Applied Cooperative Game Theory

André Casajus and Martin Kohl

University of Leipzig

November 2 2012

Overview

- The Young characterization
- Balanced Contributions
- The Myerson characterization
- Differential Monotonicity and Marginality
- Cousins of Differential Monotonicity
- The van den Brink characterization

Theorem (Young 1985)

The Shapley value is the unique solution that meets **E**, **S,** *and* **M**.

Proof. Uniqueness: Let φ satisfy **E**, **S**, and **M**. In the following, we show $\varphi(N, v)$ is uniquely determined for all $v \in \mathbb{V}(N)$ by induction on $|\mathcal{T}(v)|$, where

$$\mathcal{T}(v) := \{T \in \mathcal{K}(N) \,|\, \lambda_T(v) \neq 0\}.$$

Induction basis: If $|\mathcal{T}(v)| = 0$, then $v = \mathbf{0}$. It is easy to check that **E** and **S** imply **NG**. Hence, $\varphi_i(N, \mathbf{0}) = 0$ for all $i \in N$.

Induction hypothesis: Suppose $\varphi(N, v)$ is uniquely determined for all $v \in \mathbb{V}(N)$ such that $|\mathcal{T}(v)| \leq k$.

Induction step: Let $v \in \mathbb{V}(N)$ such that $|\mathcal{T}(v)| = k+1$. Set $T(v) := \bigcap_{T \in \mathcal{T}(v)} T$. For $i \in N \setminus T(v)$ set

$$v^{(i)} = \sum_{T \in \mathcal{T}(v): i \in T} \lambda_T(v)\, u_T.$$

Then, $\mathcal{T}\left(v^{(i)}\right) \subsetneq \mathcal{T}(v)$, hence $\left|\mathcal{T}\left(v^{(i)}\right)\right| < |\mathcal{T}(v)|$, and $T \in \mathcal{T}(v)$ and $i \in T$ imply $T \in \mathcal{T}\left(v^{(i)}\right)$ and $\lambda_T(v) = \lambda_T\left(v^{(i)}\right)$. The latter entails

$$MC_i^{v}(K) = MC_i^{v^{(i)}}(K)$$

for $K \subseteq N \setminus \{i\}$, i.e., v, $v^{(i)}$, and i satisfy the hypothesis of **M**. Since φ obeys **M**, we thus have

$$\varphi_i(N, v) = \varphi_i\left(N, v^{(i)}\right),$$

$\varphi_i(N, v)$ is uniquely determined by the induction hypothesis.

Finally, the players in $T(v)$ are symmetric: Whenever $i, j \in T(v)$, $T \in \mathcal{T}(v)$, then $i, j \in T$ entailing

$$\lambda_{K \cup \{i\}}(v) = 0 = \lambda_{K \cup \{j\}}(v), \qquad K \subseteq N \backslash \{i, j\}.$$

(Recall the characterization of symmetric players in terms of Harsanyi dividends). Since φ obeys **E** and **S**, we have

$$\varphi_i(N, v) \overset{\mathbf{S}}{=} \frac{\varphi_{T(v)}(N, v)}{|T(v)|} \overset{\mathbf{E}}{=} \frac{v(N) - \varphi_{N \backslash T(v)}(N, v)}{|T(v)|}$$

for all $i \in T(v)$, where $\varphi_{N \backslash T(v)}(N, v)$ is determined by the previous step. Done. □

- **E+S, ¬M**: $\varphi_i(N,v) = |N|^{-1} \cdot v(N)$, $i \in N$, $v \in \mathbb{V}(N)$, $\equiv$ (efficient) egalitarian solution

- **E+M, ¬S**: $w \in \mathbb{R}^N_{++}$, $\varphi_i(N,v) = \sum_{T \in \mathcal{K}(N): i \in T} \lambda_T(v) \cdot \frac{w_i}{\sum_{j \in T} w_j}$, $i \in N$, $\equiv$ simple weighted Shapley value

- **S+M, ¬E**: $\varphi_i(N,v) = 0$, $i \in N$, $v \in \mathbb{V}(N)$, $\equiv$ Null solution

- Myerson, R. B. (1977). Graphs and cooperation in games, Mathematics of Operations Research 2: 225–229.

Balanced contributions (BC) for all N, $v \in \mathbb{V}(N)$ and $i, j \in N$,

$$\varphi_i(N, v) - \varphi_i\left(N \backslash \{j\}, v|_{N \backslash \{j\}}\right) = \varphi_j(N, v) - \varphi_j\left(N \backslash \{i\}, v|_{N \backslash \{i\}}\right),$$

where $v|_K$ denotes the restriction of v to $K \subseteq N$.

- exit of j hurts/benefits i by the same amount as the exit of i hurts/benefits j
- plausible?
- in contrast to the other axiom considered so far, **BC** relates different player sets
- as we will see, **BC** is a very powerful axiom

Theorem (Myerson 1977)

The Shapley value is the unique solution that meets **E** *and* **BC**.

