Applied cooperative game theory:

Many games

Harald Wiese

University of Leipzig

June 2009

Overview "Many games"

- Simple games
- Three non-simple games
- Cost-division games
- Endowment games
- Properties of coalition functions

Simple games

Definition (monotonic game)

A coalition function $v \in \mathbb{V}_N$ is called monotonic if $\emptyset \subseteq S \subseteq S'$ implies $v(S) \leq v(S')$

Thus, monotonicity means that the worth of a coalition cannot decrease if other players join. Simple games are a special subclass of monotonic games:

Definition (simple game)

A coalition function $v \in \mathbb{V}_N$ is called simple if

- we have $v(K) = 0$ or $v(K) = 1$ for every coalition $K \subseteq N$ and.
- v is monotonic.

Thus, if S' is a superset of S (or S a subset of S'), we cannot have $v(S) = 1$ and $v(S') = 0$.

Veto players and dictators

Definition (veto player, dictator)

Let v be a simple game. A player $\in N$ is called a veto player if

$$v\left(N\backslash\left\{i\right\}\right)=0$$

holds. i is called a dictator if

$$v\left(S\right)=\begin{cases}1, & i\in S\\ 0, & \text{sonst}\end{cases}$$

holds for all $S\subseteq N$.

Problem

- *Can there be a coalition K such that $v\left(K\backslash\left\{i\right\}\right)=1$ for a veto player i or a dictator i?*
- *Is every veto player a dictator or every dictator a veto player?*

Contradictory and decidable

Definition (complement)

The set $N \backslash K := \{i \in N : i \notin K\}$ is called K's complement (with respect to N).

Definition (contradictory, decidable)

A simple game $v \in \mathbb{V}_N$ is called non-contradictory if $v(K) = 1$ implies $v(N \backslash K) = 0$.
A simple game $v \in \mathbb{V}_N$ is called decidable if $v(K) = 0$ implies $v(N \backslash K) = 1$.

Problem

- *Show that a simple game with a veto player cannot be contradictory.*
- *A simple game with two veto players cannot be decidable.*

Unanimity games

Definition (unanimity game)

For any $T \neq \emptyset$,

$$u_T(K) = \begin{cases} 1, & K \supseteq T \\ 0, & \text{otherwise} \end{cases}$$

defines a unanimity game.

- The players from T are the productive or powerful members of society.
  - Every player from T is a veto player and no player from $N \backslash T$ is a veto player.
  - In a sense, the players from T exert common dictatorship.
- For example, each player $i \in T$ possesses part of a treasure map.

Problem

Find the core and the Shapley value for $N = \{1, 2, 3, 4\}$ *and* $u_{\{1,2\}}$.

Apex games

Definition (apex game)

For $i \in N$ with $n \geq 2$, the apex game h_i is defined by

$$h_i(K) = \begin{cases} 1, & i \in K \text{ and } K \backslash \{i\} \neq \emptyset \\ 1, & K = N \backslash \{i\} \\ 0, & \text{otherwise} \end{cases}$$

Player i is called the main, or apex, player of that game.

Generally, we work with apex games for $n \geq 4$.

Problem

- *Consider h_1 for $n = 2$ and $n = 3$. How do these games look like?*
- *Is the apex player a veto player or a dictator?*
- *Show that the apex game is not contradictory and decidable.*
- *Find the Shapley value for the apex game h_1.*

Weighted voting games

Definition (weighted voting game)

A voting game v is specified by a quota q and voting weights g_i, $i \in N$, and defined by

$$v(K) = \begin{cases} 1, & \sum_{i \in K} g_i \geq q \\ 0, & \sum_{i \in K} g_i < q \end{cases}$$

In that case, the voting game is also denoted by $[q; g_1, ..., g_n]$.

The apex game h_1 for n players can be considered a weighted voting game given by

$$\left[n - 1; n - \frac{3}{2}, 1, ..., 1\right].$$

Problem

Consider the unanimity game u_T given by $t < n$ and $T = \{1, ..., t\}$. Can you express it as a weighted voting game?

UN Security Council

- 5 permanent members: China, France, Russian Federation, the United Kingdom and the United States
- 10 non-permanent members
- For substantive matters, the voting rule can be described by

$$[39; 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$$

where the weights 7 accrue to the five permanent and the weights 1 to the non-permanent members.

