Applied Cooperative Game Theory

André Casajus, Martin Kohl, and Maria Näther

University of Leipzig

November 2013

- Owen, G. (1972). Multi-linear extensions of games, Management Science 18, 64–79.

Definition

The multilinear extention of $v \in \mathbb{V}(N)$ is the mapping $f_v : [0,1]^N \to \mathbb{R}$ given by

$$f_v(x) = \sum_{S \subseteq N, S \neq \emptyset} \left[\prod_{i \in S} x_i \prod_{i \in N \setminus S} (1 - x_i) \right] \cdot v(S), \qquad x \in [0,1]^N$$

- x_i : probability that $i \in N$ is present in a coalition
- probabilities are independent
- probability that $S \subseteq N$ forms: $\prod_{i \in S} x_i \prod_{i \in N \setminus S} (1 - x_i)$

Theorem

For all $v \in \mathbb{V}(N)$ *and* $i \in N$,

$$\mathrm{Sh}_i(N, v) = \int_0^1 \frac{\partial f_v}{\partial x_i}\bigg|_{x=(t,\dots,t)} dt.$$

Proof. We have

$$
\begin{aligned}
\frac{\partial f_v}{\partial x_i} &= \sum_{S\subseteq N, i\in S} \prod_{k\in S\setminus\{i\}} x_k \prod_{k\in N\setminus S} (1-x_k)\cdot v(S) \\
&\quad - \sum_{S\subseteq N, S\neq\emptyset, i\notin S} \prod_{k\in S} x_k \prod_{k\in (N\setminus S)\setminus\{i\}} (1-x_k)\cdot v(S) \\
\left.\frac{\partial f_v}{\partial x_i}\right|_{x=(t,\ldots,t)} &= \sum_{S\subseteq N, i\in S} t^{|S|-1}(1-t)^{|N|-|S|}\cdot v(S) \\
&\quad - \sum_{S\subseteq N, S\neq\emptyset, i\notin S} t^{|S|}\cdot(1-t)^{|N|-|S|-1} v(S) \\
&= \sum_{S\subseteq N\setminus\{i\}, S\neq\emptyset} t^{|S|}(1-t)^{|N|-|S|-1}\cdot\left(v(S\cup\{i\})-v(S)\right)
\end{aligned}
$$

and

$$
\int_0^1 t^{|S|}(1-t)^{|N|-|S|-1}\,dt = \frac{|S|!\,(|N|-|S|-1)!}{|N|!}. \qquad \text{(Exercise!)}
$$

Done. □

- Banzhaf, J. F. (1965). Weighted voting does not work: A mathematical analysis, Rutgers Law Review 19: 317–343.

- simple games: (N, v), $v(K) \in \{0, 1\}$, $K \subseteq N$
- consider a simple and monotonic game (N, v)
- player $i \in N$ is **pivotal** for $K \subseteq N \backslash \{i\}$ iff

$$v(K) = 0 \quad \text{and} \quad v(K \cup \{i\}) = 1$$

- $K \subseteq N \backslash \{i\}$ is a **swing** for $i \in N$ iff

$$i \text{ is pivotal for } K$$

Definition. The Banzhaf value assigns to a simple, monotonic TU game (N, v) and $i \in N$ the payoff

$$\mathrm{Ba}_i(N, v) = \frac{\text{number of swings for } i}{\text{number of potential swings for } i} = \frac{\text{number of swings for } i}{2^{|N|-1}}$$

- extension to general TU games
- Owen, G. (1975). Multilinear extensions and the Banzhaf value, Naval Research Logistic Quarterly 22: 741–750.

Definition

The Banzhaf value assigns to any TU game (N, v) and $i \in N$ the payoff

$$\mathrm{Ba}_i(N, v) = \sum_{K \subseteq N \setminus \{i\}} \frac{1}{2^{|N|-1}} \left(v(K \cup \{i\}) - v(K)\right).$$

- compare with the Shapley value

$$\mathrm{Sh}_i(N, v) = \sum_{K \subseteq N \setminus \{i\}} \frac{|K|!\,(|N| - |K| - 1)!}{|N|!} \left(v(K \cup \{i\}) - v(K)\right)$$

