Applied Cooperative Game Theory

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- $i \in N$, $L \subseteq L^N$: $L_i := \{\lambda \in L | i \in \lambda\}$, set of player i's links in L

Balanced contributions, BC For all $i, j \in N$, $i \neq j$, $v \in \mathbb{V}(N)$, and $L \subseteq L^N$,

$$\varphi_i(N, v, L) - \varphi_i(N, v, L \backslash L_j) = \varphi_j(N, v, L) - \varphi_j(N, v, L \backslash L_i).$$

Proposition μ satisfies **BC**.

- let $\sigma, \rho \in \Sigma(N)$, $\sigma(i) = \rho(j) > \sigma(j) = \sigma(i)$, and $\sigma(\ell) = \rho(\ell)$ for $\ell \in N \backslash \{i, k\}$
- by definition of μ it suffices to show

$$\begin{aligned} & MC_i\left(\sigma, v^L\right) - MC_i\left(\sigma, v^{L \backslash L_j}\right) + MC_i\left(\rho, v^L\right) - MC_i\left(\rho, v^{L \backslash L_j}\right) \\ = \quad & MC_j\left(\sigma, v^L\right) - MC_j\left(\sigma, v^{L \backslash L_i}\right) + MC_j\left(\rho, v^L\right) - MC_j\left(\rho, v^{L \backslash L_j}\right) \end{aligned}$$

- by construction, we have ...

Balanced contributions #2

$$MC_i\left(\rho, v^L\right) - MC_i\left(\rho, v^{L\setminus L_j}\right) = 0 = MC_j\left(\sigma, v^L\right) - MC_j\left(\sigma, v^{L\setminus L_i}\right)$$

$$\begin{aligned}
& MC_i\left(\sigma, v^L\right) - MC_i\left(\sigma, v^{L\setminus L_j}\right) \\
= \quad & v^L\left(K_i\left(\sigma\right)\right) - v^L\left(K_i\left(\sigma\right)\setminus\{i\}\right) - \left(v^{L\setminus L_j}\left(K_i\left(\sigma\right)\right) - v^{L\setminus L_j}\left(K_i\left(\sigma\right)\setminus\{i\}\right)\right) \\
= \quad & v^L\left(K_i\left(\sigma\right)\right) - v^L\left(K_i\left(\sigma\right)\setminus\{i\}\right) \\
& -\left(v\left(\{j\}\right) + v^L\left(K_i\left(\sigma\right)\setminus\{j\}\right) - \left(v\left(\{j\}\right) + v^L\left(K_i\left(\sigma\right)\setminus\{i,j\}\right)\right)\right) \\
= \quad & v^L\left(K_j\left(\rho\right)\right) - v^L\left(K_j\left(\rho\right)\setminus\{i\}\right) \\
& -\left(v\left(\{i\}\right) + v^L\left(K_j\left(\rho\right)\setminus\{j\}\right) - \left(v\left(\{i\}\right) + v^L\left(K_j\left(\rho\right)\setminus\{i,j\}\right)\right)\right) \\
= \quad & v^L\left(K_j\left(\rho\right)\right) - v^L\left(K_j\left(\rho\right)\setminus\{j\}\right) \\
& -\left(v\left(\{i\}\right) + v^L\left(K_j\left(\rho\right)\setminus\{i\}\right) - \left(v\left(\{i\}\right) + v^L\left(K_j\left(\rho\right)\setminus\{i,j\}\right)\right)\right) \\
= \quad & v^L\left(K_j\left(\rho\right)\right) - v^L\left(K_j\left(\rho\right)\setminus\{j\}\right) \\
& -\left(v^{L\setminus L_i}\left(K_j\left(\rho\right)\right) - v^{L\setminus L_i}\left(K_j\left(\rho\right)\setminus\{j\}\right)\right) = MC_j\left(\rho, v^L\right) - MC_j\left(\rho, v^{L\setminus L_i}\right)
\end{aligned}$$

Theorem μ is the unique CO-value that satisfies **CE** and **BC**.

