Applied Cooperative Game Theory

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- consider a gloves game with two left-glove holders and four right-glove holders

$$N = \{\ell_1, \ell_2, r_1, r_2, r_3, r_4\}$$

- two matching pairs with the two remaining right-glove players unattached

$$\mathcal{P} = \{\{\ell_1, r_1\}, \{\ell_2, r_2\}, \{r_3\}, \{r_4\}\}$$

- How should the players in a matching-pair coalition split the worth of 1?

	AD	Wiese	Shapley	χ-value	CS-core
left with right (ℓ_1, ℓ_2)	.5000	.7167	.7333	.8000	1
right with left (r_1, r_2)	.5000	.2833	.1333	.2000	0
single right (r_3, r_4)	0	0	.1333	0	0

- problem with AD: equal split within matching pairs, though the left gloves are the scarce resource; better bargaining position due to outside options, e.g., ℓ_1 could argue that he could pool resources with r_3 instead of r_1 and also produce a worth of 1
- problem with CS-core: neglect of the productive role of the abundant resource in a matching pair

Any particular alliance describes only one particular consideration which enters the minds of the participants when they plan their behavior. Even if a particular alliance is ultimately formed, the division of the proceeds between the allies will be decisively influenced by the other alliances which each one might alternatively have entered. [...] Even if [...] one particular alliance is actually formed, the others are present in "virtual" existence: Although they have not materialized, they have contributed essentially to shaping and determining the actual reality.
[von Neumann, J., Morgenstern, O., 1944. Theory of Games and Economic Behavior. Princeton Univ. Press, p. 36]

During the course of negotiations there comes a moment when a certain coalition structure is "crystallized". The players will no longer listen to "outsiders", yet each coalition has still to adjust the final share of its proceeds. (This decision may depend on options outside the coalition, even though the chances of defection are slim).
[Maschler, M., 1992. The bargaining set, kernel, and nucleolus. In: Aumann, R.J., Hart, S. (Eds.), Handbook of Game Theory with Economic Applications, vol. I. Elsevier, Amsterdam, pp. 591–667. Chapter 34, p. 595]

- AD: recognizes productive role of matched right-glove holders, but neglects outside options
- CS-core: recognizes outside options, but neglects productive role of matched right-glove holders
- missing: component efficient CS-solutions that steer an intermediate course

- Wiese, H. (2007). Measuring the power of parties within government coalitions, International Game Theory Review 9(2): 307–322.
- outside option value
- drawback: lack of a "nice" axiomatization

- Casajus, A. (2009). Outside options, component efficiency, and stability, Games and Economic Behavior 65(1): 49–61.
- χ-value; why χ? most beautiful Greek letter

- **A**: powerful standard axiom; not in conflict with outside options
- **CE**: intended interpretation of components as productive units
- **CS**: natural restriction of **S** to CS-games
- **N**: combined with **CE** may make a CS-solution insensitive to outside options; example
  - $(N, u_T, \mathcal{P})$, $N = \{1, 2, 3\}$, $T = \{1, 2\}$, $\mathcal{P} = \{\{1\}, \{2, 3\}\}$
  - 3 is a Null player, $\varphi_3 = 0$; by **CE**, $\varphi_2 + \varphi_3 = 0$
  - hence, $\varphi_2 = 0 = \varphi_3$, even though 2 has better outside options than 3

Grand coalition Null player, GN If $i \in N$ is a Null player in (N, v), then $\varphi_i(N, v, \{N\}) = 0$.

- no outside options for $\{N\}$; **N** should/could be satisfied

- player i **dominates** player j in (N, v) if $MC_i^v(K) \geq MC_j^v(K)$ for all $K \subseteq N \backslash \{i, j\}$ and the inequality is strict for some such K

Component restricted dominance, CD If i dominates j in (N, v) and $j \in \mathcal{P}(i)$, $\varphi_i(N, v, \mathcal{P}) > \varphi_j(N, v, \mathcal{P})$.

