Applied Cooperative Game Theory

Topic 1: Basic Definitions

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Literature

- Peleg, B. /Sudhölter P.: Introduction to the theory of cooperative games; Springer 2007
  overview of the field of cooperative game theory
- Myerson, R.: Game Theory- Analysis of Conflict; Harvard 1991
  overview of the field of game theory, but main topic is noncooperative game theory
- Wiese, H.: Kooperative Spieltheorie; Oldenbourg 2005
  German text book
- Fischer, G.: Lineare Algebra- Eine Einführung für Studienanfänger
  for repetition of terms like linear mapping, linear independence, basis,...

History

- methods of game theory have been used since the 19th century (Bertrand,. Cournot), but no general theory, just applications to unique problems
- first real theory bases on work of John Neumann (1928) and his book together with Oskar Morgenstern (Theory of Games and Economic Behavior, 1944)
- 8 game theorists won Nobel Memorial Prize in Economic Sciences:
  John Nash, John Harsanyi, Reinhard Selten in 1994
  Robert Aumann and Thomas Scheling in 2005
  Roger Myerson in 2007
  Alvin Roth and Lloyd Shapley in 2012
- Game theory consists of two main theories, non-cooperative game theory and cooperative game theory
- mile stone of non-cooperative game theory: definition of Nash equilibrium by John Nash in 1950
- this theory tries to find optimal actions and strategies, cooperation must be self-enforcing
- in cooperative game theory, cooperation often is self-enforcing. Groups of players and not individual players compete

TU games

- cooperative games with **t**ransferable **u**tility

Definition

A TU game is a pair (N, v), where N is a no-empty and finite set– the **player set** – and
$v \in \mathbb{V}(N) := \{f : 2^N \to \mathbb{R} | f(\emptyset) = 0\}$ – the **characteristic function** (or **coalition function**).

- players: persons, institutions, cost centers ...
- subsets of N/ elements of 2^N: **coalitions**
- set of non-empty subsets: $\mathcal{K}(N) := 2^N \setminus \{\emptyset\}$
- set of all coalition functions on N: $\mathbb{V}(N)$
- coalition function assigns the **worth** $v(K)$ to any coalition K
  - can be distributed among the members arbitrarily
  - note, more is not necessarily better, example costs

Cost Division #1

- doctors with a common secretary or commonly used facilities
- firms organized as a collection of profit-centers
- universities with computing facilities used by several departments or faculties

Definition

For a player set N, let $c : 2^N \to \mathbb{R}_+$ be a coalition function that is called a cost function. On the basis of c, the costs-saving game is defined by $v : 2^N \to \mathbb{R}$ where for each $K \subseteq N$

$$v(K) = \sum_{i \in K} c(\{i\}) - c(K).$$

Cost Division #2

- Two towns A and B plan a water-distribution system.
- Town A could build such a system for itself at a cost of 11 million Euro
- Town B would need 7 million Euro for a system tailor-made to its needs
- The cost for a common water-distribution system is 15 million Euro.
- The cost function is then given by

$$\begin{aligned} c(\{A\}) &= 11, \\ c(\{B\}) &= 7, \\ c(\{A, B\}) &= 15. \end{aligned}$$

- The associated cost-savings game is $v : 2^{\{A,B\}} \to \mathbb{R}$ defined by

$$\begin{aligned} v(\{A\}) &= v(\{B\}) = 0, \\ v(\{A, B\}) &= 11 + 7 - 15. \end{aligned}$$

Example: German Federal Parliament (Deutscher Bundestag)

- distribution of seats (total 631)

1. CDU/CSU (Christian democrats): 311
2. SPD (social democrats): 193
3. DIE LINKE (left party): 64
4. BÜNDNIS 90/DIE GRÜNEN (green party): 63