Problem

- *Show that every permanent member is a veto player.*
- *Show also that the five permanent members need the additional support of four non-permanent ones.*
- *Is the Security Council's voting rule non-contradictory and decidable?*

UN Security Council II

- For the fifteen members of the Security Council, we have

$$15! = 1.307.674.368.000$$

  rank orders.
- The Shapley values are

$$\begin{aligned} &0,19627 \text{ for each permanent member} \\ &0,00186 \text{ für each non-permanent member.} \end{aligned}$$

Buying a car I

- Andreas (A) has a used car he wants to sell, Frank (F) and Tobias (T) are potential buyers with willingness to buy of 700 and 500, respectively.
- Coalition function:

$$
\begin{aligned}
v(A) &= v(F) = v(T) = 0, \\
v(A, F) &= 700, \\
v(A, T) &= 500, \\
v(F, T) &= 0 \text{ and} \\
v(A, F, T) &= 700.
\end{aligned}
$$

Buying a car II

The core is the set of those payoff vectors (x_A, x_F, x_T) that fulfill

$$x_A + x_F + x_T = 700$$

and

$$\begin{aligned} x_A &\geq 0, x_F \geq 0, x_T \geq 0, \\ x_A + x_F &\geq 700, \\ x_A + x_T &\geq 500 \text{ and} \\ x_F + x_T &\geq 0. \end{aligned}$$

Buying a car III

- Tobias obtains

$$\begin{aligned} x_T &= 700 - (x_A + x_F) \text{ (efficiency)} \\ &\leq 700 - 700 \text{ (by } x_A + x_F \geq 700) \\ &= 0 \end{aligned}$$

- and hence zero, $x_T = 0$.
- By $x_A + x_T \geq 500$, the seller Andreas can obtain at least 500.
- The core is the set of vectors (x_A, x_F, x_T) obeying

$$\begin{aligned} 500 &\leq x_A \leq 700, \\ x_F &= 700 - x_A \text{ and} \\ x_T &= 0. \end{aligned}$$

- Therefore, the car sells for a price between 500 and 700.

The Maschler game

Coalition function:

$$v(K) = \begin{cases} 0, & |K| = 1 \\ 60, & |K| = 2 \\ 72, & |K| = 3 \end{cases}$$

Core:

- Efficiency:

$$x_1 + x_2 + x_3 = 72$$

- and non-blockability:

$$\begin{aligned} x_1 &\geq 0, x_2 \geq 0, x_3 \geq 0, \\ x_1 + x_2 &\geq 60, x_1 + x_3 \geq 60 \text{ and } x_2 + x_3 \geq 60. \end{aligned}$$

- Summing the last three inequalities yields

$$2x_1 + 2x_2 + 2x_3 \geq 3 \cdot 60 = 180$$

  and hence a contradiction to efficiency.
- The core is empty!

The gloves game, once again I

- Gloves game with minimal scarcity:

$$\begin{aligned} L &= \{1, 2, ..., 100\} \\ R &= \{101, ..., 199\}. \end{aligned}$$

- Are the right-hand glove owners much better off?
- If

$$x = (x_1, ..., x_{100,} x_{101}, ..., x_{199}) \in core\,(v_{L,R})$$

then, by efficiency,

$$\sum_{i=1}^{199} x_i = 99.$$

The gloves game, once again II

- We now pick any left-glove holder $j \in \{1, 2, ..., 100\}$. We find

$$v(L \backslash \{j\} \cup R) = 99$$

and hence

$$\begin{aligned} x_j &= 99 - \sum_{\substack{i=1, \\ i \neq j}}^{199} x_i \text{ (efficiency)} \\ &\leq 99 - 99 \text{ (blockade by coalition } L \backslash \{j\} \cup R) \\ &= 0. \end{aligned}$$

- Therefore, we have $x_j = 0$ for every $j \in L$.
- Every right-glove owner can claim at least 1 because he can point to coalitions where he is joined by at least one left-glove owner.
- Therefore, every right-glove owner obtains the payoff 1 and every left-glove owner the payoff zero.

