Applied Cooperative Game Theory

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Overview

- Potential of the Shapley value
- Self-duality
- Marginal Contributions
- Recursion formulas
- The Shapley Value and the core

Definition

A **potential** P for TU games is an operator that assigns to any TU game (N, v) a number $P(N, v) \in \mathbb{R}$ such that
(i) $P(\emptyset, v) = 0$,
(ii) $\sum_{i \in N} \left[P(N, v) - P\left(N \backslash \{i\}, v|_{N \backslash \{i\}} \right) \right] = v(N)$.

Theorem

There is a unique potential for TU games, which satisfies
$P(N, v) - P\left(N \backslash \{i\}, v|_{N \backslash \{i\}} \right) = \mathrm{Sh}_i(N, v)$, $i \in N$.

Proof. Uniqueness: Let P, Q be two potentials. We show $P = Q$.

- *Induction basis:* $|N| = 1$. by (i+ii), $P(\{i\}, v) = Q(\{i\}, v) = v(\{i\})$.
- *Induction hypothesis (H)*: $P = Q$ for $|N| \leq k$
- *Induction step*: let $|N| = k + 1$. this implies

$$P(N, v) \overset{\text{(ii)}}{=} \frac{v(N) + \sum_{i \in N} P\left(N \backslash \{i\}, v|_{N \backslash \{i\}}\right)}{|N|} \overset{\text{H}}{=} Q(N, v)$$

The potential to the Shapley value #3

- existence: consider the operator P given by

$$P(N, v) := \sum_{T \subseteq N, T \neq \emptyset} \frac{\lambda_T(v)}{|T|}$$

- this gives $P(\emptyset, v) = 0$ and

$$\begin{aligned} & \sum_{i \in N} P(N, v) - P(N \backslash \{i\}, v|_N) \\ = & \sum_{i \in N} \sum_{T \subseteq N, T \neq \emptyset} \frac{\lambda_T(v)}{|T|} - \sum_{i \in N} \sum_{T \subseteq N, T \neq \emptyset, i \notin T} \frac{\lambda_T(v)}{|T|} \\ = & \sum_{i \in N} \sum_{T \subseteq N, T \neq \emptyset, i \in T} \frac{\lambda_T(v)}{|T|} \\ = & \sum_{i \in N} \mathrm{Sh}_i(N, v) = v(N) \end{aligned}$$

Self-duality

Definition

The dual game (N, v^*) of a TU game (N, v) is defined by

$$v^*(S) = v(N) - v(N \backslash S) \text{ for } S \subseteq N. \tag{1}$$

Interpretation: Lose of the great coalition, if the players of S leave.

Definition

A value ϕ for a TU game (N, v) is called **self-dual**, if

$$\phi(N, v) = \phi(N, v^*). \tag{2}$$

Theorem

The Shapley value is self-dual.

Proof: For $i \in N$ and $K \subseteq N \backslash \{i\}$, we have

$$\begin{aligned} MC_i^{v^*}(K) &= v^*(K \cup \{i\}) - v^*(K) \\ &= v(N) - v(N \backslash (K \cup \{i\})) - (v(N) - v(N \backslash K)) \\ &= v(N \backslash K) - v((N \backslash K) \backslash \{i\}) \\ &= MC_i^{v}((N \backslash K) \backslash \{i\}). \end{aligned}$$

Further,

$$\begin{aligned} \mathrm{Sh}_i(N, v^*) &= \sum_{K \subseteq N \backslash \{i\}} \frac{|K|!\,(|N| - |K| - 1)!}{|N|!} MC_i^{v^*}(K) \\ &= \sum_{K \subseteq N \backslash \{i\}} \frac{(|N| - |K| - 1)!\,|K|!}{|N|!} MC_i^{v}((N \backslash K) \backslash \{i\}) \\ &= \mathrm{Sh}_i(N, v). \end{aligned}$$

□

Definition

Let $S \subseteq N$ and $v \in \mathbb{V}(N)$. The TU game $(N \setminus S, v^S)$ defined by

$$v^S(T) = v(S \cup T) - v(S) \text{ for all } T \subseteq N \setminus S \tag{3}$$

is called **the** S **- marginal game of** (N, v).

Interpretation: The first players in a rank order of N are the players of coalition S. If the coalition T joins S, $v^S(T)$ describes the contribution of T to S.

Problem

Show: For any $S \subseteq N$ and any monotonic game $v \in \mathbb{V}(N)$, $v^S \in \mathbb{V}(N \backslash S)$ is nonnegative.

Marginal Contributions Property 2

Definition

A value ϕ suffices the **marginal contributions property**, if for any TU game (N, v) and any $i, j \in N, i \neq j$

$$\phi_i(N, v) - \phi_i(N \setminus \{j\}, v^j) = \phi_j(N, v) - \phi_j(N \setminus \{i\}, v^i).$$

Theorem

The Shapley value suffices the marginal contributions property.

Theorem

For all $v \in \mathbb{V}(N)$ and $i \in N$,

$$Sh_i(N, v) = \frac{1}{|N|}\left(v(N) - v(N\setminus\{i\})\right) + \frac{1}{|N|}\sum_{j\in N\setminus\{i\}} Sh_i\left(N\setminus\{j\}, v\Big|_{N\setminus\{j\}}\right).$$

Interpretation: The Shapley value of i is the sum of the marginal contribution to $N\setminus\{i\}$ (i is the last player in the rank order) and the sum of all Shapley values, such that another player is the last player.