- **CD** captures the idea that outside options as well as contributions to ones own coalition matter in a very weak sense
- however, **CD** and **CE** together are incompatible with **N**; example
  - $(N, u_T, \mathcal{P})$ as above
  - 3 is dominated by 2; by **CD,** $\varphi_2 > \varphi_3$;by **CE**, $\varphi_2 + \varphi_3 = 0$, hence, $0 > \varphi_3$; a Null player may obtain a negative payoff
  - may seem odd: 3 could avoid this negative payoff by forming $\{3\}$
  - no problem: for (N, u_T), $\mathcal{P}$ does not evolve; technically, $\mathcal{P}$ is not stable

Component independence, CI If $\mathcal{P}(i) = \mathcal{P}'(i)$, $i \in N$, then $\varphi_i(N, v, \mathcal{P}) = \varphi_i(N, v, \mathcal{P}')$.

- organization outside ones own component does not affect the payoff
- good axiom for an outside option CS-value? at least not in conflict
- not too bad: worth created by a component does not depend on the whole coalition structure
- outside options come into play when the coalitions **ultimately** have been formed
- therefore, the players are not necessarily restricted to the actual coalition structure when they bargain **within** their component on the distribution of that component's worth

Wiese outside option, WOO For all $P \in \mathcal{P}$ and $\emptyset \neq T \subseteq N$, we have $\varphi_{P \setminus T}(N, u_T, \mathcal{P}) = 0$ if $|\mathcal{P}[T]| = 1$ and

$$\varphi_{P \setminus T}(N, u_T, \mathcal{P}) = -\frac{|P \cap T|}{|T|} \frac{|P \setminus T|}{|P \cup T|}, \qquad \text{if } |\mathcal{P}[T]| > 1.$$

- technical, non-intuitive; just gives the Wiese value together with **CE**, **L, CE**, and **CS**

- $\mathcal{P}' \in \mathbb{P}(N)$ is **finer** than $\mathcal{P} \in \mathbb{P}(N)$ if $\mathcal{P}'(i) \subseteq \mathcal{P}(i)$ for all $i \in N$

Splitting, SP If $\mathcal{P}'$ is finer than $\mathcal{P}$, then for all $i \in N$ and $j \in \mathcal{P}'(i)$, we have

$$\varphi_i(N, v, \mathcal{P}) - \varphi_i(N, v, \mathcal{P}') = \varphi_j(N, v, \mathcal{P}) - \varphi_j(N, v, \mathcal{P}').$$

- splitting a component affects all players who remain together in the same way
- equal share of gains/losses of splitting/separating
- outside options and inside option are assessed to equally strong
- for $P \in \mathcal{P}$,
  - inside options: $v|_P$; outside options $v - v|_P$; measured for $\mathcal{P} = \{N\}$
  - $\varphi_i(N, v, \mathcal{P}) - \varphi_j(N, v, \mathcal{P}) = \varphi_i(N, v, \{N\}) - \varphi_j(N, v, \{N\})$
  - difference of payoffs within P depends on v, not on the decomposition $v = v|_P + (v - v|_P)$

α-splitting, αSP For $\alpha \in [0,1]$, $\mathcal{P} \in \mathbb{P}(N)$, $i \in N$, and $j \in \mathcal{P}(i)$, we have

$$\varphi_i(N, v, \mathcal{P}) - \varphi_j(N, v, \mathcal{P}) = \varphi_i(N, v|_P, \{N\}) - \varphi_j(N, v|_P, \{N\}) + \alpha \cdot \Big(\varphi_i(N, v - v|_P, \{N\}) - \varphi_j(N, v - v|_P, \{N\})\Big).$$

- for $\alpha = 1$, **αSP** becomes **SP** provided that φ obeys **A**
- for $\alpha = 0$, we have

$$\varphi_i(N, v, \mathcal{P}) - \varphi_j(N, v, \mathcal{P}) = \varphi_i(N, v|_P, \{N\}) - \varphi_j(N, v|_P, \{N\})$$

- sum up over $j \in \mathcal{P}(i)$: $\quad |\mathcal{P}(i)| \cdot \varphi_i(N, v, \mathcal{P}) - \varphi_{\mathcal{P}(i)}(N, v, \mathcal{P})$

$$= |\mathcal{P}(i)| \cdot \varphi_i(N, v|_P, \{N\}) - \varphi_{\mathcal{P}(i)}(N, v|_P, \{N\})$$

