wiese: Applying cooperative game theory to power relations, in: Quality and Quantity, forthcoming.

Payoff reflections of power-over

- Idea: We measure power-over by looking at payoff differences.
- We consider two coalition functions, v and w.
  - v stands for the actual social or economic situation where player 1 exercises power over player 2.
  - w describes what the players would get if, contrary to the actual state of affairs, player 2 were not subject to the power exerted by player 1.
- Formally, we have

$$D_1 := \varphi_1(v) - \varphi_1(w) > 0$$

and

$$D_2 := \varphi_2(v) - \varphi_2(w) < 0.$$

Example: market power

- Gloves game with one left-glove holder (player 1) and 4 right-glove holders (players 2 through 5).
- The Shapley value is $\left(\frac{4}{5}, \frac{1}{20}, \frac{1}{20}, \frac{1}{20}, \frac{1}{20}\right)$ where $\frac{1}{4}\left(1 - \frac{4}{5}\right) = \frac{1}{20}$.
- Reference point: equal splitting of gains between player 1 and player 2 to whom player 1 happens to sell the left glove.
- Then, player 1's power over player 2 is reflected by

$$D_1 = \varphi_1(v) - \varphi_1(w) = \frac{4}{5} - \frac{1}{2} = \frac{3}{10}$$

and

$$D_2 = \varphi_2(v) - \varphi_2(w) = \frac{1}{20} - \frac{1}{2} = -\frac{9}{20}.$$

Example: emotional dependence

- One player M (man) is more independent of player W (woman) than the other way around: $v(M) > v(W)$.
- The Shapley values are given by

$$
\begin{aligned}
\varphi_M &= \frac{1}{2}v(M) + \frac{1}{2}[v(M,W) - v(W)] = \frac{1}{2}v(M,W) + \frac{1}{2}[v(M) - \\
&> \frac{1}{2}v(M,W) + \frac{1}{2}[v(W) - v(M)] = \varphi_W.
\end{aligned}
$$

- Applying the egalitarian norm ($w(M) = w(W) = \frac{1}{2}v(M,W)$), we diagnose that he has power over her:

$$
\begin{aligned}
D_M &= \varphi_M(v) - \varphi_M(w) = \frac{1}{2}[v(M) - v(W)] \\
&> 0 > \frac{1}{2}[v(W) - v(M)] = D_W
\end{aligned}
$$

Withdrawing as a non-arbitrary reference point

- Fairness norms or other reference points are arbeitrary. Why not consider the differences

$$\varphi_1 (v) - \varphi_1 \left(v|_{N \backslash 2} \right)$$

and

$$\varphi_2 (v) - \varphi_2 \left(v|_{N \backslash 1} \right)$$

known from the axiom of balanced contributions.
- $\varphi_1 (v) - \varphi_1 \left(v|_{N \backslash 2} \right)$ measures the loss to player 1 if player 2 withdraws.
- How about this definition of power-over: Player 1 exerts power over player 2, if player 1 suffers less from a withdrawal by player 2 than vice versa.
- But: balanced contributions!!

Example: revisiting the gloves game

- Does not the left-glove holder 1 have power over the right-glove holders 2 through 5?
- We have the Shapley values

$$\begin{aligned} \left(\frac{4}{5},\frac{1}{20},\frac{1}{20},\frac{1}{20},\frac{1}{20}\right) \text{ for } N &= \{1,2,3,4,5\}, \\ \left(\frac{3}{4},\frac{1}{12},\frac{1}{12},\frac{1}{12}\right) \text{ for } N &= \{1,3,4,5\} \text{ and} \\ (0,0,0,0) \text{ for } N &= \{2,3,4,5\} \end{aligned}$$

- and therefore

$$\varphi_1(v) - \varphi_1\left(v|_{N\backslash 2}\right) = \frac{4}{5} - \frac{3}{4} = \frac{1}{20}$$

and

$$\varphi_2(v) - \varphi_2\left(v|_{N\backslash 1}\right) = \frac{1}{20} - 0.$$

- Therefore: Which price balances the contributions?

Example: revisiting emotional dependence

- Her payoff difference

$$\begin{aligned} & \varphi_W(v) - \varphi_W\left(v|_{N\backslash M}\right) \\ = \; & \left[\frac{1}{2}v(M,W) + \frac{1}{2}v(W) - \frac{1}{2}v(M)\right] - v(W) \\ = \; & \frac{1}{2}\left[v(M,W) - v(W) - v(M)\right]. \end{aligned}$$

- His payoff difference

$$\varphi_M(v) - \varphi_M\left(v|_{N\backslash W}\right) = \varphi_W(v) - \varphi_W\left(v|_{N\backslash M}\right).$$

- Therefore: Which actions balance the contributions? She has to wash up.
  - Then she suffers less from a break-down of the relationship and
  - his loss of her would be more serious than in a "fair" partnership.

Negative sanctions and the threat to withdraw I

- How abou the equality of the threats to withdraw in case of coercion?
- Example: a robber (player 1) points his gun to my, player 2's, head.
- What does it mean to withdraw? I cannot just quit the scene in the same way as I choose not to partake in a market game.
- Coalition function:
  - $v(1,2) = 0$: I hand over some money $c > 0$ to the robber so that his gain is my loss.
  - $v(2) = 0$? No, if I withdraw (do not hand over the money peacefully), the robber may injure me. Therefore: $v(2) = -i < 0$.
  - I can run away and force the robber to injure me. Then, he will be in fear of prosecution for injury; we have $v(1) = -f < 0$.

