Applied Cooperative Game Theory

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Overview

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- Coalition structures
- Efficiency vs Component efficiency
- The Owen value
- The Intermediate Game
- Characterizations of the Owen value
- The Aumann-Dreze value

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- so far, no groupings of the players were considered
- in reality, people often form groups in order to achieve their goals
- how to model: partitions $\mathcal{P}$ of the player set N, i.e., set of subsets of N which are pairwise disjoint and mutually exhaustive
  - for all $P, P' \in \mathcal{P}$ either $P = P'$ or $P \cap P' = \emptyset$
  - $N = \bigcup_{P \in \mathcal{P}} P$
  - **component** containing $i \in N : \mathcal{P}(i)$
  - for $K \subseteq N$: $\mathcal{P}(K) = \bigcup_{i \in K} \mathcal{P}(i)$
  - for $K \subseteq N$: $\mathcal{P}[K] = \{\mathcal{P}(i) \,|\, i \in K\}$
  - set of all partitions on N: $\mathbb{P}(N)$
  - **atomic** coalition structure: $[N] = \{\{i\} \,|\, i \in N\}$; **trivial** one: $\{N\}$
- **coalition structures** $=$ partitions of the player set
- **CS games**: $(N, v, \mathcal{P})$; TU game (N, v) , $v \in \mathbb{V}(N)$, with a coalition structure $\mathcal{P} \in \mathbb{P}(N)$
- **CS solution**, φ: assigns a vector $\varphi(N, v, \mathcal{P}) \in \mathbb{R}^N$ to any CS game $(N, v, \mathcal{P})$

Efficient versus component efficient solutions

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- basically, two interpretations of coalition structures

Efficiency (E) For all $(N, v, \mathcal{P})$, $\varphi_N (N, v, \mathcal{P}) = v(N)$.

- all players cooperate as grand coalition and bargain on the distribution of $v(N)$
- components of $\mathcal{P}$ may be viewed as bargaining blocs/units in this process
- concepts: Owen value

Component efficiency (CE) For all $(N, v, \mathcal{P})$ and $P \in \mathcal{P}$, $\varphi_P (N, v, \mathcal{P}) = v(P)$.

- components P of $\mathcal{P}$ are the productive units, create a worth of $v(P)$, respectively
- players in $P \in \mathcal{P}$ bargain on the distribution of $v(P)$
- concepts: AD value, Wiese value, χ-value

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- Owen, G. (1977). Values of games with a priori unions. In R. Henn & O. Moeschlin (Eds.), Essays in Mathematical Economics & Game Theory (pp. 76–88). Berlin: Springer

- set of orders compatible with $\mathcal{P}$:

$$\Sigma(N,\mathcal{P}) := \{\sigma \in \Sigma(N) \,|\, \forall P \in \mathcal{P},\ i,j \in P : |\sigma(i) - \sigma(j)| < |P|\}$$

  - $i,j,k \in N,\ j \in \mathcal{P}(i), \sigma(i) \leq \sigma(k) \leq \sigma(j) \Rightarrow k \in \mathcal{P}(i)$

- any $\sigma \in \Sigma(N,\mathcal{P})$ induces a unique order $\rho(\sigma) \in \Sigma(\mathcal{P})$: for all $P, P' \in \mathcal{P}$,

$$\rho(\sigma)(P) < \rho(\sigma)(P') \qquad \text{iff} \qquad \sigma(i) < \sigma(j) \text{ for some/all } i \in P \text{ and } j \in P'$$

Definition

The Owen value assigns to any CS game $(N, v, \mathcal{P})$ and $i \in N$ the payoff

$$\mathrm{Ow}_i(N,v,\mathcal{P}) := |\Sigma(N,\mathcal{P})|^{-1} \sum_{\sigma \in \Sigma(N,\mathcal{P})} MC_i^v(\sigma).$$

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- observation: $\Sigma(N, \{N\}) = \Sigma(N, [N]) = \Sigma(N)$
- entailing $\mathrm{Ow}(N, v, \{N\}) = \mathrm{Ow}(N, v, [N]) = \mathrm{Sh}(N, v)$
- A probability distribution $p \in W(\mathbb{P}(N))$ is called *symmetric* if we have $p(\mathcal{P}) = p(\pi(\mathcal{P}))$ for all $\mathcal{P} \in \mathbb{P}(N)$ and all bijections $\pi : N \to N$ where $\pi(\mathcal{P}) := \{\pi(P) \,|\, P \in \mathcal{P}\}$.

