# Applied cooperative game theory:
## Pareto optimality in microeconomics

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# Pareto optimality in microeconomics

overview

1. Introduction: Pareto improvements
2. Identical marginal rates of substitution
3. Identical marginal rates of transformation
4. Equality between marginal rate of substitution and marginal rate of transformation

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# Introduction: Pareto improvements

- Judgements of economic situations
- Ordinal utility $\leftrightsquigarrow$ comparison among different people
- Vilfredo Pareto, Italian sociologue, 1848-1923:

## Definition

- Situation 1 is called Pareto superior to situation 2 (a Pareto improvement over situation 2) if no individual is worse off in the first than in the second while at least one individual is strictly better off.
- Situations are called Pareto efficient, Pareto optimal or just efficient if Pareto improvements are not possible.

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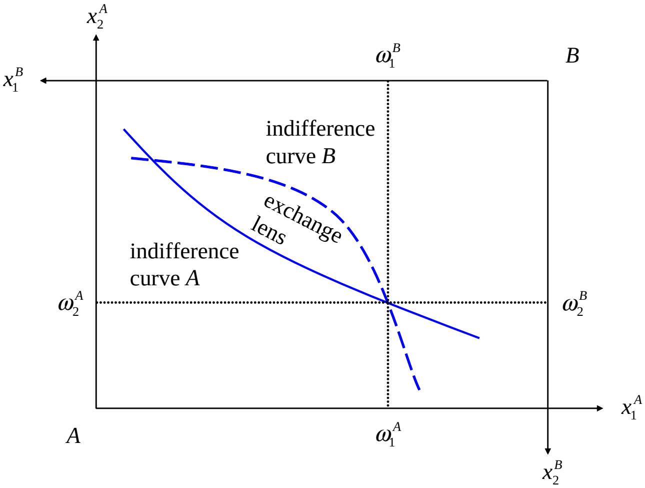
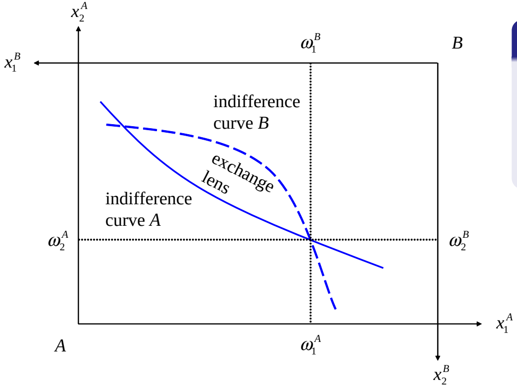
Francis Ysidro Edgeworth (1845-1926): "Mathematical Psychics"



# MRS = MRS

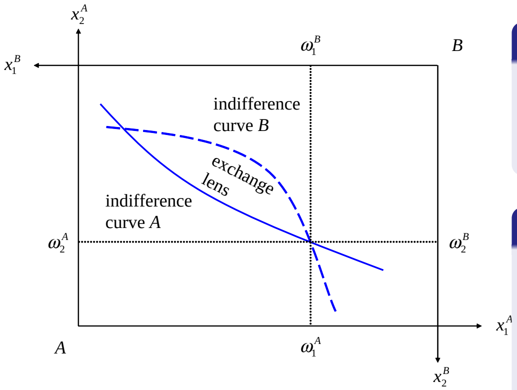



## Problem

*Which bundles of goods does individual A prefer to his endowment?*

# MRS = MRS

## The Edgeworth box for two consumers




### Problem

*Which bundles of goods does individual A prefer to his endowment?*

### Solution

*All those bundels $x_A$ to the right and above the indifference curve crossing $\omega_A$.*

