Lecture: Topics in Cooperative Game Theory

PD Dr. André Casajus

Leipzig University

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Solution concepts

Definition

A **solution concept** φ assigns to any TU game (N, v) a payoff vector

$$\varphi(N, v) \in \mathbb{R}^N.$$

- payoff of player $i \in N$: $\varphi_i(N, v)$
- sum of payoffs of players in $K \subseteq N$

$$\varphi_K(N, v) = \sum_{i \in K} \varphi_i(N, v)$$

- sometimes solution concepts are restricted
  - to a fixed player set
  - to subsets of $\mathbb{V}(N)$, e.g., the superadditive coalition functions

Marginal contributions

Definition

For $v \in \mathbb{V}(N)$, the **marginal contribution** of $i \in N$ to $K \subseteq N \backslash \{i\}$ is given by

$$MC_i^v(K) = v(K \cup \{i\}) - v(K).$$

- measure of a player's productivity
- marginal contributions are additive in the coalition function

$$MC_i^{v+w}(K) = MC_i^v(K) + MC_i^w(K)$$

- marginal contributions in terms of Harsanyi dividends

$$MC_i^v(K) = \sum_{T \in \mathcal{K}(K)} \lambda_{T \cup \{i\}}(v)$$

- An **order** on N is a bijection $\sigma : N \to \{1, \ldots, |N|\}$.
- set of all orders on N: $\Sigma(N)$
- player i's position in σ is $\sigma(i)$
- players before or equal to i in σ

$$K_i(\sigma) := \{j \in N | \sigma(j) \leq \sigma(i)\}$$

- marginal contribution of i in σ

$$MC_i^v(\sigma) := MC_i^v(K_i(\sigma) \setminus \{i\}) = v(K_i(\sigma)) - v(K_i(\sigma) \setminus \{i\})$$

- Lloyd S. Shapley (1953): A value for *n*-person games. In H. Kuhn & A. Tucker, Contributions to the theory of games (Vol. II, pp. 307–317). Princeton: Princeton University Press.

Definition

The Shapley value assigns to any TU game (N, v) and $i \in N$ the payoff

$$\mathrm{Sh}_i(N, v) := |\Sigma(N)|^{-1} \sum_{\sigma \in \Sigma(N)} MC_i^v(\sigma).$$

- alternatively

$$\begin{aligned} \mathrm{Sh}_i(N, v) &= \sum_{K \subseteq N \setminus \{i\}} p(|K|) \cdot MC_i^v(K) \\ p(k) &= \frac{k!\,(|N| - k - 1)!}{|N|!}, \qquad k = 0, \ldots, |N| - 1 \end{aligned}$$

- assigns to any player the average marginal contribution over all orders
  - average productivity = fair???

Linearity (L) if for all $v, w \in \mathbb{V}(N)$ and $\alpha, \beta \in \mathbb{R}$

$$\varphi(N, \alpha v + \beta w) = \alpha \cdot \varphi(N, v) + \beta \cdot \varphi(N, w).$$

Additivity (A) if for all $v, w \in \mathbb{V}(N)$

$$\varphi(N, v + w) = \varphi(N, v) + \varphi(N, w).$$

Homogeneity (H) for all $v \in \mathbb{V}(N)$ and $\alpha \in \mathbb{R}$

$$\varphi(N, \alpha v) = \alpha \cdot \varphi(N, v).$$

Null Game (NG) $\varphi(N, \mathbf{0}) = \mathbf{0} \in \mathbb{R}^N$.