- simple majority rule: 316 votes required
- player set: $N = \{1, 2, 3, 4\}$
- coalition function:

$$v(K) = \begin{cases} 1, & \text{parties in } K \text{ form a majority,} \\ 0, & \text{otherwise,} \end{cases} \qquad K \subseteq N$$

Example: Weighted voting games

- given by: $[q, w_1, \ldots, w_n]$, $\qquad q, w_1, \ldots, w_n \in \mathbb{R}_+$
- player set: $N = \{1, \ldots, n\}$
- coalition function:

$$v(K) = \begin{cases} 1, & \sum_{i \in K} w_i \geq q, \\ 0, & \sum_{i \in K} w_i < q, \end{cases} \qquad K \subseteq N$$

- winning coalitions: $v(K) = 1$
- loosing coalitions: $v(K) = 0$

Example: Multiple linear regression

- exotic example
- observations of the dependent variable: $(y_1, \ldots, y_T)$
- set of independent variables N
- observations of independent variables: $\left(x_1^i, \ldots, x_T^i\right)$, $i \in N$
- perform linear regression using the independent variables in $K \subseteq N$
- coalition function using the coefficient of determination R^2

$$v(K) = \begin{cases} R^2(K), & K \neq \emptyset, \\ 0 & K = \emptyset, \end{cases} \qquad K \subseteq N$$

Example: Strategic form games

- strategic form game: $\left(N, (S_i)_{i\in N}, (u_i)_{i\in N}\right)$
  - players set N, non-empty, finite
  - strategy sets S_i, $i \in N$, non-empty, finite
  - $K \subseteq N$, $S_K := \prod_{i\in K} S_i$, $S := S_N$
  - payoff functions $u_i : S \to \mathbb{R}$, $i \in N$
  - coalition function:

$$v(K) = \max_{s_K \in S_K} \min_{s_{N\setminus K} \in S_{N\setminus K}} \sum_{i\in K} u_i\left(s_K, s_{N\setminus K}\right), \qquad K \subseteq N$$

Null game, standard games, unanimity games

- Null game: $\mathbf{0} \in \mathbb{V}(N)$, $\mathbf{0}(K) = 0$ for all $K \subseteq N$
- standard game: $e_T \in \mathbb{V}(N)$, $T \in 2^N \setminus \{\emptyset\}$

$$e_T(K) = \begin{cases} 1, & K = T, \\ 0, & K \neq T, \end{cases} \qquad K \subseteq N$$

- unanimity games: $u_T \in \mathbb{V}(N)$, $T \in 2^N \setminus \{\emptyset\}$

$$u_T(K) = \begin{cases} 1, & T \subseteq K, \\ 0, & T \nsubseteq K, \end{cases} \qquad K \subseteq N$$

The vector space of coalition functions on N

- $\mathbb{V}(N)$ is a finite-dimensional real vector space
  - addition: $v + w \in \mathbb{V}(N) : (v+w)(K) = v(K) + w(K)$ for all $K \subseteq N$, $v, w \in \mathbb{V}(N)$
  - scalar multiplication: $\alpha \cdot v \in \mathbb{V}(N) : (\alpha \cdot v)(K) = \alpha \cdot v(K)$ for all $K \subseteq N$, $v \in \mathbb{V}(N)$, $\alpha \in \mathbb{R}$
- dimension: $\left|2^N \backslash \{\emptyset\}\right| = 2^{|N|} - 1$
- bases:
  - standard basis: $(e_T)_{T \in \mathcal{K}(N)}$ (obviously: $v = \sum_{T \in \mathcal{K}(N)} v(T) \cdot e_T$)
  - unanimity basis: $(u_T)_{T \in \mathcal{K}(N)}$ (to be shown!)
  - of course, there are many more, among them the following (interesting) one:

$$w_T(K) = \begin{cases} \binom{|K|}{|T|}^{-1}, & T \subseteq K, \\ 0, & T \nsubseteq K, \end{cases} \qquad K \subseteq N,\ T \in 2^N \backslash \{\emptyset\}$$

