# Applied cooperative game theory:

## The gloves game

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# Overview part B: The Shapley value and the core

- The gloves game
- Many games
- Dividends
- Axiomatizing the Shapley value

# Overview "The gloves game"

- The coalition function
- Summing and zeros
- Solution concepts
- Pareto efficiency
- The core
- The Shapley value: the formula
- The Shapoley value: the axioms

# The coalition function I

The coalition function for the gloves game is given by

$$\begin{aligned} v_{L,R} \quad : \quad & 2^N \rightarrow \mathbb{R} \\ & K \mapsto v_{L,R}(K) = \min(|K \cap L|, |K \cap R|), \end{aligned}$$

where

- $N$ is the set of players (also called the grand coalition),
- $L$ the set of players holding a left glove and $R$ the set of right-glove owners together with $L \cap R = \emptyset$ and $L \cup R = N$,
- $v_{L,R}$ denotes the coalition function for the gloves game,
- $2^N$ stands for $N$'s power set, i.e., the set of all subsets of $N$ (the domain of $v_{L,R}$),
- $\mathbb{R}$ is the set of real numbers (the range of $v_{L,R}$),
- $K$ is a coalition, i.e., d.h. $K \subseteq N$ or $K \in 2^N$,
- $|K|$ means the number of elements (players) in $K$ and
- $\min(x, y)$ is the smallest of the two numbers $x$ and $y$.

# The coalition function II

## Definition (coalition function)

For any finite and nonempty player set $N = \{1, ..., n\}$, a coalition function $v : 2^N \to \mathbb{R}$ fulfills $v(\emptyset) = 0$. $v(K)$ is called coalition $K$'s worth.

## Problem

*Assume* $N = \{1, 2, 3, 4, 5\}$, $L = \{1, 2\}$ *and* $R = \{3, 4, 5\}$. *Find the worths of the coalitions* $K = \{1\}$, $K = \emptyset$, $K = N$ *and* $K = \{2, 3, 4\}$.

- Interpretation: the gloves game is a market game where the left-glove owners form one market side and the right-glove owners the other.
- Distinguish
  - the worth (of a coalition) from
  - the payoff accruing to players.

# Summing and zeros I

## Definition

For any finite and nonempty player set $N = \{1, ..., n\}$, a payoff vector

$$x = (x_1, ..., x_n) \in \mathbb{R}^n$$

specifies payoffs for all players $i = 1, ..., n$.

- It is possible to sum coalition functions and it is possible to sum payoff vectors. Summation of vectors is easy – just sum each component individually. For example, determine the sum of the vectors

$$\begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}!$$

- Note the difference between payoff-vector summation and payoff summation $\sum_{i=1}^n x_i$.

# Summing and zeros II

Vector summation is possible for coalition functions, too. For example, the sum

$$v_{\{1\},\{2,3\}} + v_{\{1,2\},\{3\}}$$

can be seen from

$$\left( \begin{array}{c} \emptyset : 0 \\ \{1\} : 0 \\ \{2\} : 0 \\ \{3\} : 0 \\ \{1,2\} : 1 \\ \{1,3\} : 1 \\ \{2,3\} : 0 \\ \{1,2,3\} : 1 \end{array} \right) + \left( \begin{array}{c} \emptyset : 0 \\ \{1\} : 0 \\ \{2\} : 0 \\ \{3\} : 0 \\ \{1,2\} : 0 \\ \{1,3\} : 1 \\ \{2,3\} : 1 \\ \{1,2,3\} : 1 \end{array} \right) = \left( \begin{array}{c} \emptyset : 0 \\ \{1\} : 0 \\ \{2\} : 0 \\ \{3\} : 0 \\ \{1,2\} : 1 \\ \{1,3\} : 2 \\ \{2,3\} : 1 \\ \{1,2,3\} : 2 \end{array} \right)$$

# Summing and zeros III

- Mathematically speaking, $\mathbb{R}^n$ and $\mathbb{V}_N$ can be considered as vector spaces.
- Vector spaces have a zero.
  - The zero from $\mathbb{R}^n$ is
$$\underset{\in\mathbb{R}^n}{0} = \left(\underset{\in\mathbb{R}}{0}, ..., \underset{\in\mathbb{R}}{0}\right)$$
  - In the vector space of coalition functions, $0 \in \mathbb{V}_N$ is the function that attributes the worth zero to every coalition, i.e.,
$$\underset{\in\mathbb{V}_N}{0}(K) = \underset{\in\mathbb{R}}{0} \text{ for all } K \subseteq N$$