- **L** $\iff$ **A** and **H**
- **L** $\Rightarrow$ **NG**, **H** $\Rightarrow$ **NG**, **A** $\Rightarrow$ **NG**

$$\mathbf{0} = \mathbf{0} + \mathbf{0} \quad \overset{\mathbf{A}}{\Rightarrow} \quad \varphi(N, \mathbf{0}) = \varphi(N, \mathbf{0}) + \varphi(N, \mathbf{0}) = \mathbf{0} \in \mathbb{R}^N$$

- **A** $\Rightarrow$ **H** for rational scalars $\alpha = \pm\frac{\beta}{\gamma}$, $\beta, \gamma \in \mathbb{N}$, $\gamma > 0$

$$\beta \in \mathbb{N} : \beta v = \underbrace{v + \cdots + v}_{\beta \text{ summands}} \quad \overset{\mathbf{A}}{\Rightarrow} \quad \varphi(N, \beta v) = \beta \cdot \varphi(N, v)$$

$$\gamma \in \mathbb{N} : v = \underbrace{\frac{1}{\gamma} v + \cdots + \frac{1}{\gamma} v}_{\gamma \text{ summands}} \quad \overset{\mathbf{A}}{\Rightarrow} \quad \varphi\left(N, \frac{1}{\gamma} v\right) = \frac{1}{\gamma} \cdot \varphi(N, v)$$

$$\mathbf{0} = v + (-v) \quad \overset{\mathbf{A}}{\Rightarrow} \quad \varphi(N, v) + \varphi(N, -v) = \varphi(N, \mathbf{0})$$

$$\overset{\mathbf{NG}}{\Rightarrow} \quad \varphi(N, -v) = -\varphi(N, v)$$

Efficiency (E) $\varphi_N(N, v) = v(N)$ for all $v \in \mathbb{V}(N)$.

- the whole worth produced by the grand coalition is distributed among the players
  - in general, the grand coalition must cooperate in order to create this worth
  - no external subsidies, no losses
- makes sense in superadditive games, else???
- example: $N = \{1, 2\}$, $v(\{1\}) = v(\{2\}) = 2$, $v(N) = 3$

Definition

Player $i \in N$ is a **Null player** in (N, v), if $v(K \cup \{i\}) = v(K)$ for all $K \subseteq N \setminus \{i\}$.

- Null players are unproductive, $MC_i^v(K) = 0$ for all $K \subseteq N \setminus \{i\}$
- example: in $(N, \lambda u_T)$, $\emptyset \neq T \subseteq N$, $\lambda \in \mathbb{R}$ all $i \in N \setminus T$ are Null players
- characterization: i is a Null player in (N, v) iff $\lambda_T(v) = 0$ for all $T \subseteq N$, $i \in T$

Null player (N) $\varphi_i(N, v) = 0$ for all $v \in \mathbb{V}(N)$ and all Null players i in (N, v).

- excludes solidarity with unproductive players

Definition

Players $i, j \in N$ are **symmetric** in (N, v), if $v(K \cup \{i\}) = v(K \cup \{j\})$ for all $K \subseteq N \backslash \{i, j\}$.

- symmetric players have the same productivity: $MC_i^v(K) = MC_j^v(K)$ for all $K \subseteq N \backslash \{i, j\}$
- example: in $(N, \lambda u_T)$, $T \in \mathcal{K}(N)$, $\lambda \in \mathbb{R}$ all $i, j \in T$ and all $i, j \in N \backslash T$ are symmetric
- characterization: i and j are symmetric in (N, v) iff $\lambda_{T \cup \{i\}}(v) = \lambda_{T \cup \{j\}}(v)$ for all $T \subseteq N \backslash \{i, j\}$

Symmetry (S) $\varphi_i(N, v) = \varphi_j(N, v)$ for all $v \in \mathbb{V}(N)$ and all i, j who are symmetric in (N, v).

- payoffs do not depend on the players' names (given by the game theorist)
- only modelled properties of the players matter (productivity in the sense of v)

Isomorphism invariance

Definition

An **isomorphism** from (N, v) to (N', v') is a bijection $f : N \to N'$ such that

$$v(K) = v'(f(K)), \qquad K \subseteq N.$$

Definition

A **symmetry (mapping)** of (N, v) is an isomorphism of (N, v) into itself.

Isomorphism invariance (I) $\varphi_i(N, v) = \varphi_{f(i)}(N', v')$ for all $i \in N$, $v \in \mathbb{V}(N)$, $v' \in \mathbb{V}(N')$ and all isomorphisms f from (N, v) to (N', v').