- suppose **CE** holds

$$\begin{aligned} |\mathcal{P}(i)| \cdot \varphi_i(N, v, \mathcal{P}) - v(\mathcal{P}(i)) &= |\mathcal{P}(i)| \cdot \varphi_i(N, v|_P, \{N\}) - v(\mathcal{P}(i)) \\ \varphi_i(N, v, \mathcal{P}) &= \varphi_i(N, v|_P, \{N\}) \end{aligned}$$

Theorem (Casajus 2009). There is a unique CS-value that satisfies **CE**, **CS**, **A**, **GN**, and **SP**.

Proof. uniqueness:

- let φ satisfy **CE**, **CS**, **A**, **GN**, and **SP**
- for $\mathcal{P} = \{N\}$, the first four axioms become **E**, **S**, **A**, and **N**
- hence, $\varphi(N, v, \{N\}) = \mathrm{Sh}(N, v)$
- any $\mathcal{P} \in \mathbb{P}(N)$ is finer than $\{N\}$; for $j \in \mathcal{P}(i)$, we have by **SP**

$$\begin{aligned} \varphi_i(N, v, \mathcal{P}) - \varphi_i(N, v, \{N\}) &= \varphi_j(N, v, \mathcal{P}) - \varphi_i(N, v, \{N\}) \\ \varphi_i(N, v, \mathcal{P}) - \mathrm{Sh}_i(N, v) &= \varphi_j(N, v, \mathcal{P}) - \mathrm{Sh}_j(N, v) \qquad (*) \end{aligned}$$

- summing up (*) over $j \in \mathcal{P}(i)$ and applying **CE** gives

$$\begin{aligned} |\mathcal{P}(i)| (\varphi_i(N, v, \mathcal{P}) - \mathrm{Sh}_i(N, v)) &= \varphi_{\mathcal{P}(i)}(N, v, \mathcal{P}) - \mathrm{Sh}_{\mathcal{P}(i)}(N, v) \\ |\mathcal{P}(i)| (\varphi_i(N, v, \mathcal{P}) - \mathrm{Sh}_i(N, v)) &= v(\mathcal{P}(i)) - \mathrm{Sh}_{\mathcal{P}(i)}(N, v) \end{aligned}$$

- hence,

$$\varphi_i(N, v, \mathcal{P}) = \mathrm{Sh}_i(N, v) + \frac{v(\mathcal{P}(i)) - \mathrm{Sh}_{\mathcal{P}(i)}(N, v)}{|\mathcal{P}(i)|} \qquad (**)$$

- existence: let φ defined by (**)
  - **A**: inherited the Shapley value
  - **CS, SP**: $i \in \mathcal{P}(i)$:

    $$\varphi_i(N, v, \mathcal{P}) - \varphi_j(N, v, \mathcal{P}) = \mathrm{Sh}_i(N, v) - \mathrm{Sh}_j(N, v)$$

  - **GN**: for $\mathcal{P} = \{N\}$, $\varphi = \mathrm{Sh}$, which satisfies **N**
  - **SP**: by construction □

- the CS-value defined by (**) is called the χ**-value** and denoted by "χ"
- is the Shapley value made component efficient by a brute force attack

Properties

- **CI**: immediate from the definition
- **CD**: immediate from $\chi_i(N,v,\mathcal{P}) - \chi_j(N,v,\mathcal{P}) = \mathrm{Sh}_i(N,v) - \mathrm{Sh}_j(N,v)$ for $j \in \mathcal{P}(i)$ and the fact that Sh satisfies the unrestricted version

$$\chi_i(N,u_T,\mathcal{P}) = \begin{cases} \frac{1}{|T|} & , i \in T, |\mathcal{P}(T)| = 1, \\ 0 & , i \notin T, |\mathcal{P}(T)| = 1, \\ \frac{|\mathcal{P}(i)\backslash T|}{|\mathcal{P}(i)||T|} & , i \in T, |\mathcal{P}(T)| > 1, \\ -\frac{|\mathcal{P}(i)\cap T|}{|\mathcal{P}(i)||T|} & , i \notin T, |\mathcal{P}(T)| > 1. \end{cases}$$

