Applied Cooperative Game Theory

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- TU- Games: every coalition can achieve a fixed payoff, which can be allocated in any combination.
- A cooperative game (N, V) with non-transferable utility (non-TU) is described by a non-empty and finite set N (the players) and a mapping, which assigns every coalition $S \subseteq N$ a subset of $\mathbb{R}^S$, which suffices the following conditions

Definition

- $V(\emptyset) = \emptyset$
- $V(T) \neq \emptyset$ for $T \neq \emptyset$
- for any $T \subset N : V(T)$ is convex and closed

For a fixed player set N the set of all games with non-transferable utility is given by $\mathbf{V}(N)$.

Fact

Any coalition function v with transferable utility can be transformed into a coalition function V with non-transferable utility by choosing

$$V(K) = \left\{ x_K \in \mathbb{R}^K \,\middle|\, \sum_{i \in K} x_i \leq v(K) \right\}. \qquad (*)$$

Definition

Given a non-empty and finite set N, a solution for a cooperative game with non-transferable utility is a map which assigns every non-TU game a set of payoff vectors of $\mathbb{R}^{|N|}$.

Remark: Many solutions are functions!

Definition

The core of a game (N, V) with non-transferable utility is the set of all utility vectors $u = (u_i)_{i \in N} \subseteq \mathbb{R}^N$, which satisfy the following properties

- $u \in V(N)$ (feasibility)
- there is no coalition K and no other utility vector $u' = (u'_i)_{i \in N} \subseteq \mathbb{R}^N$, which satisfy $u'_K \in V(K)$ and $u_i \leq u'_i$ for all $i \in K$ and strong inequality for at least one $i \in K$. (non-blocked)

Problem

Find the core for the game (N, V) given by $N = \{1, 2\}$ and

$$V(K) = \begin{cases} \{(x_1, x_2) : x_1 \leq 2, x_2 \leq 2\} & K = \{1\} \\ \{(x_1, x_2) : x_1 \leq 1, x_2 \leq 3\} & K = \{2\} \\ \{(x_1, x_2) : 2x_1 + x_2 \leq 5\} & K = \{1, 2\} \end{cases}.$$

Theorem

Given a TU game (N, v). Consider the TU core of (N, v) and the non-TU core of the associated Non-TU game (N, V), given by (*). Then these cores coincide.

Hence, the Core of a game with non-transferable utility is the natural extension of the core of TU games.

Definition

A game is called bargaining game if there is a finite and non-empty set of players N, a disagreement point $r \in \mathbb{R}^N$ and a set of possible outcomes $S \subseteq \mathbb{R}^N$.

The players bargain for outcomes of S. If they do not reach an agreement, the outcome r is established. In most cases, the set S is chosen as a convex and compact set of outcomes.
For a given cooperative game with non-transferable utility (N, V), we can derive a bargaining game by choosing $r_i = v(i)$, $i \in N$ and $S = \bigcup_{K \subseteq N} V(K)$, where we extend the vectors by some nulls, if needed.
John Nash Jr.: The Bargaining Problem (1950) defined the bargaining solution $\Psi(N, r, S) \in \mathbb{R}^N$ by choosing the following four axioms.

Axiom 1 Pareto efficiency

Definition

For any $u \in S,\ u \geq \Psi(N, r, S) \Rightarrow u = \Psi(N, r, S)$
The players never agree on a payoff, if there is another payoff, which is preferred by all players.

The axiom can be motivated by assuming the players behave collectively rational.

Axiom 2 Invariance to equivalent utility representations

In microeconomics we learned, that an affine linear transformed utility function still represents the same preferences.

Definition

Let $\Psi(N, r, S)$ be a bargaining set. Suppose there are $a \in \mathbb{R}^{|N|},\ a_i > 0$, $b \in \mathbb{R}^{|N|}$ and S' and r' are defined by

$$\begin{aligned} S' &= \left\{ s' \in \mathbb{R}^N : s'_i = a_i s_i + b_i,\ i \in N, s \in S \right\} \\ r'_i &= a_i r_i + b_i,\ i \in N \end{aligned}$$

Then $\Psi_i(N, r', S') = a_i \Psi_i(N, r, S) + b_i,\ i \in N$.

Axiom 3 Symmetry/Anonymity

Definition

Let (N, r, S) be a bargaining set. A solution is called symmetric, if for every permutation σ of N:
$\Psi\left(N, \sigma(r), \sigma(S)\right) = \sigma(\Psi(N, r, S))$,
where $\sigma(S) = \left\{ s' \in \mathbb{R}^N : \text{ for some } s \in S : s'_i = s_{\sigma(i)} \text{ for all } i \in N \right\}$.