Strong symmetry ($\mathbf{S}^+$) $\varphi_i(N, v) = \varphi_{f(i)}(N, v)$ for all $i \in N$, $v \in \mathbb{V}(N)$ and all symmetries f of (N, v).

- $\mathbf{S}^+$ implies **S**, but not conversely

Lemma

Die Shapley value obeys **E**, **A**, **S**, *and* **N**.

Proof. Sh obeys **E**:

$$\begin{aligned}\mathrm{Sh}_N(N,v) &= \sum_{i\in N}\frac{1}{|\Sigma(N)|}\sum_{\sigma\in\Sigma(N)} MC_i^v(\sigma) = \frac{1}{|\Sigma(N)|}\sum_{\sigma\in\Sigma(N)}\sum_{i\in N} MC_i^v(\sigma)\\ &= \frac{1}{|\Sigma(N)|}\sum_{\sigma\in\Sigma(N)} v(N) = v(N);\end{aligned}$$

Sh obeys **N**: Let i be a Null player in(N,v). We have

$$\mathrm{Sh}_i(N,v) = \frac{1}{|\Sigma(N)|}\sum_{\sigma\in\Sigma(N)} MC_i^v(\sigma) = \frac{1}{|\Sigma(N)|}\sum_{\sigma\in\Sigma(N)} 0 = 0;$$

Sh satisfies **A**: Since $MC_i^{v+w}(\sigma) = MC_i^v(\sigma) + MC_i^w(\sigma)$ for all $v, w \in \mathbb{V}(N)$, $i \in N$, and $\sigma \in \Sigma(N)$, this is immediate from the definition.

Let i and j be symmetric in (N, v). From $\sigma \in \Sigma(N)$ obtain $\bar{\sigma} \in \Sigma(N)$ by changing the position of i and j: $\sigma(k) = \bar{\sigma}(k)$ for all $k \in N \backslash \{i, j\}$, $\bar{\sigma}(j) = \sigma(i)$ and $\bar{\sigma}(i) = \sigma(j)$. Let

$$\Sigma'(N) := \{\sigma \in \Sigma(N) \,|\, \sigma(i) < \sigma(j)\}.$$

This gives

$$\begin{aligned}
& |\Sigma(N)| \cdot \mathrm{Sh}_i(N, v) \\
\overset{\text{def. Sh}}{=} & \sum_{\sigma \in \Sigma'(N)} v(K_i(\sigma)) - v(K_i(\sigma) \setminus \{i\}) + v(K_i(\bar{\sigma})) - v(K_i(\bar{\sigma}) \setminus \{i\}) \\
= & \sum_{\sigma \in \Sigma'(N)} v\left(K_j(\bar{\sigma})\right) - v\left(K_j(\bar{\sigma}) \setminus \{j\}\right) + v\left(K_j(\sigma)\right) - v\left(K_j(\sigma) \setminus \{j\}\right) \\
\overset{\text{def. Sh}}{=} & |\Sigma(N)| \cdot \mathrm{Sh}_j(N, v).
\end{aligned}$$

The second equation drops from the following facts:

1. $K_i(\sigma) \setminus \{i\} = K_j(\bar{\sigma}) \setminus \{j\} \subseteq N \setminus \{i,j\}$ and i,j are symmetric
2. $K_i(\bar{\sigma}) = K_j(\sigma)$
3. $K_i(\bar{\sigma}) \setminus \{i,j\} = K_j(\sigma) \setminus \{i,j\} \subseteq N \setminus \{i,j\}$ and i,j are symmetric □

Lemma

There is at most one solution that satisfies **E**, **A**, **S**, *and* **N**.