- Casajus, A. (2008): The Shapley value, the Owen value, and the veil of ignorance, in: International Game Theory Review, forthcoming.

Theorem (2008)

If $p \in W(\mathbb{P}(N))$ is symmetric then

$$\mathrm{Sh}(N, v) = \sum_{\mathcal{P} \in \mathbb{P}(N)} p(\mathcal{P}) \, \mathrm{Ow}(N, v, \mathcal{P}).$$

- $p(\{N\}) = 1$ and $p([N]) = 1$ are symmetric, this generalizes the remarks above

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- from the definition, it is clear that the Owen value obeys **E** as well as the following axioms:

Additivity, A. For all $v, w \in \mathbb{V}(N)$ and $\mathcal{P} \in \mathbb{P}(N)$, $\varphi(N, v + w, \mathcal{P}) = \varphi(N, v, \mathcal{P}) + \varphi(N, w, \mathcal{P})$.

Null player, N. For all $v, w \in \mathbb{V}(N)$, $\mathcal{P} \in \mathbb{P}(N)$, and Null player i in (N, v), $\varphi_i(N, v, \mathcal{P}) = 0$.

Marginality, M. If $MC_i^v(K) = MC_i^w(K)$ for all $K \subseteq N \setminus \{i\}$ then $\varphi_i(N, v, \mathcal{P}) = \varphi_i(N, w, \mathcal{P})$.

- by now, it should be clear to you that the Owen value does not meet the following ones:

Symmetry, S. For all $v \in \mathbb{V}(N)$, $\mathcal{P} \in \mathbb{P}(N)$, and symmetric players i, j in (N, v), $\varphi_i(N, v, \mathcal{P}) = \varphi_j(N, v, \mathcal{P})$.

Differential marginality, DM. If $MC_i^v(K) - MC_j^v(K) = MC_i^w(K) - MC_j^w(K)$ for all $K \subseteq N \setminus \{i, j\}$ then $\varphi_i(N, v, \mathcal{P}) - \varphi_j(N, v, \mathcal{P}) = \varphi_i(N, w, \mathcal{P}) - \varphi_j(N, w, \mathcal{P})$.

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- instead of **S** and **DM**, Ow satisfies component restricted versions of these axioms

Symmetry within components, CS. For all $v \in \mathbb{V}(N)$, $\mathcal{P} \in \mathbb{P}(N)$, and symmetric players i, j in (N, v), $j \in \mathcal{P}(i)$, $\varphi_i(N, v, \mathcal{P}) = \varphi_j(N, v, \mathcal{P})$.

Differential marginality within components, CDM. If $MC_i^v(K) - MC_j^v(K)$ $= MC_i^w(K) - MC_j^w(K)$ for all $K \subseteq N \setminus \{i, j\}$, $j \in \mathcal{P}(i)$ then $\varphi_i(N, v, \mathcal{P}) - \varphi_j(N, v, \mathcal{P}) = \varphi_i(N, w, \mathcal{P}) - \varphi_j(N, w, \mathcal{P})$.

- and, of course, we have

Lemma

(a) **A** *and* **CS** *imply* **CDM.** *(b)* **NG** *and* **CDM** *imply* **CS***.*

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- for $(N, v, \mathcal{P})$ consider the TU game $\left(\mathcal{P}, v^{\mathcal{P}}\right)$, $v^{\mathcal{P}} \in \mathbb{V}(\mathcal{P})$, the **intermediate game**, i.e, the game between components
- player set: $\mathcal{P}$; the components now are the players
- the coalition function is defined as follows:

$$v^{\mathcal{P}}(\mathcal{K}) = v\left(\bigcup_{P \in \mathcal{K}} P\right), \qquad \mathcal{K} \subseteq \mathcal{P}$$