Unanimity basis (1)—Harsanyi dividends

- $(u_T)_{T\in 2^N\setminus\{\emptyset\}}$ is a basis $\mathbb{V}(N)$
- any $v \in \mathbb{V}(N)$ has a unique representation

$$v = \sum_{T\in\mathcal{K}(N)} \lambda_T(v)\cdot u_T, \qquad \lambda_T(v) \in \mathbb{R}$$

  - define the so-called **Harsanyi dividends** $\lambda_T(v)$ inductively

$$\begin{aligned} \lambda_T(v) &= v(T), & |T| = 1 \\ \lambda_T(v) &= v(T) - \sum_{S\in\mathcal{K}(T)\setminus\{T\}} \lambda_S(v), & |T| > 1 \end{aligned}$$

  - or explicitly

$$\lambda_T(v) = \sum_{S\in\mathcal{K}(T)} (-1)^{|T|-|S|}\, v(S)$$

Harsanyi dividends: Equivalence #1

- both approaches uniquely define the $\lambda_T(v)$
- **Induction basis**: for $|T| = 1$:
  $\lambda_T(v) = v(T) = (-1)^{1-1} v(T) = \sum_{S\in\mathcal{K}(T)} (-1)^{|T|-|S|} v(S)$
- **Induction hypothesis (H)**: $\lambda_T(v) = \sum_{S\in\mathcal{K}(T)} (-1)^{|T|-|S|} v(S)$ for all $T \in \mathcal{K}(N)$, $|T| \leq t$
- **Induction step**: $|T| = t+1$

$$
\begin{aligned}
\lambda_T(v) &= v(T) - \sum_{S\in\mathcal{K}(T)\setminus\{T\}} \lambda_S(v) \\
&\overset{\mathbf{H}}{=} v(T) - \sum_{S\in\mathcal{K}(T)\setminus\{T\}} \sum_{K\in\mathcal{K}(S)} (-1)^{|S|-|K|} v(K) \\
&= v(T) - \sum_{K\in\mathcal{K}(T)\setminus\{T\}} v(K) \sum_{s=|K|}^{|T|-1} \binom{|T|-|K|}{s-|K|} (-1)^{s-|K|} \\
&= v(T) - \sum_{K\in\mathcal{K}(T)\setminus\{T\}} v(K) \sum_{s=0}^{|T|-1-|K|} \binom{|T|-|K|}{s} (-1)^{s}
\end{aligned}
$$

Harsanyi dividends: Equivalence #2

- the binomial formula implies

$$0 = (1-1)^t = \sum_{k=0}^{t} \binom{t}{k} (-1)^k$$

- this gives

$$\begin{aligned} \lambda_T(v) &= v(T) - \sum_{K \in \mathcal{K}(T)\setminus\{T\}} v(K) \left(- \binom{|T|-|K|}{|T|-|K|} \right) (-1)^{|T|-|K|} \\ &= v(T) + \sum_{K \in \mathcal{K}(T)\setminus\{T\}} (-1)^{|T|-|K|} v(K) \\ &= \sum_{K \in \mathcal{K}(T)} (-1)^{|T|-|K|} v(K) \end{aligned}$$

Unanimity basis #2: System of generators

- if $w = \sum_{T \in \mathcal{K}(N)} \lambda_T(v) \cdot u_T$, for all $K \subseteq N$ we have

$$\begin{aligned} w(K) &= \sum_{T \in \mathcal{K}(N)} \lambda_T(v) \cdot u_T(K) = \sum_{T \in \mathcal{K}(K)} \lambda_T(v) \\ &= \lambda_K(v) + \sum_{T \in \mathcal{K}(K) \setminus \{K\}} \lambda_T(v) \\ &= v(K) - \sum_{T \in \mathcal{K}(K) \setminus \{K\}} \lambda_T(v) + \sum_{T \in \mathcal{K}(K) \setminus \{K\}} \lambda_T(v) \\ &= v(K) \end{aligned}$$