# MRS = MRS

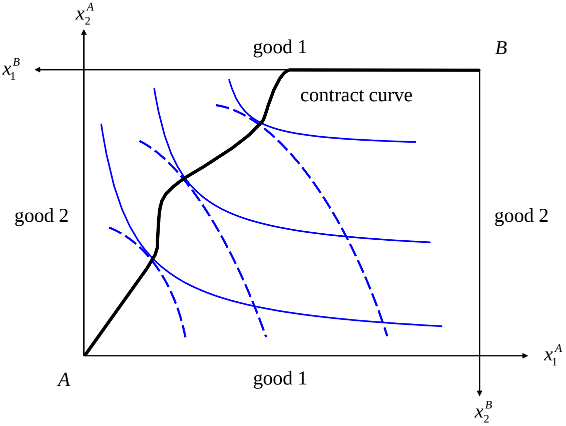
## The Edgeworth box for two consumers

- Consider

$$(3 =) \left| \frac{dx_2^A}{dx_1^A} \right| = MRS^A < MRS^B = \left| \frac{dx_2^B}{dx_1^B} \right| (= 5)$$

- If $A$ gives up a small amount of good 1,
  he needs $MRS^A$ units of good 2 in order to stay on his indifference curve.
- If individual $B$ obtains a small amount of good 1,
  she is prepared to give up $MRS^B$ units of good 2.
- $\frac{MRS^A + MRS^B}{2}$ units of good 2 given to $A$ by $B$ leave both better off
- Ergo: Pareto optimality requires $MRS^A = MRS^B$

Pareto optima in the Edgeworth box
– contract curve or exchange curve



# $\text{MRS} = \text{MRS}$

The Edgeworth box for two consumers

## Problem

*Two consumers meet on an exchange market with two goods. Both have the utility function* $U(x_1, x_2) = x_1 x_2$. *Consumer A's endowment is* $(10, 90)$, *consumer B's is* $(90, 10)$.

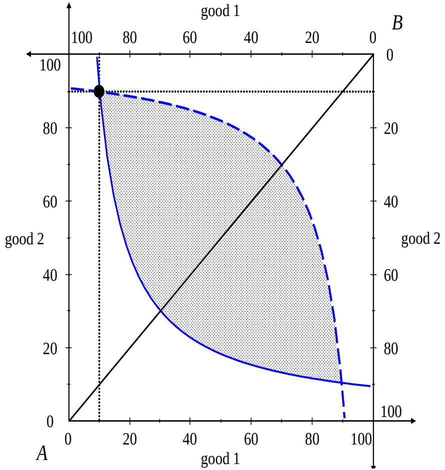
*a) Depict the endowments in the Edgeworth box!*

*b) Find the contract curve and draw it!*

*c) Find the best bundle that consumer B can achieve through exchange!*

*d) Draw the Pareto improvement (exchange lens) and the Pareto-efficient Pareto improvements!*

a)




### Solution

*b)* $x_1^A = x_2^A$,
*c)* $(70, 70)$ .
*d) The exchange lens is dotted. The Pareto efficient Pareto improvements are represented by the contract curve within this lens.*

# MR(T)S = MR(T)S

## The production Edgeworth box for two products

- Analogous to exchange Edgeworth box
- $MRTS_1 = \left|\frac{dC_1}{dL_1}\right|$
- Pareto efficiency

$$\left|\frac{dC_1}{dL_1}\right| = MRTS_1 \stackrel{!}{=} MRTS_2 = \left|\frac{dC_2}{dL_2}\right|$$

# MRS = MRS

## Two markets – one factory

- A firm that produces in one factory but supplies two markets 1 and 2.
- Marginal revenue $MR = \frac{dR}{dx_i}$ can be seen as the monetary marginal willingness to pay for selling one extra unit of good $i$.
  - Denominator good —> good 1 or 2, respectively
  - Nominator good —> "money" (revenue).
- Profit maximization by a firm selling on two markets 1 and 2 implies

$$\left|\frac{dR}{dx_1}\right| = MR_1 \stackrel{!}{=} MR_2 = \left|\frac{dR}{dx_2}\right|$$

