Applied Cooperative Game Theory

Topic 2: The Core

André Casajus, Martin Kohl and Maria Näther

University of Leipzig

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Overview

- General solution concepts
- Blockating and Pareto efficiency
- Definition of the core
- Necessary condition: Blockating Partitions
- Necessary and sufficient condition: Balanced Sets
- The core of convex games

Solution concepts

Definition

A function φ that assigns every TU-game $v \in \mathbb{V}(N)$ a payoff vector $\varphi(v) \in \mathbb{R}^{|N|}$ is called a **solution function**.

Definition

A correspondence Φ that assigns every TU-game $v \in \mathbb{V}(N)$ a set of payoff vectors $\Phi(v) \subset \mathbb{R}^{|N|}$ is called a **solution correspondence**.

Both structures are called solution concepts.

Blocking coalitions and Pareto efficiency

Definition

Let $v \in \mathbb{V}(N)$. A payoff vector $x \in \mathbb{R}^{|N|}$ is **blocked** by a coalition $S \subseteq N$, if it holds that

$$\sum_{i \in S} x_i < v(S).$$

Definition

Let $v \in \mathbb{V}(N)$. A payoff vector $x \in \mathbb{R}^{|N|}$ is called **Pareto efficient (**Vilfredo Pareto: 1848-1923, Italian Economist**)**, if it holds that

$$\sum_{i \in N} x_i = v(N).$$

The Core 1

These both properties seem quite intuitive. They can describe the well known core (compare Gillies: solutions to general non-zero-sum games 1959).

Definition

Let $v \in \mathbb{V}(N)$. The **core of the game** $v \in \mathbb{V}(N)$ is the set off all non-blocked Pareto efficient allocations, s.t.

$$core(N, v) = \left\{ x \in \mathbb{R}^{|N|} \,\middle|\, \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \forall S \subset N \right\}.$$

Example

The gloves game with two left hand gloves (player 1 and 2) and one right hand glove (player 3).

$$v(S) = \begin{cases} 0 & S \in \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}\} \\ 1 & S \in \{\{1,3\}, \{2,3\}, \{1,2,3\}\} \end{cases}$$

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_1 + x_3 &\geq 1 \\ x_2 + x_3 &\geq 1 \end{aligned}$$

$$\Longrightarrow \; x_3 \geq 1$$

$$core(N, v) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Plausibility? Scarcity of right hand gloves? Quite unfair allocation!

The Core 2

There is another argument against using the core in practise, often there is not only one allocation lying in the core.

Example

The gloves game with one left hand glove (player 1) and one right hand glove (player 2).

$$v(S) = \begin{cases} 0 & S \in \{\emptyset, \{1\}, \{2\}\} \\ 1 & S = \{1,2\} \end{cases}$$

$$core(N,v) = \left\{ \left(\begin{array}{c} t \\ 1-t \end{array} \right) \middle| \, t \in [0,1] \right\}.$$

Every payoff can be defended by using the core concept. Difficult in applications.

Biggest problem, the core can be empty.

Example

A majority game with 3 players

$$v(S) = \begin{cases} 0 & |S| \leq 1 \\ 1 & |S| > 1 \end{cases}$$

$$\begin{aligned} x_1 + x_2 &\geq 1 \\ x_1 + x_3 &\geq 1 \\ x_2 + x_3 &\geq 1 \end{aligned}$$

$$\Longrightarrow x_1 + x_2 + x_3 \geq \frac{3}{2} > 1$$

$$core(N, v) = \emptyset.$$

Question: under which condition the core is nonempty?

Conditions for emptiness 1

Problem

Consider $|N| = 3$ *and find a condition so that the core is nonempty!*

Lemma

Let $v \in \mathbb{V}(N)$*. If there is a partition* $\mathcal{P} = \{S_1, ..., S_k\}$ *of the players of* N *with*

$$\sum_{j=1}^{k} v(S_j) > v(N) \qquad (\#)$$

the core of the game (N, v) *is empty.*

Proof: Easy! Inserting $\sum_{i \in S_j} x_i \geq v(S_j)$ contradicts $\sum_{i \in N} x_j = v(N)$.

Conditions for emptiness 2

Fact

Even if the condition (#) is not fulfilled for all partitions $\mathcal{P}$, the core can be empty. Consider the apex game for three or more players

$$v(S) = \begin{cases} 1 & \begin{array}{c} 1 \in S,\ |S| > 1 \textit{ or} \\ S = N \setminus \{1\} \end{array} \\ 0 & \textit{otherwise} \end{cases}$$

For $|N| = 3$ it is the majority game $\Longrightarrow$ the core of the game is empty (compare slide EmpCo #1)

We now search for a sufficient and necessary condition for the nonemptiness of the core (compare Shapley "On balanced sets and cores" 1967)

Definition

A subset $\mathcal{D} = \{D_1, D_2, ..., D_k\}$ of 2^N is called balanced, if there are $\mu_1, ..., \mu_k \in \mathbb{R}_+$ (the nonnegative real numbers) so that for every $i \in N$

$$\sum_{j:i \in D_j} \mu_j = 1.$$

Theorem

Let $v \in \mathbb{V}(N)$. The core of the game v is nonempty, if and only if for any balanced set $\mathcal{D} = \{D_1, D_2, ..., D_k\}$ with scalars $\mu_1, ..., \mu_k \in \mathbb{R}_+$ it holds that

$$\sum_{j=1}^{k} \mu_j v(D_j) \leq v(N).$$

Proof: If the core of the game $v \in \mathbb{V}(N)$ is nonempty, the following optimization problem is solvable with minimum $v(N)$:

$$\begin{aligned} \sum_{i \in N} x_i &\quad\rightarrow\quad \text{min!} \\ \forall S &\quad\subseteq\quad N : \sum_{i \in S} x_i \geq v(S) \end{aligned}$$

The corresponding dual problem is:

$$\begin{aligned} \sum_{S \subseteq N} \lambda_S v(S) &\quad\rightarrow\quad \text{max!} \\ \text{opt} &\quad:\quad v(N) \\ \forall i &\quad\in\quad N : \sum_{S : i \in S} \lambda_S = 1 \\ \forall S &\quad\subseteq\quad N : \lambda_S \geq 0, \end{aligned}$$

which describes all balanced sets, so we are done.

Conditions for Nonemptiness 3

This necessary and sufficient condition seems not really practicable. Lets look for a different condition:
Remember: Let $v \in \mathbb{V}(N)$, $S \subset S' \subseteq N$ and $i \notin S$. If it holds that

$$v(S \cup \{i\}) - v(S) \leq v(S' \cup \{i\}) - v(S')$$

the game is called convex.

Theorem

For convex games the core is nonempty.

One payoff vector of the core for convex games will be analyzed in the next chapter.

Problems

1. Calculate the core for the gloves game with 2 left hand gloves and 3 right hand gloves.
2. Show that every partition describes a balanced set.
3. Let $N = \{1, 2, 3\}$. Is the set $\{\{1, 2\}, \{3\}, \{1, 2, 3\}\}$ balanced?
4. Let $|N| = 3$. The core of the majority game is empty. Find a balanced set $\mathcal{D} = \{D_1, D_2, ..., D_k\}$ and $\lambda = \{\lambda_1, \lambda_2, ..., \lambda_k\}$, such that

$$\sum_{i=1}^{k} \lambda_i v(D_i) > v(N).$$