The Banzhaf value: properties #1

- the Shapley value is characterized by **E**, **A**, **S**, and **N**
- so the Banzhaf value must violate (at least) one of the axioms; guess which one!
- exactly: it's **E** (for $|N| > 2$); all other axioms are met; **N** and **A** are obvious, **S** is almost obvious
- let i, j be symmetric in (N, v); then, we have

$$\begin{aligned}
& \mathrm{Ba}_i(N, v) \\
= \; & \frac{1}{2^{|N|-1}} \sum_{K \subseteq N \setminus \{i,j\}} \left[v(K \cup \{i\}) - v(K) + v(K \cup \{i,j\}) - v(K \cup \{j\}) \right] \\
= \; & \frac{1}{2^{|N|-1}} \sum_{K \subseteq N \setminus \{i,j\}} \left[v(K \cup \{j\}) - v(K) + v(K \cup \{i,j\}) - v(K \cup \{i\}) \right] \\
= \; & \mathrm{Ba}_j(N, v)
\end{aligned}$$

- payoffs for $(N, \lambda \cdot u_T)$, $T \in \mathcal{K}(N)$, $\lambda \in \mathbb{R}$
- we have

$$MC_i^{\lambda \cdot u_T}(K) = \begin{cases} 0, & i \in N \backslash T, \\ 0 & i \in T,\ T \backslash \{i\} \nsubseteq K, \\ \lambda & i \in T,\ T \backslash \{i\} \subseteq K, \end{cases} \qquad K \subseteq N \backslash \{i\}$$

- in the last case, there are $\left|2^{N \backslash T}\right| = 2^{|N|-|T|}$ such coalitions

$$\mathrm{Ba}_i(N, \lambda u_T) = \begin{cases} 0, & i \in N \backslash T, \\ \dfrac{2^{|N|-|T|}}{2^{|N|-1}}\lambda = \dfrac{\lambda}{2^{|T|-1}} & i \in T, \end{cases}$$

- since Ba is additive, this gives

$$\mathrm{Ba}_i(N, v) = \sum_{T \subseteq N : i \in T} \frac{\lambda_T(v)}{2^{|T|-1}}$$

- compare

$$\mathrm{Ba}_i(N,v) = \sum_{T \subseteq N: i \in T} \frac{\lambda_T(v)}{2^{|T|-1}} \qquad \sum_{T \subseteq N: i \in T} \frac{\lambda_T(v)}{|T|} = \mathrm{Sh}_i(N,v)$$

- observe

$$2^{|T|-1} = |T| \text{ if } |T| \leq 2 \quad \text{and} \quad 2^{|T|-1} \neq |T| \text{ if } |T| > 2$$

- implication

$$\mathrm{Ba}(N,v) = \mathrm{Sh}(N,v) \text{ if } |N| \leq 2 \quad \text{and} \quad 2^{|T|-1} \neq |T| \text{ if } |T| > 2$$

- note

$$\mathrm{Ba}_N(N,v) = \sum_{T \subseteq N} \frac{|T|}{2^{|T|-1}} \lambda_T(v) \qquad \mathrm{Sh}_N(N,v) = \sum_{T \subseteq N} \lambda_T(v) = v(N)$$

Banzhaf efficiency (BaE) For all $v \in \mathbb{V}(N)$,
$\varphi_N(N, v) = \sum_{T \subseteq N} \frac{|T|}{2^{|T|-1}} \lambda_T(v) = \mathrm{Ba}_N(N, v)$.

Theorem

The Banzhaf value is the unique value that satisfies **BaE**, **A**, **S**, *and* **N**.

Proof. We have already shown that Ba satisfies **BaE**, **A**, **S**, and **N**. Remains the uniqueness part.
Uniqueness: Homework! Hint: Follow the uniqueness proof of the standard characterization of the Shapley value!

Dummy player axiom

Definition

Player $i \in N$ is a Dummy player in (N, v) if $MC_i^v(K) = v(\{i\})$ for all $K \subseteq N \backslash \{i\}$.

Dummy player (D) For all $v \in \mathbb{V}(N)$, $\varphi_i(N, v) = v(\{i\})$ whenever i is a Dummy player in (N, v).

- Sh obeys **D** because it is the weighted average of a player's marginal contribution
- Ba obeys **D** because for any Dummy player i in (N, v)

$$\mathrm{Ba}_i(N, v) = \frac{1}{2^{|N|-1}} \sum_{K \subseteq N \backslash \{i\}} MC_i^v(K) = \frac{1}{2^{|N|-1}} \sum_{K \subseteq N \backslash \{i\}} v(\{i\}) = v(\{i\})$$

2-Efficiency #1

- Lehrer, E. (1988): An Axiomatization of the Banzhaf Value. International Journal of Game Theory 17 (2), 89–99.