- $P \in \mathcal{P}$, $P \subseteq T$ or $P \subseteq N\backslash T$: **CS**+**CE** imply equal distribution of $v(P)$
- $T \subseteq P$: $v(P) = \mathrm{Sh}_P(N,v)$; T-players get $|N|^{-1}$, non-T-players get 0
- $P \cap T \neq \emptyset$, T: $v(P) = 0$; yet, $P \cap T$ could produce $u_T(T)$ together with $T\backslash P$; so any T-player from P looses $|T|^{-1}$ by cooperating within P
- should be refunded by *all* players in P; hence, a T-player obtains $\frac{1}{|T|}$ but has to pay an amount of $\frac{|P\cap T|}{|P||T|}$, i.e. he obtains a net payoff $\frac{|P\backslash T|}{|P||T|}$
- non-T-players pay $\frac{1}{|P||T|}$ to every T-player, in total $\frac{|P\cap T|}{|P||T|}$
- both types face costs of forming P: $\frac{|P\backslash T|}{|P||T|} < \frac{1}{|T|}, -\frac{|P\cap T|}{|P||T|} < 0$

- coalition formation as in the Hart and Kurz (1993) model: χ-stability
- since χ meets **CI**, all of the Hart and Kurz (1993) stability concepts coincide and can be characterized as follows

Definition. A coalition structure $\mathcal{P}$ for (N, v) is χ**-stable** iff for all $\emptyset \neq K \subseteq N$ there is some $i \in K$ such that $\chi_i(N, v, \mathcal{P}) \geq \chi_i(N, v, \{K, N \backslash K\})$.

- equivalently: there is no $K \subseteq N$ such that $\chi_i(N, v, \{K, N \backslash K\}) > \chi_i(N, v, \mathcal{P})$ for all $i \in K$
- no deviating coalition can make all its members better off **in terms of the resulting** χ**-payoffs** (the latter is important!)

Theorem (Casajus 2009). For all TU games, there are χ-stable coalition structures.

Proof. construct $\mathcal{P} = \{K_1, K_2, \ldots, K_k\}$ as follows:

- set $P_1 = \emptyset$ and continue by induction: $P_{n+1} = P_n \cup K_n$ for $n \geq 1$ and

$$K_n \in \underset{K \subseteq N \setminus P_n}{\operatorname{argmax}} \Delta(K) \qquad , \Delta(K) := \frac{v(K) - \mathrm{Sh}_K(N, v)}{|K|}$$

  for $n > 1$ until $P_{k+1} = N$
- suppose, $\mathcal{P}$ were not χ-stable, i.e., there were some $C \subseteq N$, $C \notin \mathcal{P}$ such that $\chi_i(N, v, \{C, N \setminus C\}) > \chi_i(N, v, \mathcal{P})$ for all $i \in C \subseteq N$
- the only reason for C not being in $\mathcal{P}$ is that $\mathcal{P}$ contains a structural coalition K_j such that $C \cap K_j \neq \emptyset$ and $\Delta(C) \leq \Delta(K_j)$
- by definition of $\chi_i(N, v, \{C, N \setminus C\}) \leq \chi_i(N, v, \mathcal{P})$ for $i \in C \cap K_j$, a contradiction. □

Remark All χ-stable coalition structures can be constructed in this way.

Corollary. If $\mathcal{P}$ is χ-stable for and i a Dummy player in (N, v), then $\chi_i(N, v, \mathcal{P}) = v(\{i\})$.

Proof. let i be a Dummy player in (N, v) and $\mathcal{P}$ be a χ-stable for (N, v)

- since $\mathrm{Sh}_i(N, v) = v(\{i\})$, we have $\Delta(\{i\}) = 0$
- by definition of χ and χ-stability, $\Delta(\mathcal{P}(i)) \geq \Delta(\{i\}) = 0$
- if $\Delta(\mathcal{P}(i)) > 0$, i.e.