- extremely elegant by the use of **BC**
- characterization on the domain of all TU games, in particular, with different player sets, even player sets with different cardinality
- characterization without additivity

Proof. We have already shown that Sh obeys **E**. To see **BC**, first observe

$$\lambda_T(v) = \lambda_T\left(v|_{N\setminus\{j\}}\right), \qquad T \in \mathcal{K}(N\setminus\{j\}).$$

Hence, we have

$$\begin{aligned} & \mathrm{Sh}_i(N, v) - \mathrm{Sh}_i\left(N\setminus\{j\}, v|_{N\setminus\{j\}}\right) \\ = & \sum_{T\in\mathcal{K}(N):i\in T} |T|^{-1}\cdot\lambda_T(v) - \sum_{T\in\mathcal{K}(N\setminus\{j\}):i\in T} |T|^{-1}\cdot\lambda_T\left(v|_{N\setminus\{j\}}\right) \\ = & \sum_{T\in\mathcal{K}(N):i\in T} |T|^{-1}\cdot\lambda_T(v) - \sum_{T\in\mathcal{K}(N\setminus\{j\}):i\in T} |T|^{-1}\cdot\lambda_T(v) \\ = & \sum_{T\in\mathcal{K}(N):i,j\in T} |T|^{-1}\cdot\lambda_T(v) \end{aligned}$$

Interchanging, i and j one obtains

$$\mathrm{Sh}_j(N, v) - \mathrm{Sh}_j\left(N\setminus\{i\}, v|_{N\setminus\{i\}}\right) = \sum_{T\in\mathcal{K}(N):i,j\in T} |T|^{-1}\cdot\lambda_T(v)$$

and we are done.

Uniqueness: Let φ obey **E** and **BC**. In the following, we show that $\varphi(N, v)$ is uniquely determined for all N and $v \in \mathbb{V}(N)$ by induction on $|N|$.

Induction basis: For $|N| = 1$, i.e., $N = \{i\}$, the claim drops from **E**:
$\varphi_i(\{i\}, v) = \varphi_{\{i\}}(\{i\}, v) = v(\{i\})$.

Induction hypothesis: Suppose $\varphi(N, v)$ is uniquely determined for all N such that $|N| \leq k$ and $v \in \mathbb{V}(N)$.

Induction step: Let now $|N| = k + 1$ and $v \in \mathbb{V}(N)$. By **BC**, for all $i, j \in N$, we have

$$\varphi_i\left(N\setminus\{j\}, v|_{N\setminus\{j\}}\right) - \varphi_j\left(N\setminus\{i\}, v|_{N\setminus\{i\}}\right) = \varphi_i(N, v) - \varphi_j(N, v).$$

Summing up over $j \in N\setminus\{i\}$ gives

$$\begin{aligned}
&\sum_{j\in N\setminus\{i\}}\left(\varphi_i\left(N\setminus\{j\}, v|_{N\setminus\{j\}}\right) - \varphi_j\left(N\setminus\{i\}, v|_{N\setminus\{i\}}\right)\right)\\
&= (|N|-1)\cdot\varphi_i(N, v) - \varphi_{N\setminus\{i\}}(N, v)\\
&= |N|\cdot\varphi_i(N, v) - \varphi_N(N, v)\\
&= |N|\cdot\varphi_i(N, v) - v(N),
\end{aligned}$$

where the last equation drops from **E**. By the induction hypothesis, the first line is uniquely determined. Since i was arbitrarily chosen, $\varphi(N, v)$ is uniquely determined. □

The Myerson characterization: Independence

- **E,** $\neg$**BC**: $\varphi_i(N, v) = |N|^{-1} \cdot v(N)$, $i \in N$, $v \in \mathbb{V}(N)$, $\equiv$ (efficient) egalitarian solution

- **BC,** $\neg$**E**: $\varphi_i(N, v) = 0$, $i \in N$, $v \in \mathbb{V}(N)$, $\equiv$ Null solution

Differential monotonicity (DMo) for all $v, w \in \mathbb{V}(N)$ and $i, j \in N$ such that

$$v(K \cup \{i\}) - v(K \cup \{j\}) \geq w(K \cup \{i\}) - w(K \cup \{j\})$$

for all $K \subseteq N \setminus \{i, j\}$, we have

$$\varphi_i(N, v) - \varphi_j(N, v) \geq \varphi_i(N, w) - \varphi_j(N, w).$$

- two player's payoff differential (in different situations) does not decrease with nowhere non-decreasing productivity differential

- Casajus, A. (2009). Differential marginality, van den Brink fairness, and the Shapley value. Theory and Decision, forthcoming.