Definition

For $i, j \in N$, $i \neq j$, the **amalgamated game** $\left(N_{ij}, v_{ij}\right)$ is given by
$N_{ij} = N \backslash \{j\}$, $v_{ij} \in \mathbb{V}\left(N \backslash \{j\}\right)$,

$$v_{ij}(K) = \begin{cases} v(K), & i \notin K, \\ v(K \cup \{j\}), & i \in K, \end{cases} \qquad K \subseteq N_{ij}.$$

Superadditivity (SA) $\varphi_i\left(N_{ij}, v_{ij}\right) \geq \varphi_i(N, v) + \varphi_j(N, v)$ for all $i, j \in N$.

2-Efficiency (2E) $\varphi_i\left(N_{ij}, v_{ij}\right) = \varphi_i(N, v) + \varphi_j(N, v)$ for all $i, j \in N$.

- Sh fails both **SA** and **2E**: $|N| > 2$, $i, j \in N$, $i \neq j$, $v = u_N$:
  $v_{ij} = u_{N \backslash \{j\}} = u_{N_{ij}}$

$$\varphi_i\left(N_{ij}, v_{ij}\right) = \varphi_i\left(N_{ij}, u_{N_{ij}}\right) = \frac{1}{\left|N_{ij}\right|} = \frac{1}{|N| - 1} < \varphi_i(N, u_N) + \varphi_j(N, u_N) = \frac{2}{|N|}$$

- Ba obeys both **SA** and **2E**: (very pedantic)

$$
\begin{aligned}
& \mathrm{Ba}_i (N, v) + \mathrm{Ba}_j (N, v) \\
= \; & \frac{1}{2^{|N|-1}} \sum_{K \subseteq N \setminus \{i\}} MC_i^v (K) + \frac{1}{2^{|N|-1}} \sum_{K \subseteq N \setminus \{j\}} MC_j^v (K) \\
= \; & \frac{1}{2^{|N|-1}} \sum_{K \subseteq N \setminus \{i,j\}} MC_i^v (K) + MC_i^v (K \cup \{j\}) + MC_j^v (K) + MC_j^v (K \cup \{i\} \\
= \; & \frac{1}{2^{|N|-1}} \sum_{K \subseteq N \setminus \{i,j\}} MC_i^v (K) + MC_j^v (K \cup \{i\}) + MC_j^v (K) + MC_i^v (K \cup \{j\} \\
= \; & \frac{1}{2^{|N|-1}} \sum_{K \subseteq N \setminus \{i,j\}} 2 \left(v (K \cup \{i, j\}) - v (K) \right) \\
= \; & \frac{1}{2^{|N_{ij}|-1}} \sum_{K \subseteq N_{ij} \setminus \{i\}} MC_i^{v_{ij}} (K) \\
= \; & \mathrm{Ba}_i \left(N_{ij}, v_{ij} \right)
\end{aligned}
$$

Theorem (Lehrer 1988)

The Banzhaf value is the unique value that satisfies **D**, **S**, **A**, *and* **SA**.

Proof. We have already shown that Ba obeys **D**, **S**, **A**, **SA**, and **2E**.
Uniqueness: See the paper. Possible theme for the *Seminar.* □

Theorem (Lehrer 1988)

The Banzhaf value is the unique value that satisfies **2E** *and that coincides with* Ba *for* $|N| = 2$.

Proof. We have already shown that Ba obeys **2E**. Obviously, Ba coincides with itself, in particular for $|N| = 2$.
Uniqueness: Let φ obey **2E** and let $\varphi = \mathrm{Ba}$ for $|N| = 2$. $|N| = 1$, w.l.o.g., $N = \{1\}$. Let $v \in \mathbb{V}(N)$. Set $N' = \{1, 2\}$ and $v' \in \mathbb{V}(N')$, $v'(N') = v(N)$ and $v(K) = 0$ for $K \subsetneq N'$. We than have $N = N'_{12}$ and $v = v'_{12}$. Hence by **2E**,

$$\begin{aligned}\varphi_1(N, v) &= \varphi_1\left(N'_{12}, v'_{12}\right) = \varphi_1\left(N', v'\right) + \varphi_2\left(N', v'\right) \\ &= \mathrm{Ba}_1\left(N', v'\right) + \mathrm{Ba}_2\left(N', v'\right) = v'(N) = v(N) = \mathrm{Ba}_1(N, v),\end{aligned}$$

