Applied cooperative game theory:
Dividends

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Overview "Dividends"

- Definition and interpretation
- Coalition functions as vectors
- Spanning and linear independence
- The basis of unanimity games

Definition and interpretation of dividends I

Definition (Harsanyi dividend)

Let $v \in \mathbb{V}_N$ be a coalition function. The dividend (also called Harsanyi dividend) is a coalition function d^v on N defined by

$$d^v(T) = \sum_{K \subseteq T} (-1)^{|T|-|K|} v(K).$$

Theorem (Harsanyi dividend)

For any coalition function $v \in \mathbb{V}_N$, its Harsanyi dividends are defined by the induction formula

$$\begin{aligned} d^v(S) &= v(S) \text{ for } |S| = 1, \\ d^v(S) &= v(S) - \sum_{K \subset S} d^v(K) \text{ for } |S| > 1 \end{aligned}$$

Definition and interpretation of dividends II

- Consider a player i who is a member of 2^{n-1} coalitions $T \subseteq N$.
- Player i "owns" coalition T together with the other players from T where his ownership fraction is $\frac{1}{|T|}$.
- Each coalition T brings forth a dividend $d^v(T)$.
- Player i should obtain

$$\sum_{i \in T \subseteq N} \frac{d^v(T)}{|T|} = Sh_i(v).$$

Coalition functions as vectors I

- We need $2^n - 1$ entries to describe any game $v \in \mathbb{V}_N$. (The worth of $\emptyset$ is always zero!)
- For example, $u_{\{1,2\}} \in V(\{1,2,3\})$ can be identified with the vector from $\mathbb{R}^7$

$$\left(\underbrace{0}_{\{1\}}, \underbrace{0}_{\{2\}}, \underbrace{0}_{\{3\}}, \underbrace{1}_{\{1,2\}}, \underbrace{0}_{\{1,3\}}, \underbrace{0}_{\{2,3\}}, \underbrace{1}_{\{1,2,3\}} \right).$$

Problem

Write down the vector that describes the Maschler game

$$v(K) = \begin{cases} 0, & |K| = 1 \\ 60, & |K| = 2 \\ 72, & |K| = 3 \end{cases}$$

Coalition functions as vectors II

You know how to sum vectors. We can also multiply a vector by a real number (scalar multiplication). Both operations proceed entry by entry:

Problem

Consider $v = (1, 3, 3)$, $w = (2, 7, 8)$ *and* $\alpha = \frac{1}{2}$ *and determine* $v + w$ *and* αw.

Spanning and linear independence I

Spanning

$\mathbb{R}^m$, $m \geq 1$, is a prominent class of vector spaces some of which obey $m = 2^n - 1$.

Definition (linear combination, spanning)

A vector $w \in \mathbb{R}^m$ is called a linear combination of vectors $v_1, ..., v_k \in \mathbb{R}^m$ if there exist scalars (also called coefficients) $\alpha_1, ..., \alpha_k \in \mathbb{R}$ such that

$$w = \sum_{\ell=1}^{k} \alpha_\ell v_\ell$$

holds. The set of vectors $\{v_1, ..., v_k\}$ is said to span $\mathbb{R}^m$ if every vector from $\mathbb{R}^m$ is a linear combinations of the vectors $v_1, ..., v_k$.

Consider, for example, $\mathbb{R}^2$ and the set of vectors

$$\{(1,2),(0,1),(1,1)\}.$$

Any vector (x_1, x_2) is a linear combination of these vectors by

$$\begin{aligned} & 2x_1(1,2) - (3x_1 - x_2)(0,1) - x_1(1,1) \\ = & (2x_1 - x_1, 4x_1 - (3x_1 - x_2) - x_1) \\ = & (x_1, x_2). \end{aligned}$$

Problem

Show that $(0,1)$ *is a linear combination of the other two vectors,* $(1,2)$ *and* $(1,1)$*!*

Using the result of the above exercise, we have

$$\begin{aligned}
& 2x_1(1,2) - (3x_1 - x_2)(0,1) - x_1(1,1) \\
= \; & 2x_1(1,2) - (3x_1 - x_2)\left[(1,2) - (1,1)\right] - x_1(1,1) \\
= \; & \left[2x_1 - (3x_1 - x_2)\right](1,2) - \left[x_1 + (3x_1 - x_2)\right](1,1)
\end{aligned}$$

so that any vector from $\mathbb{R}^2$ is a linear combination of just $(1,2)$ and $(1,1)$.