# Solution concepts I

## Definition (solution function, solution correspondence)

A function $\sigma$ that attributes, for each coalition function $v$ from $G$, a payoff to each of $v$'s players,

$$\sigma(v) \in \mathbb{R}^{|N(v)|},$$

is called a solution function. Player $i$'s payoff is denoted by $\sigma_i(v)$. In case of $N(v) = \{1, ..., n\}$, we also write $(\sigma_1(v), ..., \sigma_n(v))$ for $\sigma(v)$ or $(\sigma_i(v))_{i \in N(v)}$. A correspondence that attributes a set of payoff vectors to every coalition function $v$,

$$\sigma(v) \subseteq \mathbb{R}^{|N(v)|}$$

is called a solution correspondence. Solution functions and solution correspondences are also called solution concepts.

# Solution concepts II

- Solution concepts can be described algorithmically. An algorithm is some kind of mathematical procedure (a more less simple function) that tells how to derive payoffs from the coalition functions. Examples:
  - player 1 obtains $v(N)$ and the other players zero,
  - every player gets 100,
  - every player gets $v(N)/n$,
  - every player $i$'s payoff set is given by $[v(\{i\}), v(N)]$ (which may be the empty set).
- Alternatively, solution concepts can be defined by axioms. For example, axioms might demand that
  - all the players obtain the same payoff,
  - no more than $v(N)$ is to be distributed among the players,
  - player 1 is to get twice the payoff obtained by player 2,
  - every player gets $v(N) - v(N \backslash \{i\})$.
- Ideally, solution concepts can be described both algorithmically and axiomatically.

# Pareto efficiency I

- Arguably, Pareto efficiency is the single most often applied solution concept in economics – rivaled only by Nash equilibrium from noncooperative game theory.
- For any game $v \in \mathbb{V}_N$, Pareto efficiency is defined by

$$\sum_{i \in N} x_i = v(N).$$

- Equivalently, Pareto efficiency means

$$\begin{aligned} \sum_{i \in N} x_i &\leq v_{L,R}(N) \text{ (feasibility) and} \\ \sum_{i \in N} x_i &\geq v_{L,R}(N) \text{ (the grand coalition cannot block } x\text{).} \end{aligned}$$

# Pareto efficiency II

In case of $\sum_{i=1}^{n} x_i < v_{L,R}(N)$, the players would leave "money on the table". All players together could block (or contradict) the payoff vector $x$ by proposing (for example) the (feasible!) payoff vector $y = (y_1, ..., y_n)$ defined by

$$y_i = x_i + \frac{1}{n}\left(v_{L,R}(N) - \sum_{i=1}^{n} x_i\right), i \in N.$$

**Problem**

*Find the Pareto-efficient payoff vectors for the gloves game $v_{\{1\},\{2\}}$!*

For the gloves game, the solution concept "Pareto efficiency" has two important drawbacks:

- We have very many solutions and the predictive power is weak.
- The payoffs for a left-glove owner does not depend on the number of left and right gloves in our simple economy.

# The core I

## Definition

### Definition (blockability and core)

Let $v \in \mathbb{V}_N$ be a coalition function. A payoff vector $x \in \mathbb{R}^n$ is called blockable by a coalition $K \subseteq N$ if

$$\sum_{i \in K} x_i < v(K)$$

holds. The core is the set of all those payoff vectors $x$ fulfilling

$$\begin{aligned} \sum_{i \in N} x_i &\leq v(N) \text{ (feasibility) and} \\ \sum_{i \in K} x_i &\geq v(K) \text{ for all } K \subseteq N \text{ (no blockade by any coalition).} \end{aligned}$$

Do you see that every payoff vector from the core is also Pareto efficient? Just take $K := N$.

# The core II

## Example

Consider $v_{\{1\},\{2\}}$!

- Pareto efficiency requires $x_1 + x_2 = 1$.
- Furthermore, $x$ may not be blocked by one-man coalitions:

$$\begin{aligned} x_1 &\geq v_{L,R}\left(\{1\}\right) = 0 \text{ and} \\ x_2 &\geq v_{L,R}\left(\{2\}\right) = 0. \end{aligned}$$

- Hence, the core is the set of payoff vectors $x = (x_1, x_2)$ obeying

$$x_1 + x_2 = 1, x_1 \geq 0, x_2 \geq 0.$$

- Are we not forgetting about $K := \emptyset$? No:

$$\sum_{i \in \emptyset} x_i = 0 \geq 0 = v\left(\emptyset\right).$$

# The core III:

## Exercises

- Determine the core for the gloves game $v_{L,R}$ with $L = \{1, 2\}$ and $R = \{3\}$.
- Why does the Pareto-efficient payoff vector

$$y = \left(\frac{1}{10}, \frac{1}{10}, \frac{8}{10}\right)$$

not lie in the core?