where the last equation drops from $|N'| = 2$. We proceed by induction on $|N|$.
Induction basis: By assumption, $\varphi = \mathrm{Ba}$ for $|N| = 2$.
Induction hypothesis (H): $\varphi = \mathrm{Ba}$ for $|N| = k$.
Induction step: Let $|N| = k + 1$. By **2E**, the $\varphi_i(N, v)$, $i \in N$ are a solution to the following system of linear equations

$$\varphi_i(N, v) + \varphi_j(N, v) = \varphi_i\left(N_{ij}, v_{ij}\right) \overset{\mathrm{H}}{=} \mathrm{Ba}_i\left(N_{ij}, v_{ij}\right), \qquad i, j \in N, i \neq j. \quad (*)$$

Since Ba obeys **2E**, we know that (*) has a solution, $\mathrm{Ba}_i(N, v)$, $i \in N$.
Remains to show that this solution is unique. From the structure of (*), however, this is not too difficult. □

- Nowak, A. S. (1997). On an axiomatization of the Banzhaf value without the additivity axiom, International Journal of Game Theory 26: 137–141.

Theorem (Nowak 1997)

The Banzhaf value is the unique value that satisfies **2E**, **S**, **D**, *and* **M**.

- in this characterization, **2E** cannot be relaxed into **SA**
- Casajus (2010a) below provides a short proof using the second Lehrer characterization

- Casajus, A. (2010a). Marginality, differential marginality, and the Banzhaf value, Theory and Decision, forthcoming.

Theorem (Casajus 2010a)

The Banzhaf value is the unique value that satisfies **SA**, **D**, *and* **DM**.

- does not hold within the class of superadditive games, for example

- Casajus, A. (2010b). Amalgamating players, symmetry, and the Banzhaf value, working paper, LSI and IMW.

- the characterizations of Lehrer (1988), Nowak (1997) and Casajus (2009) are redundant. one can drop **S**

Theorem (Casajus 2010b)

The Banzhaf value is the unique value that satisfies

1. **D**, **A**, and **SA**.
2. **D** and **2E**.

- Owen, G. (1975). Multi-linear extensions and the Banzhaf value, Management ScienceNaval Research Logistics Quarterly, 22, 741–750.

Theorem

For all $v \in \mathbb{V}(N)$ and $i \in N$,

$$\mathrm{Ba}_i(N,v) = \int_0^1 \left.\frac{\partial f_v}{\partial x_i}\right|_{x=\left(\frac{1}{2},\ldots,\frac{1}{2}\right)} dt.$$

Proof. We have

$$\begin{aligned}
\left.\frac{\partial f_v}{\partial x_i}\right|_{x=(t,\ldots,t)} &= \sum_{S\subseteq N\setminus\{i\}, S\neq\emptyset} t^{|S|}(1-t)^{|N|-|S|-1}\cdot\left(v(S\cup\{i\}) - v(S\right. \\
\left.\frac{\partial f_v}{\partial x_i}\right|_{x=\left(\frac{1}{2},\ldots,\frac{1}{2}\right)} &= \sum_{S\subseteq N\setminus\{i\}, S\neq\emptyset} \frac{1}{2^{|N|-1}}\left(v(S\cup\{i\}) - v(S)\right) \\
\int_0^1 \left.\frac{\partial f_v}{\partial x_i}\right|_{x=\left(\frac{1}{2},\ldots,\frac{1}{2}\right)} dt &= \frac{1}{2^{|N|-1}} \sum_{S\subseteq N\setminus\{i\}, S\neq\emptyset} \left(v(S\cup\{i\}) - v(S)\right) \\
&= \mathrm{Ba}_i(N,v)
\end{aligned}$$

Definition

A player $i \in N$ is **nullifying** in (N, v) if $v(K) = 0$ for all $K \subseteq N$ such that $i \in K$.

- Consider the following axiom. Is it plausible?

Nullifying player (Nf) If i is nullifying in (N, v), then $\varphi_i(N, v) = 0$.

Try to prove the following claim. Make use of the standard basis of $\mathbb{V}(N)$ and follow the idea of the proof of the standard characterization of the Shapley value.

Theorem

There is a unique solution concept that satisfies **E**, **A**, **S**, *and* **Nf**.

Problem

Prove or disprove the following claim: If φ is additive for two-player games and meets **2E**, *then φ satisfies* **A**.