Proof. Let φ obey **E**, **A**, **S** and **N**. Any TU game (N, v) can be uniquely represented by unanimity games: $v = \sum_{T \in \mathcal{K}(N)} \lambda_T(v) \cdot u_T$. By **A**, we thus have

$$\varphi(N, v) = \sum_{T \in \mathcal{K}(N)} \varphi(N, \lambda_T(v) \cdot u_T).$$

In $(N, \lambda_T(v) \cdot u_T)$ all $i \in N \backslash T$ are Null players. By **N**, $\varphi_i(N, \lambda_T(v) \cdot u_T) = 0$. Therefore and by **E**, we have

$$\begin{aligned} \varphi_T(N, \lambda_T(v) \cdot u_T) &= \varphi_N(N, \lambda_T(v) \cdot u_T) - \varphi_{N \backslash T}(N, \lambda_T(v) \cdot u_T) \\ &= \lambda_T(v) \cdot u_T(N) - |N \backslash T| \cdot 0 = \lambda_T(v). \end{aligned}$$

Finally, the players in T are pairwise symmetric in $(N, \lambda_T(v) \cdot u_T)$. By **S**, we have for $i \in T$

$$\lambda_T(v) = \varphi_T(N, \lambda_T(v) \cdot u_T) = |T| \cdot \varphi_i(N, \lambda_T(v) \cdot u_T),$$

i.e.,

$$\varphi_i(N, \lambda_T(v) \cdot u_T) = \frac{\lambda_T(v)}{|T|}.$$

If there is a solution concept that satisfies **E**, **A**, **S**, and **N**, then it must be given by

$$\varphi_i(N, v) = \sum_{T \subseteq N, i \in T} \frac{\lambda_T(v)}{|T|}, \qquad i \in N,$$

i.e., it is unique. □

Shapley value: The standard characterization

Theorem (Shapley 1953)

The Shapley value is the unique solution that meets **E**, **A**, **S** *and* **N**.

- the Shapley value is completely determined by the axioms **E**, **A**, **S** and **N**
- if you buy **E**, **A**, **S** and **N** as plausible properties, then you buy the Shapley value

- axiomatic characterizations provide insights in the driving forces behind solution concepts given by some computational rule
- facilitates comparison of solution concepts

- a good characterization should be non-redundant, i.e., the axiom should be independent
- i.e., the characterization should not work with a subset of the axioms
- i.e., no axiom is implied by the other axioms
- how to prove?
- find examples (solutions) that satisfy all but one axiom
- **E+A+S, ¬N**: $\varphi_i(N, v) = |N|^{-1} \cdot v(N)$, $i \in N$, $v \in \mathbb{V}(N)$, $\equiv$ (efficient) egalitarian solution
- **A+S+N, ¬E**: $\varphi_i(N, v) = 0$, $i \in N$, $v \in \mathbb{V}(N)$, $\equiv$ Null solution
- **E+A+N, ¬S**: $w \in \mathbb{R}^N_{++}$, $\varphi_i(N, v) = \sum_{T \in \mathcal{K}(N): i \in T} \lambda_T(v) \cdot \frac{w_i}{\sum_{j \in T} w_j}$, $\equiv$ simple weighted Shapley value
- **E+S+N, ¬A**: $N_0(v) := \{i \in N | i \text{ is a Null player in } (N, v)\}$,

$$\varphi_i(N, v) = \begin{cases} |N \backslash N_0(v)|^{-1} v(N), & i \in N \backslash N_0(v), \\ 0, & i \in N_0(v) \end{cases}$$

  $\equiv$ (efficient) egalitarian solution on non-Null players

- H. P. Young (1985). Monotonic solutions of cooperative games. International Journal of Game Theory 14, 65–72.

Monotonicity (Mo) φ obeys **Mo**, if for all $v, w \in \mathbb{V}(N)$ and $i \in N$ such that $v(K \cup \{i\}) - v(K) \geq w(K \cup \{i\}) - w(K)$ for all $K \subseteq N \backslash \{i\}$, we have $\varphi_i(N, v) \geq \varphi_i(N, w)$.

- a player's payoff (in different situations) does not decrease with increasing productivity

Marginality (M) φ obeys **M**, if for all $v, w \in \mathbb{V}(N)$ and $i \in N$ such that $v(K \cup \{i\}) - v(K) = w(K \cup \{i\}) - w(K)$ for all $K \subseteq N \backslash \{i\}$, we have $\varphi_i(N, v) = \varphi_i(N, w)$.