Intermediate game, IG. For all $v \in \mathbb{V}(N)$, $\mathcal{P} \in \mathbb{P}(N)$, and $P \in \mathcal{P}$,

$$\varphi_P(N, v, \mathcal{P}) = \varphi_P\left(\mathcal{P}, v^{\mathcal{P}}, \{\mathcal{P}\}\right).$$

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Proposition. The Owen value satisfies the **IG**.
Proof. $\mathrm{Ow}_P(N, v, \mathcal{P}) = \sum_{i \in P} \mathrm{Ow}_i(N, v, \mathcal{P})$

$$
\begin{aligned}
&= |\Sigma(N, \mathcal{P})|^{-1} \sum_{i \in P} \sum_{\sigma \in \Sigma(N, \mathcal{P})} MC_i^v(\sigma) \\
&= |\Sigma(N, \mathcal{P})|^{-1} \sum_{\sigma \in \Sigma(N, \mathcal{P})} \sum_{i \in P} MC_i^v(\sigma) \text{ (changing the finite sums)} \\
&= |\Sigma(N, \mathcal{P})|^{-1} \sum_{\sigma \in \Sigma(N, \mathcal{P})} MC_P^{v^{\mathcal{P}}}(\rho(\sigma)),
\end{aligned}
$$

we now sum over the orders of $\rho \in \Sigma(\mathcal{P})$ and count how often the same order ρ with respect to $\Sigma(N, \mathcal{P})$ appears. This is $\prod_{P' \in \mathcal{P}} |P'|!$, independent of the order $\rho(\sigma)$, because we can permute any elements of $P' \in \mathcal{P}$. So we have:

$$
\begin{aligned}
&= |\Sigma(N, \mathcal{P})|^{-1} \prod_{P' \in \mathcal{P}} |P'|! \sum_{\rho \in \Sigma(\mathcal{P})} MC_P^{v^{\mathcal{P}}}(\rho) \\
&= |\Sigma(\mathcal{P})|^{-1} \sum_{\rho \in \Sigma(\mathcal{P})} MC_i^{v^{\mathcal{P}}}(\rho) = \mathrm{Sh}_P\left(\mathcal{P}, v^{\mathcal{P}}\right) = \mathrm{Ow}_P\left(\mathcal{P}, v^{\mathcal{P}}, \{\mathcal{P}\}\right) \qquad \square
\end{aligned}
$$

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Definition. Components $P, P' \in \mathcal{P}$ are symmetric in $(N, v, \mathcal{P})$ iff they are symmetric players in the intermediate game, i.e., $v(\mathcal{P}(K) \cup P) = v(\mathcal{P}(K) \cup P')$ for all $K \subseteq N \backslash (P \cup P')$.

Symmetry between components (SC) If $P, P' \in \mathcal{P}$ are symmetric in $(N, v, \mathcal{P})$, then $\varphi_P(N, v, \mathcal{P}) = \varphi_{P'}(N, v, \mathcal{P})$.

- roughly speaking, **SC** is something like **S** in the intermediate game
- since Ow meets **IG** and Sh obeys **S**, it should be clear that Ow satisfies **SC**

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Theorem

The Owen value is the unique CS solution that satisfies **E**, **A**, **N, CS**, *and* **SC**.

Proof.

- We have already seen that Ow meets **E**, **A**, **N, CS**, and **SC**.
- Let φ satisfy **E**, **A**, **N, CS**, and **SC**.
- By **A**, $\varphi(N, v, \mathcal{P}) = \sum_{T \in \mathcal{K}(N)} \varphi(N, \lambda_T(v) \cdot u_T, \mathcal{P})$.
- In $(N, \lambda_T(v) \cdot u_T, \mathcal{P})$ all $i \in N \backslash T$ are Null players. By **N**, $\varphi_i(N, \lambda_T(v) \cdot u_T, \mathcal{P}) = 0$.
- all $P, P' \in \mathcal{P}[T]$ are symmetric in the intermediate game
- since φ meets **SC,** $\varphi_P(N, v, \mathcal{P}) = \varphi_{P'}(N, v, \mathcal{P})$, $P, P' \in \mathcal{P}[T]$
- for $P \in \mathcal{P}[T]$

$$\sum_{P' \in \mathcal{P}[T]} \varphi_{P'}(N, \lambda_T(v) u_T, \mathcal{P}) \overset{\mathbf{E}}{=} \lambda_T(v) u_T(N) - \sum_{P \in \mathcal{P} \backslash \mathcal{P}[T]} \varphi_P(N, \lambda_T(v) u_T, \mathcal{P})$$

$$|\mathcal{P}[T]| \cdot \varphi_P(N, \lambda_T(v) u_T, \mathcal{P}) \overset{\mathbf{SC,N}}{=} \lambda_T(v) + 0$$