# MRS = MRS

## Two firms in a cartel

- The monetary marginal willingness to pay for producing *and* selling one extra unit of good $y$ is a marginal rate of substitution.
- Two firms in a cartel maximize

$$\Pi_{1,2}\left(x_1, x_2\right) = \Pi_1\left(x_1, x_2\right) + \Pi_2\left(x_1, x_2\right)$$

  with FOCs

$$\frac{\partial \Pi_{1,2}}{\partial x_1} \overset{!}{=} 0 \overset{!}{=} \frac{\partial \Pi_{1,2}}{\partial x_2}$$

- If $\frac{\partial \Pi_{1,2}}{\partial x_2}$ were higher than $\frac{\partial \Pi_{1,2}}{\partial x_1}$ ...

How about the Cournot duopoly with FOCs

$$\frac{\partial \Pi_1}{\partial x_1} \overset{!}{=} 0 \overset{!}{=} \frac{\partial \Pi_2}{\partial x_2}?$$

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# MRT = MRT

## Two factories – one market

- Marginal cost $MC = \frac{dC}{dy}$ is a monetary marginal opportunity cost of production

$$MRT = \left| \frac{dx_2}{dx_1} \right|^{\text{transformation curve}}$$

- One firm with two factories or a cartel in case of homogeneous goods:

$$MC_1 \stackrel{!}{=} MC_2.$$

- Pareto improvements (optimality) have to be defined relative to a specific group of agents!

David Ricardo (1772–1823)

- "comparative cost advantage"
- $$4 = MRT^P = \left|\frac{dW}{dCl}\right|^P > \left|\frac{dW}{dCl}\right|^E = MRT^E = 2$$

## Lemma

*Assume that $f$ is a differentiable transformation function $x_1 \mapsto x_2$. Assume also that the cost function $C(x_1, x_2)$ is differentiable. Then, the marginal rate of transformation between good 1 and good 2 can be obtained by*

$$MRT(x_1) = \left|\frac{df(x_1)}{dx_1}\right| = \frac{MC_1}{MC_2}.$$

## Proof.

- Assume a given volume of factor endowments and given factor prices. Then, the overall cost for the production of goods 1 and 2 are constant along the transformation curve:

$$C(x_1, x_2) = C(x_1, f(x_1)) = \text{constant}.$$

- Forming the derivative yields

$$\frac{\partial C}{\partial x_1} + \frac{\partial C}{\partial x_2}\frac{df(x_1)}{dx_1} = 0.$$

- Solving for the marginal rate of transformation yields

$$MRT = -\frac{df(x_1)}{dx_1} = \frac{MC_1}{MC_2}.$$

- Before Ricardo:
  England exports cloth and imports wine if

$$\begin{array}{rcl} MC_{Cl}^{E} & < & MC_{Cl}^{P} \text{ and} \\ MC_{W}^{E} & > & MC_{W}^{P} \end{array}$$

  hold.
- Ricardo:

$$\frac{MC_{Cl}^{E}}{MC_{W}^{E}} < \frac{MC_{Cl}^{P}}{MC_{W}^{P}}$$

  suffices for profitable international trade.

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# MRS = MRT

## Base case

- Assume

$$MRS = \left|\frac{dx_2}{dx_1}\right|^{\text{indifference curve}} < \left|\frac{dx_2}{dx_1}\right|^{\text{transformation curve}} = MRT$$

- If the producer reduces the production of good 1 by one unit ...
- Inequality points to a Pareto-inefficient situation
- Pareto-efficiency requires

$$MRS \stackrel{!}{=} MRT$$

# MRS = MRT

## Perfect competition - output space

- FOC output space

$$p \overset{!}{=} MC$$

- Let good 2 be money with price 1
  - *MRS* is
    - consumer's monetary marginal willingness to pay for one additional unit of good 1
    - equal to $p$ for marginal consumer
  - *MRT* is the amount of money one has to forgo for producing one additional unit of good 1, i.e., the marginal cost
- Thus,

$$\text{price} = \text{marginal willingness to pay} \overset{!}{=} \text{marginal cost}$$

which is also fulfilled by first-degree price discrimination.