$$\begin{aligned} 0 &< \Delta(\mathcal{P}(i)) = \frac{v(\mathcal{P}(i)) - \mathrm{Sh}_{\mathcal{P}(i)}(N, v)}{|\mathcal{P}(i)|} \\ &= \frac{v(\mathcal{P}(i) \setminus \{i\}) + v(\{i\}) - \mathrm{Sh}_{\mathcal{P}(i)\setminus\{i\}}(N, v) - v(\{i\})}{|\mathcal{P}(i)|} \\ &= \frac{v(\mathcal{P}(i) \setminus \{i\}) - \mathrm{Sh}_{\mathcal{P}(i)\setminus\{i\}}(N, v)}{|\mathcal{P}(i)|} \\ &< \frac{v(\mathcal{P}(i) \setminus \{i\}) - \mathrm{Sh}_{\mathcal{P}(i)\setminus\{i\}}(N, v)}{|\mathcal{P}(i) \setminus \{i\}|} = \Delta(\mathcal{P}(i) \setminus \{i\}) \end{aligned}$$

contradicting $\mathcal{P}$ being χ-stable. □

- simple: $v(K) \in \{0,1\}$, $K \subseteq N$
- monotonic: $S \subseteq T$ implies $v(S) \leq v(T)$, $S, T \subseteq N$
- non-contradictory: $v(S) = 1$ implies $v(N \backslash S) = 0$, $S \subseteq N$
- winning coalitions: $\mathbb{W} = \{K \subseteq N | v(K) = 1\}$; minimal winning coalitions: $\mathbb{W}_{\text{min}} = \{K \in \mathbb{W} | \forall S \subsetneq K : v(S) = 0\}$
- $\Delta(K) = -\frac{\text{Sh}_K}{|K|}$ if $K \notin \mathbb{W}$ and $\Delta(K) = \frac{1-\text{Sh}_K}{|K|}$ if $K \in \mathbb{W}$
- $\text{Sh}_i(N,v) \geq 0$; $\text{Sh}_i(N,v) = 0$ iff i is a Null player; $i \notin T$ for some $T \in \mathbb{W}$
- $\Delta(K) < 0$ if $K \notin \mathbb{W}$, but contains non-Null players
- $\mathbb{W}_{\text{min}} = \{T\}$; $T \subseteq K$ or $T \cap K = \emptyset$ entails $\Delta(K) = 0$; $\mathcal{P}$ is χ-stable iff $|\mathcal{P}[T]| = 1$
- $|\mathbb{W}_{\text{min}}| > 1$; $\text{Sh}_K < 1$, hence $\Delta(K) > 0$ for all $K \in \mathbb{W}_{\text{min}}$
- if $S \in \mathbb{W}_{\text{min}}$, $S \subseteq T$, $T \in \mathbb{W} \backslash \mathbb{W}_{\text{min}}$, then $\Delta(S) > \Delta(T)$
- hence, a χ-stable $\mathcal{P}$ contains some $K \in \mathbb{W}_{\text{min}}$ that maximizes $\Delta(K) = \frac{1-\text{Sh}_K}{|K|}$
- since v is non-contradictory, $v(S) = 0$ for all $S \subseteq N \backslash K$; the players in $N \backslash K$ form components containing players with the same Shapley payoff; the latter follows from $\Delta(K) = -\frac{\text{Sh}_K}{|K|}$ if $K \notin \mathbb{W}$

Theorem. In simple monotonic non-contradictory games, we have the following χ-stable coalition structures:

1. If $\mathbb{W}_{\text{min}} = \{T\}$, $T \subseteq N$ then $\mathcal{P}$ is χ-stable iff $|\mathcal{P}[T]| = 1$.
2. If $|\mathbb{W}_{\text{min}}| > 1$ then $\mathcal{P}$ is χ-stable iff there is some $T \in \mathbb{W}_{\text{min}} \cap \mathcal{P}$ such that $\frac{1-\text{Sh}_T}{|T|} \geq \frac{1-\text{Sh}_K}{|K|}$ for all $K \in \mathbb{W}_{\text{min}}$, and for all $i, j \in N \backslash T$, $j \in \mathcal{P}(i)$ implies $\text{Sh}_i = \text{Sh}_j$.