Spanning and linear independence IV

Linear independence

If we want to span $\mathbb{R}^2$ (or any $\mathbb{R}^m$), we try to find a minimal way to do so. Any vector in a spanning set that is a linear combination of other vectors in that set, can be eliminated.

Definition (linear independence)

A set of vectors $\{v_1, ..., v_k\}$ is called linearly independent if no vector from that set is a linear combination of other vectors from that set.

Problem

Are the vectors $(1,3,3)$, $(2,1,1)$ *and* $(8,9,9)$ *linearly independent?*

Spanning and linear independence V

Basis

Definition (basis)

A set of vectors $\{v_1, ..., v_k\}$ is called a basis for $\mathbb{R}^m$ if it spans $\mathbb{R}^m$ and is linearly independent.

An obvious basis for $\mathbb{R}^m$ consists of the m unit vectors

$$
\begin{aligned}
&(1, 0, ..., 0)\,,\\
&(0, 1, 0, ...,)\,,\\
&...,\\
&(0, ..., 0, 1)\,.
\end{aligned}
$$

Spanning and linear independence VI

Basis

Any $x = (x_1, ..., x_m)$ is a linear combination of these vectors by

$$\begin{aligned} & x_1 (1, 0, ..., 0) + x_2 (0, 1, 0, ...,) + ... + x_m (0, ..., 0, 1) \\ = \; & (x_1, 0, ..., 0) + (0, x_2, 0, ...,) + ... + (0, ..., 0, x_m) \\ = \; & (x_1, ..., x_m) . \end{aligned}$$

This proves that the unit vectors do indeed span $\mathbb{R}^m$.
In order to show linear independence, consider any linear combination of $m - 1$ unit vectors, for example

$$\alpha_1 (1, 0, ..., 0) + \alpha_2 (0, 1, 0, ...,) + ... + \alpha_{m-1} (0, ..., 0, 1, 0)$$

which is equal to $(\alpha_1, ..., \alpha_{m-1}, 0)$ and unequal to $(0, ..., 0, 1)$ for any coefficients $\alpha_1, ..., \alpha_{m-1}$.

Spanning and linear independence VII

Theorem (basis criterion)

Every basis of the vector space $\mathbb{R}^m$ has m elements. Any set of m elements of the vector space $\mathbb{R}^m$ that span $\mathbb{R}^m$ form a basis. Any set of m elements of the vector space $\mathbb{R}^m$ that are linearly independent form a basis.

Theorem (uniquely determined coefficients)

Let $\{v_1, ..., v_m\}$ be a basis of $\mathbb{R}^m$ and let x be any vector such that

$$x = \sum_{i=1}^{m} \alpha_i v_i = \sum_{i=1}^{m} \beta_i v_i .$$

Then $\alpha_i = \beta_i$ for all $i = 1, .., m$.

The basis of unit games

Unit vectors form a basis. For the vector space $\mathbb{V}_N$, this means that the $2^n - 1$ coalition functions v_T, $T \neq \emptyset$, given by

$$v_T(S) = \begin{cases} 1, & S = T \\ 0, & S \neq T \end{cases}$$

form a basis.

The basis of unanimity games I

Lemma (unanimity games form basis)

The $2^n - 1$ *unanimity games* u_T, $T \neq \emptyset$, *form a basis of the vector space* $\mathbb{V}_N$.

Thus, there exist uniquely determined coefficients $\lambda^v(T)$ such that

$$v = \sum_{T \in 2^N \backslash \{\emptyset\}} \lambda^v(T)\, u_T$$

or

$$v(S) = \sum_{T \in 2^N \backslash \{\emptyset\}} \lambda^v(T)\, u_T(S),\, S \subseteq N.$$

holds.