The solution does not depend on which player is called player one.

Axiom 4 Independence of irrelevant alternatives

Definition

Let (N, r, S) be a bargaining set. A solution satisfies independence of irrelevant alternatives if for another NTU game (N, r, S'), $S' \subset S$, $\Psi(N, r, S) \in S'$, then $\Psi(N, r, S) = \Psi(N, r, S')$.

Theorem

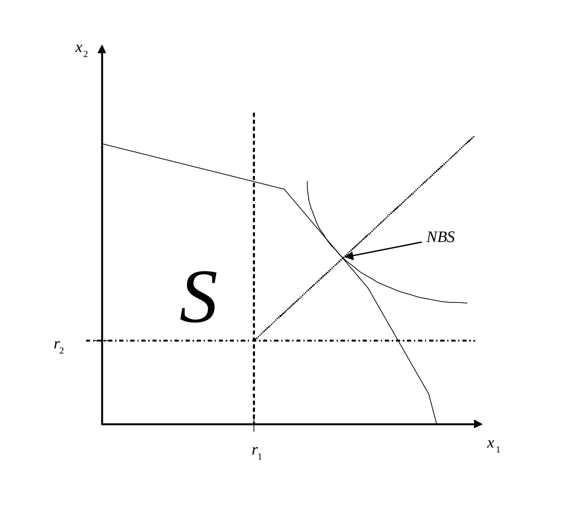
Let (N, r, S) be a bargaining set. There is one bargaining solution, which satisfies

- *Pareto efficiency,*
- *Invariance to equivalent utility representations,*
- *Symmetry and*
- *Independence of irrelevant alternatives.*

The solution is given by the optimization problem

$$NB(N, r, S) = \arg\max_{s \in S} \prod_{i \in N} (s_i - r_i) .$$

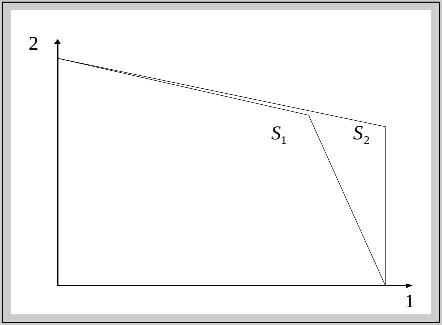
Example



Kalai/Smorodinsky: Other solutions to Nash's bargaining problem (1975)
The Nash bargaining solution does not seem fair in some situations.

Example

$N = 2,\ r = 0,$
$S_1 = conv((1,0)^t, (0,1)^t, (0.75, 0.75)^t)$
$S_2 = conv((1,0)^t, (0,1)^t, (1, 0.7)^t)$



$S_1 \subset S_2$ and player 2 should get more by playing (N, r, S_2) than in the game (N, r, S_1). But the Nash bargaining solutions are:

$$\begin{aligned} NB(N, r, S_1) &= (0.75, 0.75)^t \\ NB(N, r, S_2) &= (1, 0.7)^t. \end{aligned}$$

By playing (N, r, S_2) player 1 gets his maximal payoff and player 2 has to resign some payoff.
Instead of using axiom 4 Kalai and Smorodinsky suggested another axiom:

Definition

A solution Ψ satisfies the Monotonicity axiom, if for any (N, r, S_1) and (N, r, S_2), $S_1 \subseteq S_2$

$$\Psi_i(N, r, S_1) \leq \Psi_i(N, r, S_2) \text{ for every } i \in N.$$

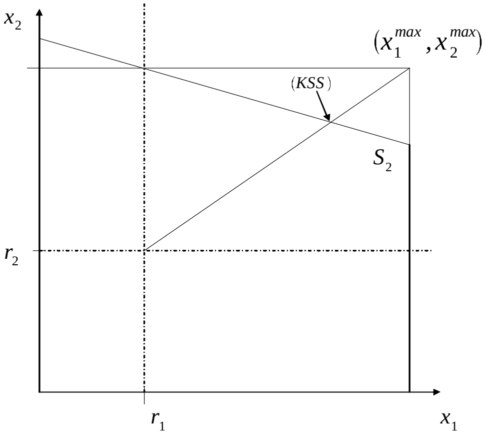
Kalai Smorodinsky Solution

Theorem

Let (N, r, S) be a bargaining set. There is one bargaining solution, which satisfies

- *Pareto efficiency,*
- *Invariance to equivalent utility representations,*
- *Symmetry and*
- *Monotonicity*

The Kalai Smorodinsky solution can be derived graphically.



, where $x_i^{\max} = \max_{s \in S, s \geq r} s_i$.