Applied Cooperative Game Theory

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Basic example

- Man owns estate worth 200 euro
- Man dies, leaving debts of 100, 200, 300 euro
- How should the estate be divided?
- First assumptions:
  - The whole estate should be divided
  - Nobody gets more than his claim
  - Everybody gets at least 0.

Classical Solutions

- Aristotle proposes $(100/3, 200/3, 100)$ if the estate is worth 200 euro and $(50, 100, 150)$ if the estate is worth 300 euro
- Talmud proposes $(50, 75, 75)$ if the estate is worth 200 euro and $(50, 100, 150)$ if the estate is worth 300 euro

Basic model proposed by O'Neill

Barry O'Neill: "A problem of rights arbitration from the Talmud" 1982
The set of bankruptcy problems is given by

$$B = \left\{ \left(N, E, (c_i)_{i \in N}\right) \middle| C := \sum_{i \in N} c_i > E \right\},$$

where

- N is the finite player set,
- E is the worth of the estate,
- c_i is the claim of player i.
  A solution to a bankruptcy problem is a mapping $S : B \to \mathbb{R}^N$ such that

$$\begin{aligned} 0 &\leq S_i\left(N, E, (c_j)_{j \in N}\right) \leq c_i, i \in N \\ \sum_{j \in N} S_j\left(N, E, (c_k)_{k \in N}\right) &= E. \end{aligned}$$

Axioms #1

Definition (Equal Treatment of Equals)

For every bankruptcy problem $\left(N, E, (c_k)_{k \in N}\right)$:If $c_i = c_j, i, j \in N, i \neq j$, we then have $S_i\left(N, E, (c_k)_{k \in N}\right) = S_j\left(N, E, (c_k)_{k \in N}\right)$.

Similar to Symmetry in cooperative games.

Definition (Homogenity)

For every bankruptcy problem $\left(N, E, (c_j)_{j \in N}\right)$ and every $a \in \mathbb{R}_{++}$: $S_i\left(N, a \cdot E, (a \cdot c_j)_{j \in N}\right) = a \cdot S_i\left(N, E, (c_j)_{j \in N}\right) \forall i \in N$.

Axioms #2

Definition (Consistency)

For every bankruptcy problem $\left(N, E, (c_j)_{j\in N}\right)$:For every $T \subseteq N, S_i\left(N\backslash T, E - \sum_{j\in S} S_j\left(N, E, (c_k)_{k\in N}\right), (c_j)_{j\in N\backslash T}\right) = S_i\left(N, E, (c_j)_{j\in N}\right) \forall i \in N\backslash S$.

If a subset of players leave the bargaining process with their rewards, the remaining player's award does not change.

Axioms #3- Reasonable Bounds

D.Dominguez, W.Thomson: "A new solution to the problem of adjudicating conflicting claims" 2006

Definition (Reasonable lower bounds on awards)

For every bankruptcy problem $\left(N, E, (c_j)_{j\in N}\right)$ and all $i \in N : S_i\left(N, E, (c_j)_{j\in N}\right) \geq \frac{1}{n}\min\{c_i, E\}$.

Definition (Reasonable lower bounds on loses)

For every bankruptcy problem $\left(N, E, (c_j)_{j\in N}\right)$ and all $i \in N : c_i - S_i\left(N, E, (c_j)_{j\in N}\right) \geq \frac{1}{n}\min\{c_i, C - E\}$.

Axioms #4- Composition

Definition (Composition up)

For every bankruptcy problem $\left(N, E, (c_j)_{j \in N}\right)$ and every $E < E'$and all $i \in N : S_i\left(N, E', (c_j)_{j \in N}\right) =$
$S_i\left(N, E, (c_j)_{j \in N}\right) + S_i\left(N, E' - E, \left(c_j - S_j\left(N, E, (c_k)_{k \in N}\right)\right)_{j \in N}\right)$

Definition (Composition down)

For every bankruptcy problem $\left(N, E, (c_j)_{j \in N}\right)$ and every $E > E'$ and all $i \in N : S_i\left(N, E', (c_j)_{j \in N}\right) = S_i\left(N, E', \left(S_i\left(N, E, (c_j)_{j \in N}\right)\right)\right)$,

Solution #1: The Proportional rule

The easiest way to solve the problem is given by the proportional rule.