Cost division games I

- doctors with a common secretary or commonly used facilities
- firms organized as a collection of profit-centers
- universities with computing facilities used by several departments or faculties

Definition (cost-division game)

For a player set N, let $c : 2^N \to \mathbb{R}_+$ be a coalition function that is called a cost function. On the basis of c, the cost-savings game is defined by $v : 2^N \to \mathbb{R}$ and

$$v(K) = \sum_{i \in K} c(\{i\}) - c(K), K \subseteq N.$$

The idea behind this definition is that cost savings can be realized if players pool their resources so that $\sum_{i \in K} c(\{i\})$ is greater than $c(K)$ and $v(K)$ is positive.

Cost division games II

- Two towns A and B plan a water-distribution system.
- Cost:
  - Town A could build such a system for itself at a cost of 11 million Euro and
  - town B would need 7 million Euro for a system tailor-made to its needs.
  - The cost for a common water-distribution system is 15 million Euro.
- The cost function is given by

$$\begin{aligned} c(\{A\}) &= 11, c(\{B\}) = 7 \text{ and} \\ c(\{A, B\}) &= 15. \end{aligned}$$

- The associated cost-savings game is $v : 2^{\{A,B\}} \to \mathbb{R}$ defined by

$$\begin{aligned} v(\{A\}) &= 0, c(\{B\}) = 0 \text{ and} \\ v(\{A, B\}) &= 7 + 11 - 15 = 3. \end{aligned}$$

Cost division games III

- v's core is obviously given by

$$\left\{(x_A, x_B) \in \mathbb{R}^2_+ : x_A + x_B = 3\right\}.$$

- The cost savings of $3 = 11 + 7 - 15$ can be allotted to the towns such that no town is worse off compared to going alone. Thus, the set of undominated cost allocations is

$$\left\{(c_A, c_B) \in \mathbb{R}^2 : c_A + c_B = 15, c_A \leq 11, c_B \leq 7\right\}.$$

Cost division games IV

Problem

Calculate the Shapley values for c and v! Comment!

Endowment games I

Definition (endowment economy)

An endowment economy is a tuple

$$\mathcal{E} = \left(N, \mathcal{L}, \left(\omega^i\right)_{i \in N}, f\right)$$

consisting of

- the set of agents $N = \{1, 2, ..., n\}$,
- the finite set of goods $\mathcal{L} = \{1, ..., \ell\}$,
- for every agent $i \in N$, an endowment $\omega^i = \left(\omega_1^i, ..., \omega_\ell^i\right) \in \mathbb{R}_+^\ell$ where

$$\omega := \sum_{i \in N} \omega^i = \left(\sum_{i \in N} \omega_1^i, ..., \sum_{i \in N} \omega_\ell^i\right)$$

  is the economy's total endowment, and ...

Endowment games II

Definition

...and

- an aggregation function $f : \mathbb{R}^\ell \to \mathbb{R}$.

The aggregation function aggregates the different goods' amounts into a specific real number in the same way as the min-operator does in the gloves game.

Definition (endowment game)

Consider an endowment economy $\mathcal{E}$. An endowment game $v^{\mathcal{E}} : 2^N \to \mathbb{R}$ is defined by

$$v^{\mathcal{E}}(K) := f\left(\sum_{i \in K} \omega_1^i, ..., \sum_{i \in K} \omega_\ell^i\right).$$

Endowment games III

Definition (summing of endowment games)

For a player set N, consider two endowment economies $\mathcal{E}$ and $\mathcal{F}$ with endowments $\omega_{\mathcal{E}}$ and $\omega_{\mathcal{F}}$ and the derived endowment games $v_{\mathcal{E}}$ and $v_{\mathcal{F}}$. The endowment-based sum of these games is denoted by $v_{\mathcal{E}} \oplus v_{\mathcal{F}}$ and defined by

$$\begin{aligned} \omega_j^i &= \left(\omega^{\mathcal{E}}\right)_j^i + \left(\omega^{\mathcal{F}}\right)_j^i \text{ and} \\ \left(v_{\mathcal{E}} \oplus v_{\mathcal{F}}\right)(K) &:= f\left(\sum_{i \in K} \omega_1^i, ..., \sum_{i \in K} \omega_\ell^i\right). \end{aligned}$$

This summation operation is not (!) the summation defined in the vector space of coalition functions!