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- all $i, j \in P \cap T$ are symmetric in (N, v); hence, for $i \in P \cap T$

$$|P \cap T| \cdot \varphi_i(N, \lambda_T(v) u_T, \mathcal{P}) + \varphi_{P \setminus T}(N, \lambda_T(v) u_T, \mathcal{P}) = \frac{\lambda_T(v)}{|\mathcal{P}[T]|}$$

$$|P \cap T| \cdot \varphi_i(N, \lambda_T(v) u_T, \mathcal{P}) + 0 = \frac{\lambda_T(v)}{|\mathcal{P}[T]|}$$

$$\varphi_i(N, \lambda_T(v) u_T, \mathcal{P}) = \frac{\lambda_T(v)}{|\mathcal{P}[T]| \, |\mathcal{P}(i) \cap T|}$$

- i.e., φ is unique □

- from this we know for $T \in \mathcal{K}(N)$ and

$$\mathrm{Ow}_i(N, \lambda u_T, \mathcal{P}) = \begin{cases} 0, & i \in N \setminus T, \\ \dfrac{\lambda}{|\mathcal{P}[T]| \, |\mathcal{P}(i) \cap T|}, & i \in T \end{cases}$$

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- Khmelnitskaya, A. B., & Yanovskaya, E. B. (2007). Owen coalitional value without additivity axiom. Mathematical Methods of Operations Research, 66 (2), 255–261.

Theorem

The Owen value is the unique CS solution that satisfies **E**, **M, CS**, *and* **SC**.

- using **IG**, one has

Theorem

The Owen value is the unique CS solution that satisfies **E**, **A**, **N, CS**, *and* **IG**.

Disadvantages of the Owen-value

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- the Owen-value obeys the **E,** but what if the components are the productive units?
- efficiency does not seem plausible in any application, if we have coalition structures
- sometimes a concept is required, which uses the axiom **CE** and is geared to the Shapley value

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- Aumann, R.J., Drèze, J.H., 1974. Cooperative games with coalition structures. Int. J. Game Theory 3, 217–237.
- in fact, they consider this concept as the Shapley value

Definition. The AD-value assigns to any CS game $(N, v, \mathcal{P})$ and $i \in N$ the payoff

$$\mathrm{AD}_i(N, v, \mathcal{P}) := \mathrm{Sh}_i\left(\mathcal{P}(i), v|_{\mathcal{P}(i)}\right).$$

- i.e., the AD-value is the Shapley value restricted to the components of the coalition structure
- since Sh obeys **E**, it is immediate that AD meets **CE**
- further most characterizations of Sh have an analogon for AD: just replace **E** by **CE** and—if appropriate—employ component restricted versions of the other axioms

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Theorem The AD-value is the unique CS-solution that obeys any of the following systems of axioms:

1. **CE**, **A**, **N**, and **CS**,
2. **CE**, **M**, and **CS**,
3. **CE** and **CBC** (**CBC** "is" **BC** restricted to $j \in \mathcal{P}(i)$)
4. **CE**, **N**, and **CDM** (or **CBF** which "is" **BF** restricted to $j \in \mathcal{P}(i)$)

- proof of (1) and (3) is roughly as for Sh: fix $P \in \mathcal{P}$ and mimic the original arguments
- for (4), follow Remark 2 in Casajus (2009). Another characterization of the Owen value without the additivity axiom. Theory and Decision (forthcoming)
- for (2), combine the idea in the last paragraph of the proof of Theorem 2 in the paper just mentioned and the proof of the Young characterization of Sh