Negative sanctions and the threat to withdraw II

- We have $v(1) = v|_{N\backslash 2}(1) = -f$ and $\varphi_1\left(v|_{N\backslash 2}\right) = v(1)$ and therefore

$$\varphi_1(v) - \varphi_1\left(v|_{N\backslash 2}\right) = \underbrace{c}_{\substack{\text{money}\\ \text{robbed}}} - \underbrace{-f}_{\substack{\text{disutility from fear of}\\ \text{prosecution for injury}}}$$

- We have $v(2) = v|_{N\backslash 1}(2) = -i$ and $\varphi_2\left(v|_{N\backslash 1}\right) = v(2)$ and

$$\varphi_2(v) - \varphi_2\left(v|_{N\backslash 1}\right) = \underbrace{-c}_{\substack{\text{money given}\\ \text{to robber}}} - \underbrace{-i}_{\text{disutility from injury}}$$

- The equality between these two differences can now be used to calculate the money I will have to hand over to the robber:

$$c = \frac{i-f}{2}.$$

Research program

- For the Shapley value, the threat of withdrawal from a cooperative agreement has to be symmetric:
  - gloves game: price of glove
  - emotional dependence: who does the washing up
  - robbery: how much does the robber gain
- However, the holder of the scarce commodity, the man in the dependency example and the robber manage to "realize their own will ... against the resistance" of the other party.
- Whenever we have a seemingly asymmetric power-over relationship, we can look out for Weberian power by equalizing the payoff differences with respect to the threat of withdrawal.
- Examples: (symmetric!) power-over relationships may exist between parents and children, God and humans, a king and his subjects, a bureaucrat and people obtaining permission, master and slave, etc.

The Banzhaf formula I

The Banzhaf formula is given by

$$Bh_i(v) = \frac{1}{2^{n-1}} \sum_{\substack{K \subseteq N, \\ i \notin K}} \left[v(K \cup \{i\}) - v(K) \right], i \in N.$$

Again, an average of marginal contributions is calculated.

- Shapley: rank orders
- Banzhaf: coalitions which (do not) contain a given player i of which there are

$$\left| 2^{N \backslash \{i\}} \right| = 2^{|N \backslash \{i\}|} = 2^{n-1}$$

Problem

Given $N = \{1, 2, 3\}$, write down the coalitions that do not contain player i.

The Banzhaf formula II

Definition (pivotal coalition)

For a simple game v, $K \subseteq N$ is a pivotal coalition for $i \in N$ if $v(K) = 0$ and $v(K \cup \{i\}) = 1$. The number of i's pivotal coalitions is denoted by $\eta_i(v)$.

Problem

Find η_i for a null player and for a dictator.

Now, the Banzhaf index for player i can be rewritten as

$$\beta_i(v) = \frac{\eta_i}{2^{n-1}}.$$

The Banzhaf formula III

Problem

Calculate the Banzhaf payoffs for player 1 *in case of* $N = \{1,2,3\}$ *and* $u_{\{1,2\}}$. *What do you find for* $N = \{1,2,3,4\}$ *and* $u_{\{1,2,3\}}$?

Problem

Find the Banzhaf payoffs for $N = \{1,2,3,4\}$ *and the apex game* h_1 *defined by*

$$h_1(K) = \begin{cases} 1, & 1 \in K \text{ and } K \setminus \{1\} \neq \emptyset \\ 1, & K = N \setminus \{1\} \\ 0, & \text{sonst} \end{cases}$$

Does the Banzhaf solution fulfill Pareto efficiency?

Theorem (axiomatization of the Banzhaf value)

The Banzhaf formula is axiomatized by null-player axiom, the symmetry axiom, the marginalism axiom and the merging axiom.

The Banzhaf axioms I

Definition (merging players)

For a game (N, v) and two players $i, j \in N$, $i \neq j$, the merged game (N_{ij}, v_{ij}) is given by $N_{ij} = (N \backslash \{i, j\}) \cup \{ij\}$ and

$$v_{ij}(K) = \begin{cases} v(K), & K \subseteq N \backslash \{ij\} \\ v((K \backslash \{ij\}) \cup \{i, j\}), & ij \in K \end{cases}$$

for all $K \subseteq N_{ij}$.

Definition (merging axiom)

A solution function σ is said to obey the merging axiom if we have

$$\sigma_i(v) + \sigma_j(v) = \sigma_{ij}(N_{ij}, v_{ij})$$

for any merged game in the sense of the definition above.

The Banzhaf axioms II

Consider the gloves game $v_{\{1,2\},\{3\}}$.

- $Sh\left(v_{\{1,2\},\{3\}}\right) = \left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right)$, $Bh\left(v_{\{1,2\},\{3\}}\right) = \left(\frac{1}{4}, \frac{1}{4}, \frac{3}{4}\right)$
- Players 1 and 2 merge.
  - The new player 12 obtains the Shapley payoff $\frac{1}{2} > \frac{1}{6} + \frac{1}{6}$. No more competition.
  - The Banzhaf payoffs are $\frac{1}{2}$ for both 12 and 3.
- Players 2 and 3 merge. The new player 23 is a dictator with Shapley value 1 and Banzhaf value 1.