# MRS = MRT

## Perfect competition - input space

- FOC output space

$$MVP = p\frac{dy}{dx} \stackrel{!}{=} w$$

  where

- the marginal value product $MVP$ is the monetary marginal willingness to pay for the factor use and
- $w$, the factor price, is the monetary marginal opportunity cost of employing the factor.

For the Cournot monopolist, the $MRS \stackrel{!}{=} MRT$ can be rephrased as the equality between

- the monetary marginal willingness to pay for selling – this is the marginal revenue $MR = \frac{dR}{dy}$ – and
- the monetary marginal opportunity cost of production, the marginal cost $MC = \frac{dC}{dy}$

Consuming household "produces" goods by using his income to buy them, $m = p_1 x_1 + p_2 x_2$, which can be expressed with the transformation function

$$x_2 = f(x_1) = \frac{m}{p_2} - \frac{p_1}{p_2} x_1.$$

Hence,

$$MRS \overset{!}{=} MRT = MOC = \frac{p_1}{p_2}$$

# Sum of MRS = MRT

## Public goods

- Definition: non-rivalry in consumption
- Setup:
  - $A$ and $B$ consume a private good $x$ ($x^A$ and $x^B$)
  - and a public good $G$
- Optimality condition

$$\begin{aligned} & MRS^A + MRS^B \\ = \; & \left| \frac{dx^A}{dG} \right|^{\text{indifference curve}} + \left| \frac{dx^B}{dG} \right|^{\text{indifference curve}} \\ \stackrel{!}{=} \; & \left| \frac{d\left(x^A + x^B\right)}{dG} \right|^{\text{transformation curve}} = MRT \end{aligned}$$

- Assume $MRS^A + MRS^B < MRT$. Produce one additional unit of the public good ...

# Sum of MRS = MRT

## Public goods

- Good $x$ as the numéraire good (money with price 1)
- Then, the optimality condition simplifies: sum of the marginal willingness' to pay equals the marginal cost of the good.

# Sum of MRS = MRT

## Public goods

### Problem

*In a small town, there live 200 people $i = 1, ..., 200$ with identical preferences. Person $i$'s utility function is $U_i(x_i, G) = x_i + \sqrt{G}$, where $x_i$ is the quantity of the private good and $G$ the quantity of the public good. The prices are $p_x = 1$ and $p_G = 10$, respectively. Find the Pareto-optimal quantity of the public good.*

### Solution

- $MRT = \left| \frac{d\left(\sum_{i=1}^{200} x_i\right)}{dG} \right|$ *equals* $\frac{p_G}{p_x} = \frac{10}{1} = 10$.
- *MRS for inhabitant $i$ is* $\left| \frac{dx^i}{dG} \right|^{\text{indifference curve}} = \frac{MU_G}{MU_{x^i}} = \frac{\frac{1}{2\sqrt{G}}}{1} = \frac{1}{2\sqrt{G}}$.
- *Hence:* $200 \cdot \frac{1}{2\sqrt{G}} \overset{!}{=} 10$ *and* $G = 100$.

# Further exercises: Problem 1

Agent $A$ has preferences on $(x_1, x_2)$, that can be represented by $u^A(x_1^A, x_2^A) = x_1^A$. Agent $B$ has preferences, which are represented by the utility function $u^B(x_1^B, x_2^B) = x_2^B$. Agent $A$ starts with $\omega_1^A = \omega_2^A = 5$, and $B$ has the initial endowment $\omega_1^B = 4, \omega_2^B = 6$.

(a) Draw the Edgeworth box, including

- $\omega$,
- an indifference curve for each agent through $\omega$!

(b) Is $(x_1^A, x_2^A, x_1^B, x_2^B) = (6, 0, 3, 11)$ a Pareto-improvement compared to the initial allocation?

(c) Find the contract curve!