Endowment games IV

Definition (gains from trade)

For a player set N, consider two endowment economies $\mathcal{E}$ and $\mathcal{F}$. The gains from trade are defined by

$$GfT\left(\mathcal{E},\mathcal{F}\right)=\left(v_{\mathcal{E}}\oplus v_{\mathcal{F}}\right)\left(N\right)-v_{\mathcal{E}}\left(N\right)-v_{\mathcal{F}}\left(N\right).$$

Problem

Show that the gains from trade are zero for any gloves game $v_{\mathcal{E}}:=v_{L,R}$ *and* $v_{\mathcal{F}}:=v_{\mathcal{E}}$.

Superadditivity

Nearly all the coalition functions we work with in this book are superadditive. Roughly, superadditivity means that cooperation pays.

Definition (superadditivity)

A coalition function $v \in \mathbb{V}_N$ is called superadditive if for any two coalitons R and S

$$R \cap S = \emptyset$$

implies

$$v(R) + v(S) \leq v(R \cup S).$$

$v(R \cup S) - (v(R) + v(S)) \geq 0$ is called the gain from cooperation.

Problem

Are gloves games superadditive? How about the apex game and unanimity games?

Convexity I

Definition (convexity)

A coalition function $v \in \mathbb{V}_N$ is called convex if for any two coalitons S and S' with $S \subseteq S'$ and for all players $i \in N \backslash S'$, we have

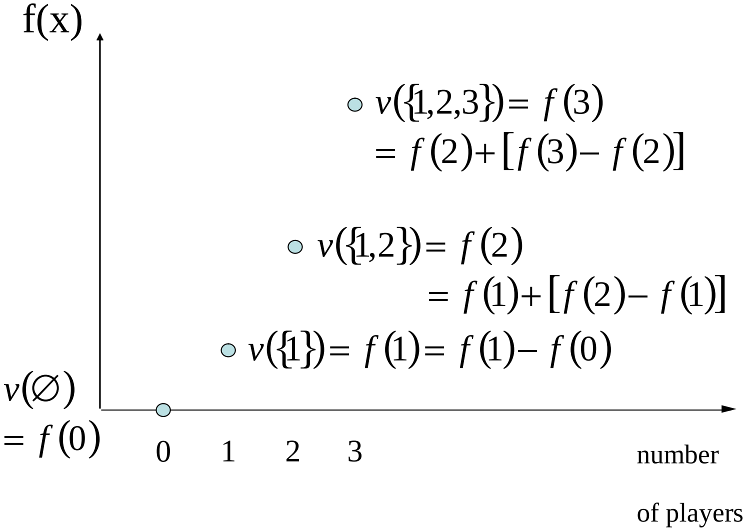
$$v\left(S \cup \{i\}\right) - v\left(S\right) \leq v\left(S' \cup \{i\}\right) - v\left(S'\right).$$

v is called strictly convex if the inequality is strict.

Problem

Is the unanimity game u_T convex? Is u_T strictly convex? Hint: Distinguish between $i \in T$ and $i \notin T$.

Convexity: Illustration of the term



Convexity III

Theorem (criterion for convexity)

A coalition function v is convex if and only if for all coalitions R and S, we have

$$v(R \cup S) + v(R \cap S) \geq v(R) + v(S).$$

v is strictly convex if and only if

$$v(R \cup S) + v(R \cap S) > v(R) + v(S)$$

holds for all coalitions R and S with $R \backslash S \neq \emptyset$ and $S \backslash R \neq \emptyset$.

Problem

Is the Maschler game convex? Is it superadditive?

Theorem

Every convex coalition function is superadditive.

The Shapley value and the core

The Shapley value need not be in the core even if the core is nonempty.

Problem

Consider the coalition function given by $N = \{1, 2, 3\}$ and

$$v(K) = \begin{cases} 0, & |K| = 1 \\ \frac{1}{2}, & K = \{1,3\} \text{ or } K = \{2,3\} \\ \frac{8}{10}, & K = \{1,2\} \\ 1, & K = \{1,2,3\} \end{cases}$$

Show that $\left(\frac{4}{10}, \frac{4}{10}, \frac{2}{10}\right)$ belongs to the core but that the Shapley value does not.

Theorem

If a coalition function v is convex, the Shapley value $Sh(v)$ lies in the core.