Proof: For $|N| = 1$ the claim is immediate. Let now $|N| > 1$.The Shapley value satisfies BC and E. By BC we get

$$Sh_i(N,v) - Sh_j(N,v) = Sh_i\left(N\setminus\{j\}, v\Big|_{N\setminus\{j\}}\right) - Sh_j\left(N\setminus\{i\}, v\Big|_{N\setminus\{i\}}\right).$$

By summing up over $j \in N\setminus\{i\}$ we get

$$\sum_{j\in N\setminus\{i\}} Sh_i\left(N\setminus\{j\}, v\Big|_{N\setminus\{j\}}\right) - Sh_{N\setminus\{i\}}\left(N\setminus\{i\}, v\Big|_{N\setminus\{i\}}\right) =$$

$$(|N|-1)\, Sh_i(N,v) - Sh_{N\setminus\{i\}}(N,v)$$

Adding a zero term leads to

$$\begin{aligned} & |N|\, Sh_i(N,v) - Sh_N(N,v) \\ = \quad & \sum_{j \in N\setminus\{i\}} Sh_i\left(N\setminus\{j\}, v\Big|_{N\setminus\{j\}}\right) - Sh_{N\setminus\{i\}}\left(N\setminus\{i\}, v\Big|_{N\setminus\{i\}}\right). \end{aligned}$$

Using efficiency

$$|N|\, Sh_i(N,v) - v(N) = \sum_{j \in N\setminus\{i\}} Sh_i\left(N\setminus\{j\}, v\Big|_{N\setminus\{j\}}\right) - v\left(N\setminus\{i\}\right)$$

Hence

$$Sh_i(N,v) = \frac{1}{|N|}\left(v(N) - v(N\setminus\{i\})\right) + \frac{1}{|N|}\sum_{j \in N\setminus\{i\}} Sh_i\left(N\setminus\{j\}, v\Big|_{N\setminus\{j\}}\right).$$

and we are done.

Theorem

For all $v \in \mathbb{V}(N)$ and $i \in N$,

$$Sh_i(N, v) = \frac{1}{|N|} v(\{i\}) + \frac{1}{|N|} \sum_{j \in N \setminus \{i\}} Sh_i\left(N \setminus \{j\}, v^j\right),$$

Proof For $j \in N$ and $S \in N \setminus \{j\}$:

$$\begin{aligned}
\left(v^j\right)^* (S) &= v^j(N \setminus j) - v^j(N \setminus \{j\} \setminus S) \\
&= v(N) - v(\{j\}) - v(N \setminus S) + v(\{j\}) \\
&= v(N) - v(N \setminus S) \\
&= v^*(S).
\end{aligned}$$

By using this equation and the self-duality of the Shapley value we get

$$\begin{aligned}
Sh(N,v) &= Sh(N,v^*) \\
&= \frac{1}{|N|}\left(v^*(N) - v^*(N\setminus\{i\})\right) + \\
&\quad \frac{1}{|N|}\sum_{j\in N\setminus\{i\}} Sh_i\left(N\setminus\{j\}, v^*\Big|_{N\setminus\{j\}}\right) \\
&= \frac{1}{|N|}\left(v(N) - v(\emptyset) - v(N) + v(\{i\})\right) + \\
&\quad \frac{1}{|N|}\sum_{j\in N\setminus\{i\}} Sh_i\left(N\setminus\{j\}, \left(v^j\right)^*\right) \\
&= \frac{1}{|N|}v(\{i\}) + \frac{1}{|N|}\sum_{j\in N\setminus\{i\}} Sh_i\left(N\setminus\{j\}, v^j\right)
\end{aligned}$$

and we are done.

Theorem

Let (N, v) be a convex TU-game. Then the Shapley payoff $(Sh_i(N, v))_{i \in N}$ lies in the core of the game (N, v)

Proof:

Efficiency: The Shapley payoffs fulfill the efficiency axiom, therefore $\sum_{i \in N} Sh_i(N, v) = v(N)$, Non-blockability: Let $S \subseteq N$, we have to show

$$\sum_{i \in S} Sh_i(N, v) \geq v(S).$$

Let $\sigma \in \Sigma(N)$ be an order of N and define $\tau : S \to S$ the induced order of S which is defined by

$$\sigma(i) > \sigma(j) \implies \tau(i) > \tau(j).$$

Because of convexity of v and $K_i(\tau) \subseteq K_i(\sigma)$ we have:

$$MC_i(\tau) \leq MC_i(\sigma).$$

Therefore we have

$$\sum_{i \in S} MC_i(\sigma) \geq \sum_{i \in S} MC_i(\tau) = v(S).$$

Now we obtain

$$\begin{aligned}\sum_{i\in S} Sh_i(N,v) &= \sum_{i\in S}\frac{1}{|\Sigma(N)|}\sum_{\sigma\in\Sigma(N)} MC_i(\sigma)\\ &= \frac{1}{|\Sigma(N)|}\sum_{\sigma\in\Sigma(N)}\sum_{i\in S} MC_i(\sigma)\\ &\geq \frac{1}{|\Sigma(N)|}\sum_{\sigma\in\Sigma(N)} v(S)\\ &\geq v(S).\end{aligned}$$

Therefore for convex games the Shapley payoffs fulfills non-blockability.