The basis of unanimity games II

Lemma

The coefficients for the linear combination of v are

$$\lambda^v(T) := d^v(T).$$

The coefficients: an example I

Consider $N := \{1, 2, 3\}$ and the coalition function v given by

$$v(S) = \begin{cases} 0, & |S| = 1 \\ 60, & S = \{1,2\} \\ 48, & S = \{1,3\} \\ 30, & S = \{2,3\} \\ 72, & S = N \end{cases}$$

This coalition function can also be expressed by the vector

$$\begin{pmatrix} 0 \ (\{1\}) \\ 0 \ (\{2\}) \\ 0 \ (\{3\}) \\ 60 \ (\{1,2\}) \\ 48 \ (\{1,3\}) \\ 30 \ (\{2,3\}) \\ 72 \ (\{1,2,3\}) \end{pmatrix}$$

The coefficients: an example II

$$\begin{aligned}
d^v(\{1\}) &= d^v(\{2\}) = d^v(\{3\}) = 0, \\
d^v(\{1,2\}) &= \sum_{K \in 2^{\{1,2\}} \backslash \{\emptyset\}} (-1)^{|\{1,2\}|-|K|} v(K) \\
&= (-1)^{2-1} v(\{1\}) + (-1)^{2-1} v(\{2\}) + (-1)^{2-2} v(\{1,2\}) \\
&= 0 + 0 + 60, \\
d^v(\{1,3\}) &= (-1)^{2-1} v(\{1\}) + (-1)^{2-1} v(\{3\}) + (-1)^{2-2} v(\{1,3\}) \\
&= 48, \\
d^v(\{2,3\}) &= (-1)^{2-1} v(\{2\}) + (-1)^{2-1} v(\{3\}) + (-1)^{2-2} v(\{2,3\}) \\
&= 30 \text{ and} \\
d^v(\{1,2,3\}) &= \sum_{K \in 2^N \backslash \{\emptyset\}} (-1)^{|N|-|K|} v(K) \\
&= \ldots = -66
\end{aligned}$$

The coefficients: an example III

$$d^v(\{1,2\})\,u_{\{1,2\}} + d^v(\{1,3\})\,u_{\{1,3\}} + d^v(\{2,3\})\,u_{\{2,3\}} + d^v(N)\,u_N$$

$$= 60\begin{pmatrix}0\\0\\0\\1\\0\\0\\1\end{pmatrix} + 48\begin{pmatrix}0\\0\\0\\0\\1\\0\\1\end{pmatrix} + 30\begin{pmatrix}0\\0\\0\\0\\0\\1\\1\end{pmatrix} - 66\begin{pmatrix}0\\0\\0\\0\\0\\0\\1\end{pmatrix}$$

$$= \begin{pmatrix}0\\0\\0\\60\\48\\30\\72\end{pmatrix}$$

The coefficients: exercises

Problem

Calculate the coefficients for the following games on $N = \{1, 2, 3\}$:

- $v \in \mathbb{V}_N$ *is defined by* $v(\{1,2\}) = v(\{2,3\}) = v(\{1,2,3\}) = 1$ *and* $v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1,3\}) = 0.$
- $v \in \mathbb{V}_N$ *is defined by*

$$v(S) = \begin{cases} 0, & |S| \leq 1 \\ 8, & |S| = 2 \\ 9, & S = N \end{cases}$$

The proof of the lemma I

Lemma (unanimity games form basis)

The $2^n - 1$ unanimity games u_T, $T \neq \emptyset$, form a basis of the vector space $\mathbb{V}_N$.

- It is sufficient to show that the unanimity games are linearly independent.
- We use a proof by contradiction and assume that there is a unanimity game u_T that is a linear combination of the others:

$$u_T = \sum_{\ell=1}^{k.} \beta_\ell u_{T_\ell}$$

where

  - the coalitions T, T_1, ..., T_k are all pairwise different,
  - $k \leq 2^n - 2$ holds and
  - $\beta_\ell \neq 0$ holds for all $\ell = 1, ..., k$.

The proof of the lemma II

- Let us assume $|T| \leq |T_\ell|$ for all $\ell = 1, .., k$. If not, rearrange and rename.
- Using the coalition T as an argument, we now obtain

$$\begin{aligned} 1 &= u_T(T) \\ &= \sum_{\ell=1}^{k} \beta_\ell u_{T_\ell}(T) \\ &= \sum_{\ell=1}^{k} \beta_\ell \cdot 0 \\ &= 0 \end{aligned}$$

and hence the desired contradiction.

Problem

In the above proof, do you see why $u_{T_\ell}(T) = 0$ holds for all $\ell = 1, ..., k$?