- indeed, $(u_T)_{T \in \mathcal{K}(N)}$ generates $\mathbb{V}(N)$

Unanimity basis #3: Linear independence

- $\mathbb{V}(N)$ has dimension $2^{|N|} - 1$ and the generating system $(u_T)_{T \in \mathcal{K}(N)}$ also has $2^{|N|} - 1$ elements $\Rightarrow$ $(u_T)_{T \in \mathcal{K}(N)}$ is a basis of $\mathbb{V}(N)$
- direct proof: $(u_T)_{T \in \mathcal{K}(N)}$ is linearly independent
- to show: $\mathbf{0} = \sum_{T \in \mathcal{K}(N)} \alpha_T \cdot u_T$, $\alpha_T \in \mathbb{R}$ implies $\alpha_T = 0$ for all $T \in \mathcal{K}(N)$
- induction on $|T|$
  - **Induction basis**: for $|T| = 1$, i.e., $T = \{i\}$ for some $i \in N$ we have

$$0 = \mathbf{0}(\{i\}) = \sum_{T' \in \mathcal{K}(N)} \alpha_{T'} \cdot u_{T'}(\{i\}) = \sum_{T' \in \mathcal{K}(\{i\})} \alpha_{T'} = \alpha_{\{i\}}$$

  - **Induction hypothesis (H)**: $\alpha_T = 0$ for all $T \in 2^N \backslash \{\emptyset\}$, $|T| \leq t$
  - **Induction step**: $|T| = t + 1$

$$0 = \mathbf{0}(T) = \sum_{T' \in \mathcal{K}(N)} \alpha_{T'} \cdot u_{T'}(T) = \sum_{T' \in \mathcal{K}(T)} \alpha_{T'} \overset{\mathbf{H}}{=} \alpha_T$$

Properties of TU games #1

- **Simplicity**: $v(K) \in \{0, 1\}$ for all $K \subseteq N$
- **Non-negativity**: $v(K) \geq 0$ for all $K \subseteq N$
- **Monotonicity**: $v(K) \geq v(S)$ for all $K, S \subseteq N$, $S \subseteq K$
  - monotonicity $\Rightarrow$ non-negativity
- **0-normedness**: $v(\{i\}) = 0$ for all $i \in N$
- **0-normalization**: for $v \in \mathbb{V}(N)$ define $v^0 \in \mathbb{V}(N)$,

$$\begin{aligned} v^0(K) &= v(K) - \sum_{i \in K} v(\{i\}), \qquad K \subseteq N \\ v^0 &= v - \sum_{i \in N} \lambda_{\{i\}}(v) \cdot u_{\{i\}} \end{aligned}$$

Properties of TU games #2

- **Superadditivity**: $v(S \cup K) \geq v(S) + v(K)$ for all $K, S \subseteq N$, $S \cap K = \emptyset$
- **Subadditivity**: $v(S \cup K) \leq v(S) + v(K)$ for all $K, S \subseteq N$, $S \cap K = \emptyset$
  - $v \in \mathbb{V}(N)$ it subadditive $\iff$ $-v \in \mathbb{V}(N)$ is superadditive
- **Modularity**: $v \in \mathbb{V}(N)$ is superadditive and subadditive
  - $\iff$ for all $K \subseteq N$: $v(K) = \sum_{i \in K} v(\{i\})$
- **Symmetry**: $v(K) = v(S)$ for all $S, K \subseteq N$, $|S| = |K|$
  - exists $f : \mathbb{R} \to \mathbb{R}$ such that $v(K) = f(|K|)$ for all $K \subseteq N$