Differential marginality (DM) for all $v, w \in \mathbb{V}(N)$ and $i, j \in N$ such that

$$v(K \cup \{i\}) - v(K \cup \{j\}) = w(K \cup \{i\}) - w(K \cup \{j\})$$

for all $K \subseteq N \backslash \{i, j\}$, we have

$$\varphi_i(N, v) - \varphi_j(N, v) = \varphi_i(N, w) - \varphi_j(N, w).$$

- obviously, **DMo** $\Rightarrow$ **Mo**
- **DM** requires a two player's payoff differential to depend on the differentials of *their* productivities (measured by marginal contributions) only
- the hypothesis of **DM** is satisfied iff

$$\lambda_{K\cup\{i\}}(v) - \lambda_{K\cup\{j\}}(v) = \lambda_{K\cup\{i\}}(w) - \lambda_{K\cup\{j\}}(w)$$

for all $K \subseteq N \backslash \{i, j\}$. Homework! Hint: Use the characterization of marginal contributions by Harsanyi dividends and proceed by induction on K.

- van den Brink, R. (2001). An axiomatization of the Shapley value using a fairness property. International Journal of Game Theory 30, 309–319.

van den Brink fairness (BF) for all $v, w \in \mathbb{V}(N)$ and $i, j \in N$ such that i and j are symmetric in (N, w), we have $\varphi_i(N, v+w) - \varphi_i(N, v) = \varphi_j(N, v+w) - \varphi_j(N, v)$.

- observation: **DM** and **BF** are equivalent. Homework!

Lemma

NG *and* **DM** *imply* **S.**

Proof.

- Let φ meet **NG** and **DM**, and let $i, j \in N$ be symmetric in (N, v).
- For $\mathbf{0} \in \mathbb{V}(N)$, we have $MC_i^v(K) - MC_j^v(K) = 0$
  $= MC_i^{\mathbf{0}}(K) - MC_j^{\mathbf{0}}(K)$ for all $K \subseteq N \setminus \{i\}$.
- Hence by **DM**, $\varphi_i(N, v) - \varphi_j(N, v) = \varphi_i(N, \mathbf{0}) - \varphi_j(N, \mathbf{0})$.
- Further by **NG**, $\varphi_i(N, \mathbf{0}) = 0 = \varphi_j(N, \mathbf{0})$, which proves the claim. □

- **S** does not imply **DM**: $\varphi_i(N, v) = [v(\{i\})]^2$, $i \in N$, $v \in \mathbb{V}(N)$

Lemma

A *and* **S** *imply* **DM**.

Proof.

- Let φ meet **A** and **S**, and let $i, j \in N$, $v, w \in \mathbb{V}(N)$ be such that $MC_i^v(K) - MC_j^v(K) = MC_i^w(K) - MC_j^w(K)$ for all $K \subseteq N \setminus \{i, j\}$.
- Since the marginal contributions are additive in the coalition function, we have $MC_i^{v-w}(K) - MC_j^{v-w}(K) = 0$ for all $K \subseteq N \setminus \{i, j\}$.
- Hence, i and j are symmetric in $(N, v - w)$.
- By **S**, we thus have $\varphi_i(N, v - w) = \varphi_j(N, v - w)$.
- Finally, **A** entails

$$\begin{aligned} \varphi_i(N, v) - \varphi_i(N, w) &= \varphi_i(N, v - w) = \varphi_j(N, v - w) \\ &= \varphi_j(N, v) - \varphi_j(N, w), \end{aligned}$$

i.e.,

$$\varphi_i(N, v) - \varphi_j(N, v) = \varphi_i(N, w) - \varphi_j(N, w),$$

and we are done. □

- **DM** does not imply **S**: $g \in \mathbb{R}^N$, $g_i \neq g_j$ for some $i, j \in N$, $\varphi_i(N, v) = \mathrm{Sh}_i(N, v) + g_i$
- **DM** does not imply **A**: $\varphi_i(N, v) = 1$, $i \in N$, $v \in \mathbb{V}(N)$
- **DM** does not imply **M**: $\varphi_i(N, v) = |N|^{-1} \cdot v(N)$, $i \in N$, $v \in \mathbb{V}(N)$, $\equiv$ (efficient) egalitarian solution

- **M** does not imply **DM**: $\varphi_i(N, v) = [v(\{i\})]^2$, $i \in N$, $v \in \mathbb{V}(N)$

Theorem (van den Brink 2001)

The Shapley value is the unique solution that meets **E**, **N,** *and* **BF**.

Theorem (Casajus 2009)

The Shapley value is the unique solution that meets **E**, **N,** *and* **DM**.

- another characterizations without additivity

Proof. See van den Brink (2001) or Casajus (2009) or Casajus (2009b) below. Possible theme for the *Seminar*. □

- Casajus, A. (2009b). Another characterization of the Owen value without the additivity axiom. Theory and Decision 69 (4), 523–536.