Applied cooperative game theory

Introduction

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Overview "Introduction"

- The players, the coalitions, and the coalition functions
- The Shapley value
- The outside option value
- The network value
- Cooperative and noncooperative game theories
- The book/The course

The players, the coalitions, and the coalition functions I

- player set $N = \{1, ..., n\}$
- subsets of N which are also called coalitions
- the coalitions of $N := \{1, 2, 3\}$ include
  - $\{1, 2\}$,
  - $\{2\}$,
  - $\emptyset$ (the empty set – no players at all) and
  - N (all players taken together – the grand coalition).

The players, the coalitions, and the coalition functions II

Example gloves game for $N = \{1, 2, 3\}$ where

- the two players 1 and 2 hold a left glove and
- player 3 holds a right glove.
- complementarity – pairs of gloves have a worth of 1.
- the coalition function for that gloves game is given by

$$\begin{aligned} v(\emptyset) &= 0, \\ v(\{1\}) &= v(\{2\}) = v(\{3\}) = 0, \\ v(\{1,2\}) &= 0, \\ v(\{1,3\}) &= v(\{2,3\}) = 1, \\ v(\{1,2,3\}) &= 1. \end{aligned}$$

- two sides of a market – demand and supply

The players, the coalitions, and the coalition functions III

Two pillars:

- Coalition functions describe the economic, social or political situation of the agents while
- solution concepts determine the payoffs for all the players from N taking a coalition function as input.

Thus,

$$\begin{aligned} & \text{coalition functions} \\ + \quad & \text{solution concepts} \\ \text{yield} \quad & \text{payoffs.} \end{aligned}$$

The Shapley value I

The Shapley value is the most useful solution concept in cooperative game theory.
In the gloves game,

$$Sh_1(v) = \frac{1}{6}, Sh_2(v) = \frac{1}{6}, Sh_3(v) = \frac{2}{3}.$$

- The Shapley value distributes the worth of the grand coalition $v(N) = 1$ among the three players $(Sh_1(v) + Sh_2(v) + Sh_3(v) = 1)$,
- allots the same payoff to the symmmetric (!) players 1 and 2,
- awards the lion's share to player 3 who possesses the scarce resource of a right glove.

The Shapley value II

Notation: x_i = payoff to player i

- The players bargain on how to divide the worth of the grand coalition, $v(N) = 1$, between them:

$$x_1 + x_2 + x_3 = 1.$$

- "Where would you be without me" arguments:
  - Player 3's threat against player 1:
    Without me, there would be only two left gloves and you, player 1, would lose

    $$x_1 - 0$$

  - Player 1's counter-threat against player 3:
    Without me, you, player 3 would lose

    $$x_3 - \frac{1}{2}$$

The Shapley value III

1. Equal bargaining power between 1 and 3:

$$\underbrace{x_1 - 0}_{\substack{\text{loss to player 1} \\ \text{if player 3 withdraws}}} = \underbrace{x_3 - \frac{1}{2}}_{\substack{\text{loss to player 3} \\ \text{if player 1 withdraws}}}.$$

2. equal bargaining power between 2 and 3 and
3. $x_1 + x_2 + x_3 = 1$

yield

$$(x_1, x_2, x_3) = \left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right).$$

The outside option value I

Assume that glove traders 1 (left glove) and 3 (right glove) agree to cooperate
$\longrightarrow$ partition on N

$$\{\{1,3\},\{2\}\}$$

into two components
Shapley value for the individual components
$=$ AD value (where A stands for Aumann and D for Dreze):

$$AD_1(v) = AD_3(v) = \frac{1}{2}, AD_2(v) = 0.$$

However, player 2 is an “outside option” for player 3.

The outside option value II

- Players 1 and 3 share the value of a glove:

$$x_1 + x_3 = 1$$

- Both are necessary to form the component $\{1, 3\}$.
- The gain from leaving player 2 out should be divided equally:

$$\underbrace{x_1 - Sh_1(v)}_{\substack{\text{gain for player 1} \\ \text{from forming component } \{1,3\}}} = \underbrace{x_3 - Sh_3(v)}_{\substack{\text{gain for player 3} \\ \text{from forming} \\ \text{component } \{1,3\}}} .$$

- The Casajus outside-option value is

$$(x_1, x_2, x_3) = \left(\frac{3}{4}, 0, \frac{1}{4}\right).$$

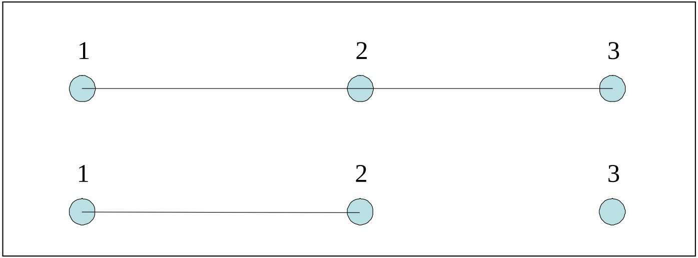
The network value I

- Links between two players = they can communicate or cooperate
- Example: unanimity game where 1 and 3 are the productive or powerful players:
  $N = \{1, 2, 3\}$ and

$$v(K) = \begin{cases} 1, & K \supseteq \{1, 3\} \\ 0, & \text{otherwise} \end{cases}$$

- Full network —> Shapley payoffs $\left(\frac{1}{2}, 0, \frac{1}{2}\right)$
  - Player 2 is unimportant (a null player, as we will say later) and obtains nothing.
  - The two players 1 and 3 are symmetric and share the worth of 1.

The network value II



The removal of the link 23 should harm both players equally:

$$\underbrace{x_2 - 0}_{\substack{\text{loss to player 2} \\ \text{if link is removed}}} = \underbrace{x_3 - 0}_{\substack{\text{loss to player 3} \\ \text{if link is removed}}}$$

Hence,

$$(x_1, x_2, x_3) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

Cooperative and noncooperative game theories

- Noncooperative game theory
  = strategy-oriented game theory
- Strategies, payoff functions
- Detailed specification of actions, knowledge and preferences
- Nash equilibrium, ...

- Cooperative game theory
  = payoff-oriented game theory
- Coalition functions
- Bird's eye view on payoffs (actions are implied)
- Core, Shapley value, ...

But:

- Nash program
- mixed models (biform games)

The book/The course I

- Part A: careful and slow introduction
  - Shapley value
  - core
  - Banzhaf solution
- Part B: partitional values
  - AD value
  - outside-option values (price of a glove, power of political parties in government coalition)
  - union value
- Part C: network values
  - Granovetter thesis
  - hierarchies

The book/The course II

- Part D: solidarity value, fighting value
- Part E: values for a continuum of players
  - working part-time
  - growth theory and distribution
  - evolutionary cooperative game theory
- Part F: Non-transferable utility
  - the allocation of goods within the Edgeworth box
  - the Nash bargaining solution.