- we already know that μ satisfies **CE** and **BC**
- let φ, ψ both satisfy **CE** and **BC**, but $\varphi \neq \psi$
- there is some smallest L such that $\varphi_i(N, v, L) \neq \psi_i(N, v, L)$ for some $i \in N$
- by **CE**, $|C_i(N, L)| > 1$
- obviously, $\left|L \backslash L_j\right| < |L|$ and $|L \backslash L_i| < |L|$
- by the minimality of L, for all $i, j \in C := C_i(N, L)$, we have

$$\begin{aligned} \varphi_i(N, v, L) - \varphi_j(N, v, L) &\overset{\textbf{BC}}{=} \varphi_i\left(N, v, L \backslash L_j\right) - \varphi_j(N, v, L \backslash L_i) \\ &= \psi_i\left(N, v, L \backslash L_j\right) - \psi_j(N, v, L \backslash L_i) \\ &\overset{\textbf{BC}}{=} \psi_i(N, v, L) - \psi_j(N, v, L) \qquad (*) \end{aligned}$$

- summing up (*) over $j \in C$ gives

$$|C| \cdot \varphi_i(N, v, L) - \varphi_C(N, v, L) = |C| \cdot \psi_i(N, v, L) - \psi_C(N, v, L)$$

- by **CE**, $\varphi_C(N, v, L) = \psi_C(N, v, L) = v(C)$, hence,

$$\varphi_i(N, v, L) = \psi_i(N, v, L)$$

- contradiction □

The position value: Motivation

- consider the CO-game (N, u_N, L), $N = \{1, 2, 3\}$, $L = \{12, 23\}$, i.e.

$$L = \quad \overset{1}{\bullet} \;—\; \overset{2}{\bullet} \;—\; \overset{3}{\bullet}$$

- this gives the Myerson payoffs

$$\mu(N, u_N, L) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

- the central/connecting role of player 2 is not accounted for by μ
- both links are necessary to "create" the worth of $u_N(N) = 1$
- hence any link should earn $\frac{1}{2}$, which should be divided equally among the players forming this link
- this gives the payoffs

$$\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$$

- this gives rise to the position value

- Meessen, R. (1988): Communication games, Master's thesis, Department of Mathematics. University of Nijmegen, the Netherlands (in Dutch)
- Borm, P., Owen, G., Tijs, S. (1992): On the position value for communication situations. SIAM Journal of Discrete Mathematics 5:305–320
- definition refers to 0-normalized games, $v(\{i\}) = 0$, $i \in N$, but can easily be extended to arbitrary games
- any $v \in \mathbb{V}(N)$ can be 0-normalized as follows
  - $\mathbb{V}_0(N) = \{v \in \mathbb{V}(N) \,|\, v \text{ is 0-normalized}\}$
  - define $v_0 \in \mathbb{V}_0(N)$ by

$$v_0 = v - \sum_{i \in N} v(\{i\}) \cdot u_{\{i\}} = v - \sum_{i \in N} \lambda_{\{i\}}(v) \cdot u_{\{i\}}$$

  - define: $\pi(N, v, L) = \pi(N, v_0, L) + v(\{i\})$

- for (N, v, L), $v \in \mathbb{V}_0(N)$ define the TU game (link game) $\left(L, v^N\right)$
  - the links are now the players
  - for $L' \subseteq L$, define $v^N(L') = v^{L'}(N)$
  - note, since $v \in \mathbb{V}_0(N)$, $v^N(\emptyset) = v^{\emptyset}(N) = \sum_{i \in N} v(\{i\}) = 0$

Definition. The position value assigns to any CO-game (N, v, L), $v \in \mathbb{V}_0(N)$ and $i \in N$ the payoff

$$\pi_i(N, v, L) := \sum_{\lambda \in L_i} \frac{1}{2} \mathrm{Sh}_\lambda(L, v^N).$$

- the players get half of the Shapley payoffs of their links in the link game
- from the leading example, it is clear that $\pi \neq \mu$

Proposition. π satisfies **CE**.