# The Shapley formula I: general idea

- In contrast to Pareto efficiency and the core, the Shapley value is a point-valued solution concept, i.e., a solution function.
- Shapley's (1953) article is famous for pioneering the twofold approach of algorithm and axioms.
- The idea behind the Shapley value is that every player obtains
  - an average of
  - his marginal contributions.

# The Shapley formula II: marginal contribution

The marginal contribution of player $i$ with respect to coalition $K$ is

"the value with him" minus "the value without him".

## Definition (marginal contribution)

Player $i$'s marginal contribution with respect to a coalition $K$ is denoted by $MC_i^K(v)$ and given by

$$MC_i^K(v) := v(K \cup \{i\}) - v(K \setminus \{i\}).$$

## Problem

*Determine the marginal contributions for* $v_{\{1,2,3\},\{4,5\}}$ *and*

- $i = 1, K = \{1, 3, 4\}$,
- $i = 1, K = \{3, 4\}$,
- $i = 4, K = \{1, 3, 4\}$,
- $i = 4, K = \{1, 3\}$.

# The Shapley formula III: rank orders

- Imagine three players outside the door who enter this room, one after the other.
- We have the rank orders:

$$(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1).$$

- Why 6 rank orders?
  - For a single player 1, we have just one rank order $(1)$.
  - The second player 2 can be placed before or after player 1 so that we obtain the $1 \cdot 2$ rank orders

$$\begin{gathered}(1,2), \\ (2,1).\end{gathered}$$

  - For each of these two, the third player 2 can be placed before the two players, in between or after them:

$$\begin{gathered}(3,1,2),(1,3,2),(1,2,3), \\ (3,2,1),(2,3,1),(2,1,3).\end{gathered}$$

  - Therefore, we have $2 \cdot 3 = 6$ rank orders.

# The Shapley formula IV: rank orders and marginal contributions

## Definition (rank order)

Bijective functions $\rho : N \to N$ are called rank orders or permutations on $N$. The set of all permutations on $N$ is denoted by $RO_N$. The set of all players "up to and including player $i$ under rank order $\rho$" is denoted by $K_i(\rho)$ and given by

$$\rho(j) = i \text{ and } K_i(\rho) = \{\rho(1), .., \rho(j)\}.$$

Player $i$'s marginal contribution with respect to rank order $\rho$:

$$MC_i^{\rho}(v) := MC_i^{K_i(\rho)}(v) = v(K_i(\rho)) - v(K_i(\rho) \setminus \{i\}).$$

# The Shapley formula V: an example

For rank order $(3,1,2)$, one finds the marginal contributions

$$v(\{3\}) - v(\emptyset), v(\{1,3\}) - v(\{3\}) \text{ and } v(\{1,2,3\}) - v(\{1,3\}).$$

They add up to $v(N) - v(\emptyset) = v(N)$.

**Problem**

*Find player 2's marginal contributions for the rank orders $(1,3,2)$ and $(3,1,2)$!*

# The Shapley formula VI: algorithm

- We first determine all the possible rank orders.
- We then find the marginal contributions for every rank order.
- For every player, we add his marginal contributions.
- Finally, we divide the sum by the number of rank orders.

**Definition (Shapley value)**

The Shapley value is the solution function $Sh$ given by

$$Sh_i(v) = \frac{1}{n!} \sum_{\rho \in RO_N} MC_i^{\rho}(v).$$

# The Shapley formula VII: the gloves game

- Consider the gloves game $v_{\{1,2\},\{3\}}$. We find

$$Sh_1\left(v_{\{1,2\},\{3\}}\right) = \frac{1}{6}.$$

- The Shapley values of the other two players can be obtained by the same procedure.
- However, there is a more elegant possibility.
  - The Shapley values of players 1 and 2 are identical because they hold a left glove each and are symmetric (in a sense to be defined shortly).
  - Thus, we have $Sh_2\left(v_{\{1,2\},\{3\}}\right) = \frac{1}{6}$.
  - Also, the Shapley value satisfies Pareto efficiency:

$$\sum_{i=1}^{3} Sh_i\left(v_{\{1,2\},\{3\}}\right) = v\left(\{1,2,3\}\right) = 1$$

  - Thus, we find

$$Sh\left(v_{\{1,2\},\{3\}}\right) = \left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right).$$