- (N, u_T); unique minimal winning coalition T
- by the Theorem, the coalition structures $\mathcal{P}$ satisfying $|\mathcal{P}(T)| = 1$ are the χ-stable ones

- $A_n = (N, v)$, $n \geq 2$, $N = \{0, 1, \ldots, n\}$, $v(K) = 1$ if $0 \in K$ and $|K| > 1$ or $N \setminus \{0\} \subseteq K$, else $v(K) = 0$
- $\mathrm{Sh}_0 = \frac{n-1}{n+1}$ and $\mathrm{Sh}_i = \frac{2}{n(n+1)}$, $i \neq 0$

$$\chi_0(A_n, \mathcal{P}) = \begin{cases} 0 & , \mathcal{P}(0) = \{0\} \\ \frac{n-2}{n} + \frac{2}{n|\mathcal{P}(I)|} & , \mathcal{P}(0) \neq \{0\} \end{cases}$$

$$\chi_i(A_n, \mathcal{P}) = \begin{cases} \frac{2}{n|\mathcal{P}(i)|} & , 0 \in \mathcal{P}(i) \\ \frac{1}{n} & , \mathcal{P}(i) = N \setminus \{0\} \\ 0 & , \mathcal{P}(i) \subsetneq N \setminus \{0\} \end{cases}$$

- A_n is simple monotonic non-contradictory, $|\mathbb{W}_{\min}| > 1$
- $K \in \mathbb{W}_{\min}$ if $K = \{0, i\}$, $i \neq 0$ or $K = N \setminus \{0\}$
- $\Delta(\{0, i\}) = \frac{n-1}{n(n+1)} = \Delta(N \setminus \{0\})$
- χ-stable coalition structures: (a) the apex player forms a coalition with one minor player and the other players are organized arbitrarily
- (b) all minor players form a coalition excluding the apex player

Theorem The χ-value is the unique CS-solution that obeys any of the following systems of axioms:

1. **CE**, **GN**, **CDM** (or **CBF** which "is" **BF** restricted to $j \in \mathcal{P}(i)$), and **SP**,
2. **CE**, **A**, **CS**, **GN**, **MO** and **OSP**.
3. **CE**, **CDM**, **GN**, **MO** and **OSP**.

Order preserving splitting, OSP. If $\mathcal{P}'$ is finer than $\mathcal{P}$ and $j \in \mathcal{P}(i)$, then $\varphi_i(N, v, \mathcal{P}) \geq \varphi_j(N, v, \mathcal{P})$ iff $\varphi_i(N, v, \mathcal{P}') \geq \varphi_j(N, v, \mathcal{P}')$.

Equality preserving splitting, ESP. If $\mathcal{P}'$ is finer than $\mathcal{P}$ and $j \in \mathcal{P}(i)$, then $\varphi_i(N, v, \mathcal{P}) = \varphi_j(N, v, \mathcal{P})$ iff $\varphi_i(N, v, \mathcal{P}') = \varphi_j(N, v, \mathcal{P}')$.

Modularity, MO. For all $x \in \mathbb{R}^N$, $\varphi(N, m_x, \mathcal{P}) = x$, where $m_x \in \mathbb{V}(N)$ and $m_x(K) = \sum_{i \in K} x_i$ for all $K \subseteq N$.

1. Consider the following theorem:
   Theorem For $\alpha \in [0, 1]$, there is a unique CS-value that satisfies **CE**, **CS**, **A**, **GN**, and $\boldsymbol{\alpha}$**SP**.
   (a) Prove the Theorem for $\alpha = 1$! Which CS-value is characterized?
   (b) Prove the Theorem!
   (c) Which CS-value is characterized for $\alpha \neq 1$?
   (d) Interpret!
2. Determine all χ-stable coalition structures for the gloves games! Use the following facts:
   - $[\lambda, \rho]$, $\rho \geq \lambda$; $\mathrm{Sh}_\ell(\lambda, \rho) > \mathrm{Sh}_r(\lambda, \rho) > 0$ if $\rho > \lambda$; $\mathrm{Sh}_\ell(\lambda, \rho) = \mathrm{Sh}_r(\lambda, \rho) = \frac{1}{2}$ if $\lambda = \rho$
   - $\rho > \lambda$: $\mathrm{Sh}_\ell(\lambda, \rho) + \mathrm{Sh}_r(\lambda, \rho) < 1$