Definition (Proportional Rule)

For every bankruptcy problem $\left(N, E, (c_j)_{j \in N}\right)$:

$$P_i\left(N, E, (c_j)_{j \in N}\right) := \frac{E}{\sum_{j \in N} c_j} c_i \forall i \in N.$$

Solution #2: The constrained equal award rule

Definition (constrained equal award rule)

For every bankruptcy problem $\left(N, E, (c_j)_{j\in N}\right)$

$$\begin{aligned} A_i\left(N, E, (c_j)_{j\in N}\right) \quad &: \quad = \min\{c_i, \lambda\} \, \forall i \in N \text{ such that} \\ \sum_{j\in N} \min\{c_j, \lambda\} \quad &= \quad E. \end{aligned}$$

Solution #3: The constrained equal lose rule

Definition (Constrained equal lose rule)

For every bankruptcy problem $\left(N, E, (c_j)_{j\in N}\right)$

$$L_i\left(N, E, (c_j)_{j\in N}\right) := \max\{0, c_i - \lambda\}\ \forall i \in N \text{ such that}$$
$$\sum_{j\in N} \max\{0, c_j - \lambda\} = E.$$

Solution #4: Talmud rule

R.J. Aumann & M.Maschler: Game Theoretic Analysis of a Bankruptcy Problem from the Talmud, 1985.

Definition (Talmud rule)

For every bankruptcy problem $\left(N, E, (c_j)_{j \in N}\right)$

$$T_i\left(N, E, (c_j)_{j \in N}\right) \quad : \quad = \begin{cases} \min\{c_i/2, \lambda\} & \text{if } E \leq C/2 \\ \max\{c_i/2, c_i - \mu\} & \text{if } E \geq C/2 \end{cases}$$

such that $\sum_{i \in N} T_i\left(N, E, (c_j)_{j \in N}\right) \quad = \quad E.$

Uniqueness P,L,A

Theorem

The solutions P, L, and A are the only ones that satisfy

- *Equal Treatment of Equals*
- *Homogeneity*
- *Consistency*
- *Composition Up*
- *Composition Down*

Uniqueness T

Theorem

The solution T is the only one that satisfies

- *Consistency*
- *Reasonable Lower Bound On Awards*
- *Reasonable Lower Bound On Loses*

The nucleolus of a cooperative game #1

Definition (excess demand)

Let $x \in \mathbb{R}^N$ and $v \in V(N)$. The excess demand of a coalition S is given by $x(S) := v(S) - \sum_{i \in N} x_i$.

Definition (order)

Let $(x(S))_{S \subseteq N}$ be the vector of excess demands. A non-increasing order of the excess demand x is a function $\theta : 2^N \rightarrow \mathbb{R}^{2^N}$ such that

$$\theta_i(x) \geq \theta_j(x) \, \forall i < j.$$

Definition (lexicographic order)

Let $x, y \in \mathbb{R}^{2^N}$ and $\theta : 2^N \rightarrow \mathbb{R}^{2^N}$ be a non-increasing order. We say x is lexicographically smaller than y if there is an index $k \in N$ such that

$$\begin{aligned} \theta_i(x) &= \theta_i(y) \, \forall i < k \text{ and} \\ \theta_k(x) &< \theta_k(y). \end{aligned}$$

The nucleolus of a cooperative game #2

Definition

Let $v \in V(N)$ be a cooperative game. The nucleolus is the efficient payoff vector which lexicographically minimizes the excess demand, i.e.

$$\text{nucleolus}(N, v) = lex \min_{\substack{x \in \mathbb{R}^N, \\ \sum_{i \in N} x_i = v(N)}} (x(S))_{S \subseteq N}$$

Corollary

If the core of the cooperative game (N, v) is non-empty, the nucleolus of the game (N, v) lies within the core.

The bankruptcy game

Definition

Let $\left(N, E, (c_j)_{j \in N}\right)$ be a bankruptcy problem. The associated bankruptcy game is given by

$$v_{E,c}(S) = \max\left\{0, E - \sum_{i \in N \setminus S} c_i\right\}.$$

Theorem (Aumann & Maschler)

Let $\left(N, E, (c_j)_{j \in N}\right)$ be a bankruptcy problem. The nucleolus of the associated bankruptcy game coincides with the Talmud rule of the bankruptcy problem.

Definition (Random Arrival Rule)

Let $\left(N, E, (c_j)_{j \in N}\right)$ be a bankruptcy problem. The Shapley Value of the associated bankruptcy game is called Random Arrival Rule.