- let $C \in \mathcal{C}(N, v)$. then, $\pi_C(N, v, L) = \sum_{i\in C}\sum_{\lambda\in L_i}\frac{1}{2}\mathrm{Sh}_\lambda(L, v^N)$

$$
\begin{aligned}
&= \sum_{\lambda\in L|_C}\mathrm{Sh}_\lambda(L, v^N) = \sum_{\lambda\in L|_C}\frac{1}{|\Sigma(L)|}\sum_{\sigma\in\Sigma(L)} MC_\lambda^{v^N}(\sigma)\\
&= \frac{1}{|\Sigma(L)|}\sum_{\sigma\in\Sigma(L)}\sum_{\lambda\in L|_C} MC_\lambda^{v^N}(\sigma)\\
&= \frac{1}{|\Sigma(L)|}\sum_{\sigma\in\Sigma(L)}\sum_{\lambda\in L|_C}\left(v^N(K_\lambda(\sigma)) - v^N(K_\lambda(\sigma)\setminus\{\lambda\})\right)\\
&= \frac{1}{|\Sigma(L)|}\sum_{\sigma\in\Sigma(L)}\sum_{\lambda\in L|_C}\left(v^{K_\lambda(\sigma)}(N) - v^{K_\lambda(\sigma)\setminus\{\lambda\}}(N)\right)\\
&= \frac{1}{|\Sigma(L)|}\sum_{\sigma\in\Sigma(L)}\sum_{\lambda\in L|_C}\left(v^{K_\lambda(\sigma)}(C) - v^{K_\lambda(\sigma)\setminus\{\lambda\}}(C)\right)\\
&= \frac{1}{|\Sigma(L)|}\sum_{\sigma\in\Sigma(L)} v(C) = v(C) \qquad \square
\end{aligned}
$$

- from the leading example, it is easy to check that π fails **F** as well as **BC**

Balanced link contributions, BLC For all $i, j \in N$, $i \neq j$, $v \in \mathbb{V}_0(N)$, and $L \subseteq L^N$,

$$\sum_{\lambda \in L_j} \left(\varphi_i(N, v, L) - \varphi_i(N, v, L - \lambda)\right) = \sum_{\lambda \in L_i} \varphi_j(N, v, L) - \varphi_j(N, v, L - \lambda).$$

- Slikker, M. (2005): A characterization of the position value, International Journal of Game Theory 33: 505–514

Proposition (Slikker 2005) π satisfies **BLC**.

Theorem (Slikker 2005) π is the unique CO-value that satisfies **CE** and **BLC**.

- we already know that π satisfies **CE** and **BLC**
- let φ, ψ both satisfy **CE** and **BLC**, but $\varphi \neq \psi$
- there is some smallest L such that $\varphi_i(N, v, L) \neq \psi_i(N, v, L)$ for some $i \in N$
- by **CE**, $|C_i(N, L)| > 1$
- hence, $|L \backslash L_j| < |L|$ and $|L \backslash L_i| < |L|$
- by the minimality of L, for all $i, j \in C := C_i(N, L)$, we have

$$\begin{aligned}
& \sum_{\lambda \in L_j} \varphi_i(N, v, L) - \sum_{\lambda \in L_i} \varphi_j(N, v, L) \\
\overset{\textbf{BLC}}{=} \; & \sum_{\lambda \in L_j} \varphi_i(N, v, L - \lambda) - \sum_{\lambda \in L_i} \varphi_j(N, v, L - \lambda) \\
= \; & \sum_{\lambda \in L_j} \psi_i(N, v, L - \lambda) - \sum_{\lambda \in L_i} \psi_j(N, v, L - \lambda) \\
\overset{\textbf{BLC}}{=} \; & \sum_{\lambda \in L_j} \psi_i(N, v, L) - \sum_{\lambda \in L_i} \psi_j(N, v, L) \qquad (*)
\end{aligned}$$

- summing up (*) over $j \in C$ gives

$$\begin{aligned} & |C| \cdot \sum_{\lambda \in L_j} \varphi_i(N, v, L) - \sum_{\lambda \in L_i} \varphi_C(N, v, L) \\ = \quad & |C| \cdot \sum_{\lambda \in L_j} \psi_i(N, v, L) - \sum_{\lambda \in L_i} \psi_C(N, v, L) \end{aligned}$$

- by **CE**, $\varphi_C(N, v, L) = \psi_C(N, v, L) = v(C)$, hence,

$$\varphi_i(N, v, L) = \psi_i(N, v, L)$$

- contradiction □