# The Shapley formula VIII: results for the gloves game

The following table reports the Shapley values for an owner of a right glove in a market with $r$ right-glove owners and $l$ left-glove owners:

| | | number $l$ of left-glove owners | | | | |
|---|---|---|---|---|---|---|
| | | 0 | 1 | 2 | 3 | 4 |
| number $r$ | 1 | 0 | 0,500 | **0,667** | 0,750 | 0,800 |
| of | 2 | 0 | 0,167 | 0,500 | 0,650 | 0,733 |
| right-glove | 3 | 0 | 0,083 | 0,233 | 0,500 | 0,638 |
| owners | 4 | 0 | 0,050 | 0,133 | 0,271 | 0,500 |

This table clearly shows how the payoff increases with the number of players on the other market side. The payoff $Sh_3\left(v_{\{1,2\},\{3\}}\right) = \frac{2}{3}$ is highlighted.

# The Shapley axioms I: overview

The Shapley value fulfills four axioms:

- the efficiency axiom: the worth of the grand coalition is to be distributed among all the players,
- the symmetry axiom: players in similar situations obtain the same payoff,
- the null-player axiom: a player with zero marginal contribution to every coalition, obtains zero payoff, and
- additivity axiom: if players are subject to two coalition functions, it does not matter whether we apply the Shapley value to the sum of these two coalition functions or apply the Shapley value to each coalition function separately and sum the payoffs.

# The Shapley axioms II: equality and efficiency

Let us consider a player set $N = \{1, ..., n\}$ and a solution function $\sigma$. It may or may not obey one or several of these axioms:

**Definition (equality axiom)**

A solution function $\sigma$ is said to obey the equality axiom if

$$\sigma_i(v) = \sigma_j(v)$$

holds for all players $i, j \in N$.

**Definition (efficiency axiom)**

A solution function $\sigma$ is said to obey the efficiency axiom or the Pareto axiom if

$$\sum_{i \in N} \sigma_i(v) = v(N)$$

holds.

# The Shapley axioms III: symmetry

## Definition (symmetry)

Two players $i$ and $j$ are called symmetric (with respect to $v \in G$) if we have

$$v(K \cup \{i\}) = v(K \cup \{j\})$$

for any coalition function $v \in G$ and for every coalition $K$ that does not contain $i$ or $j$.

## Problem

*Are left-glove holders symmetric? Show $MC_i^K = MC_j^K$ for two symmetric players $i$ and $j$ fulfilling $i \notin K$ and $j \notin K$.*

## Definition (symmetry axiom)

A solution function $\sigma$ is said to obey the symmetry axiom if we have

$$\sigma_i(v) = \sigma_j(v)$$

# The Shapley axioms IV: null player

### Definition (null player)

A player $i \in N$ is called a null player (with respec to $v \in G$) if

$$v(K \cup \{i\}) = v(K)$$

holds for any coalition function $v \in G$ and every coalition $K$.

Can a left-glove holder be a null player? Shouldn't a null player obtain nothing?

### Definition (null-player axiom)

A solution function $\sigma$ is said to obey the null-player axiom if we have

$$\sigma_i(v) = 0$$

for any null player $i$.

# The Shapley axioms V: additivity

## Definition (additivity axiom)

A solution function $\sigma$ is said to obey the additivity axiom if we have

$$\sigma(v+w) = \sigma(v) + \sigma(w)$$

for any two coalition functions $v, w \in \mathbb{V}$, $N(v) = N(w)$.

- On the left-hand side, we add the coalition functions first and then apply the solution function.
- On the right-hand side we apply the solution function to the coalition functions individually and then add the payoff vectors.

## Problem

*Can you deduce $\sigma(0) = 0$ from the additivity axiom? Hint: use $v = w := 0$.*

# The Shapley axioms VI: equivalence

## Theorem (Shapley theorem)

*The Shapley formula is the unique solution function that fulfills the symmetry axiom, the efficiency axiom, the null-player axiom and the additivity axiom.*

- The Shapley formula fulfills the four axioms.
- The Shapley formula is the only solution function to do so.
- Differently put, the Shapley formula and the four axioms are equivalent – they specify the same payoffs.
- Cooperative game theorists say that she Shapley formula is "axiomatized" by the set of the four axioms.

## Problem

*Determine the Shapley value for the gloves game for* $L = \{1\}$ *and* $R = \{2, 3, 4\}$! *Hint: You do not need to write down all* 4! *rank orders. Try to find the probability that player* 1 *does not complete a pair.*