- **Convexity**:
  - Def (i): $v(S \cup K) + v(S \cap K) \geq v(S) + v(K)$ for all $K, S \subseteq N$
  - Def (ii): $v(K \cup \{i\}) - v(K) \geq v(S \cup \{i\}) - v(S)$ for all $i \in N \backslash K$ and $K, S \subseteq N$, $S \subseteq K$
  - We have to check that both definitions are equivalent:
  - (i) $\Rightarrow$ (ii): $S' \subseteq K'$, set in (i) $K = K'$ and $S = S' \cup \{i\}$:

$$\begin{aligned} v\left(S' \cup \{i\} \cup K'\right) + v\left(S' \cup \{i\} \cap K'\right) &\geq v\left(S' \cup \{i\}\right) + v\left(K'\right) \\ v\left(\{i\} \cup K'\right) + v\left(S'\right) &\geq v\left(S' \cup \{i\}\right) + v\left(K'\right) \end{aligned}$$

  - (ii) $\Rightarrow$ (i): $S \backslash K = \{i_1, \ldots, i_n\}$

$$\begin{array}{rrcl} & v(K \cup \{i_1\}) - v(K) & \geq & v((S \cap K) \cup \{i_1\}) - v(S \cap K) \\ & v(K \cup \{i_1, i_2\}) - v(K \cup \{i_1\}) & \geq & v((S \cap K) \cup \{i_1, i_2\}) \\ & & & \vdots \qquad\qquad - v((S \cap K) \cup \{i_1\}) \\ & v(K \cup S \backslash K) - \ldots & \geq & v((S \cap K) \cup S \backslash K) - \ldots \\ \text{sum up:} & v(K \cup S) - v(K) & \geq & v(S) - v(S \cap K) \end{array}$$

Convexity #2

- let $f : \mathbb{R} \to \mathbb{R}$ be convex, i.e., $f(\alpha \cdot x + (1-\alpha) \cdot y) \leq \alpha \cdot f(x) + (1-\alpha) \cdot f(y)$ for all $x, y \in \mathbb{R}$ and $\alpha \in [0,1]$
- let $v \in \mathbb{V}(N)$, $v(K) = f(|K|)$, $K \subseteq N$, i.e., v is symmetric
- then v is convex
- to show: v exhibits non-decreasing "marginal contributions"
- to see it suffices to establish

$$f(x+2) - f(x+1) \geq f(x+1) - f(x), \qquad x \in \mathbb{R}$$

- proof:

$$\begin{aligned}
\frac{1}{2} f(x+2) + \frac{1}{2} f(x) &\geq f\left(\frac{1}{2}(x+2) + \frac{1}{2}x\right) = f(x+1) \\
f(x+2) + f(x) &\geq 2 \cdot f(x+1) \\
f(x+2) - f(x+1) &\geq f(x+1) - f(x)
\end{aligned}$$

Problems

1. Show that $(w_T)_{T\in 2^N\setminus\{\emptyset\}}$,

$$w_T(K) = \begin{cases} \binom{|K|}{|T|}^{-1}, & T \subseteq K, \\ 0, & T \not\subseteq K, \end{cases} \qquad K \subseteq N,\ T \in 2^N \setminus \{\emptyset\},$$

is a basis of $\mathbb{V}(N)$!

2. For $v \in \mathbb{V}(N)$ and $K \subseteq N$, express $v(K)$ in terms of Harsanyi dividends!
3. Show the following statements are equivalent:
   1. $v \in \mathbb{V}(N)$ is modular.
   2. $v(K) = \sum_{i\in K} v(\{i\})$ for all $K \subseteq N$.
   3. $v = \sum_{i\in N} x_i \cdot u_{\{i\}}$ for some $x \in \mathbb{R}^N$.
4. Show that superadditivity does not imply monotonicity!
5. Show that superadditivity and non-negativity imply monotonicity!
6. Show that for any $v \in \mathbb{V}(N)$ one can find a modular game $w \in \mathbb{V}(N)$ such that $v + w$ is monotonic!
7. Show that for any $v \in \mathbb{V}(N)$ one can find a convex game $w \in \mathbb{V}(N)$ such that $v + w$ is convex!