- obviously, **Mo** $\Rightarrow$ **M**
- **M** requires a player's payoff to depend on his *own* productivity (measured by marginal contributions) only

- Chun, Y. (1989). A new axiomatization of the Shapley value. Games and Economic Behavior 1, 119–130.

Coalitional strategic equivalence (CSE) φ obeys **CSE**, if for all $v \in \mathbb{V}(N)$, $T \in \mathcal{K}(N)$, $\lambda \in \mathbb{R}$, and $i \in N \backslash T$, we have $\varphi_i(N, v) = \varphi_i(N, v + \lambda u_T)$.

- van den Brink, R. (2007). Null or nullifying players: The difference between the Shapley value and equal division solutions. Journal of Economic Theory, 136, 767–775.

van den Brink Null player (BN) φ obeys **BN**, if for all $v, w \in \mathbb{V}(N)$ such that i is a Null player in (N, w), we have $\varphi_i(N, v) = \varphi_i(N, v + w)$.

- observation: **M**, **CSE**, and **BN** are equivalent. Homework! Partly not too easy. Hint: Use the presentation of coalition functions by unanimity games and the characterization marginal contributions by Harsanyi dividends.

Shapley
SolConcept
MC
Order
Shapley
L/H/A #1
L/H/A #2
E
N
S
I
ShProp #1
ShProp #2
ShProp #3
ShUniq #1
ShUniq #2
ShStChar
ShIndep
Mo+M
Cousins
M&N #1
M&N #2
ShYoung

Lemma

NG *and* **M** *imply* **N.**

Proof. Let φ meet **NG** and **M**, and let $i \in N$ be a Null player in (N, v). For $\mathbf{0} \in \mathbb{V}(N)$, we have $MC_i^v(K) = 0 = MC_i^{\mathbf{0}}(K)$ for all $K \subseteq N \setminus \{i\}$. Hence by **M**, $\varphi_i(N, v) = \varphi_i(N, \mathbf{0})$. Further by **NG**, $\varphi_i(N, \mathbf{0}) = 0$, which proves the claim. □

- **N** does not imply **M**: $\varphi_i(N, v) = v(\{i\}) \cdot v(N)$

Marginality and the Null player axiom #2

Lemma

A *and* **N** *imply* **M**.

Proof. Let φ meet **A** and **N**, and let $i \in N$, $v, w \in \mathbb{V}(N)$ be such that $MC_i^v(K) = MC_i^w(K)$ for all $K \subseteq N \setminus \{i\}$.
Since the marginal contributions are additive in the coalition function, we have $MC_i^{v-w}(K) = MC_i^v(K) - MC_i^w(K) = 0$ for all $K \subseteq N \setminus \{i\}$. Hence, i is a Null player in $(N, v-w)$.
By **N**, we thus have $\varphi_i(N, v-w) = 0$. Finally, **A** entails $\varphi_i(N, v) = \varphi_i(N, v-w) + \varphi_i(N, w)$, i.e., $\varphi_i(N, v) = \varphi_i(N, w)$, and we are done. □

- **M** does not imply **N**: $\varphi_i(N, v) = \mathrm{Sh}_i(N, v) + 1$
- **M** does not imply **A**: $\varphi_i(N, v) = [v(\{i\})]^2$

Theorem (Young 1985)

The Shapley value is the unique solution that meets **E**, **S,** *and* **M**.

- very elegant by the use of **M**
- characterization without additivity

Proof. We have already shown that Sh obeys **E** and **S**. Since Sh_i "is" a player's average marginal contribution, it is immediate that Sh meets **M.**

- Uniqueness: Next lecture. If you are up for a great challenge try it yourself. Hint: Induction on the cardinality of the set

$$\mathcal{T}(v) := \{T \in \mathcal{K}(N) \,|\, \lambda_T(v) \neq 0\}.$$